1. a) $\underbrace{d\sigma}_{GeV^{-2}} = \underbrace{\frac{1}{2s}}_{GeV^{-2}} |A(i \to f_n)|^2 d\Pi_n$

Dimension of the phase space element:

$$d\Pi_n = \left[\prod_{m=1}^{n} \underbrace{\frac{d^4 p_m}{(2\pi)^4} 2\pi \, \delta(p_m^2 - m_m^2)}_{\frac{d^3 p_m}{(2\pi)^3 2E_m}} \right] (2\pi)^4 \delta^4 \left(p_a + p_b - \sum_{i=1}^{n} p_n \right)$$

GeV^2

[note: $\int d^4 k \, \delta(k^2 - m^2) = \int d^3 \bar{k} \, d\tilde{E} \, \delta(\tilde{E}^2 - \bar{k}^2 - m^2)$
$= \int d^3 \bar{k} \frac{1}{2E}$
where $E = \sqrt{\bar{k}^2 + m^2}$, thus
$d^4 p_m = \frac{d^3 p_m}{2E_m}$]

Thus dimension of $d\Pi_n$ is

$[GeV^2]^n \; GeV^{-4}$,

as dim. of $\delta^4(p_a + p_b - \sum_n p_n)$ is GeV^{-4}

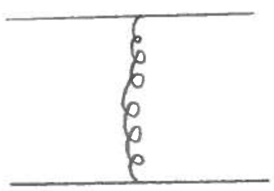
($\delta(ax) = \frac{1}{|a|}\delta(x)$)

thus dim. of $d\Pi_n$ is GeV^{2n-4}

Let d be dim. of $|A|^2$ Now

$GeV^{-2} = GeV^{-2} \cdot d \cdot GeV^{2n-4} \Rightarrow \underline{d = GeV^{4-2n}}$, dim. of A: $\underline{GeV^{2-n}}$

b) $qq \to qq$



External leg: spinor u/v $\sim \text{GeV}^{1/2}$

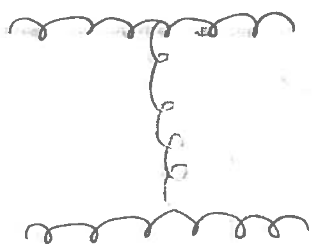
qqg - vertex: 1

Gluon propagator $\sim \text{GeV}^{-2}$

Dimension of A: $(\text{GeV}^{1/2})^4 \, \text{GeV}^{-2} = 1$

Dimensionless.

$gg \to gg$



External legs: polarisation tensor, dimensionless

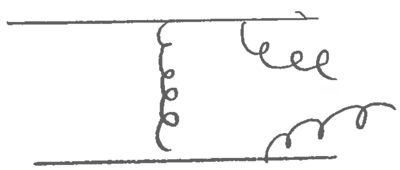
Gluon propagator $\sim \text{GeV}^{-2}$

ggg-vertex $\sim \text{GeV}$

Dimension of A: $1^4 \cdot (\text{GeV})^2 \cdot \text{GeV}^{-2} = 1$

Dimensionless.

$qq \to qqgg$



External quarks: $\text{GeV}^{1/2}$

External gluons 1

Gluon propagator GeV^{-2}

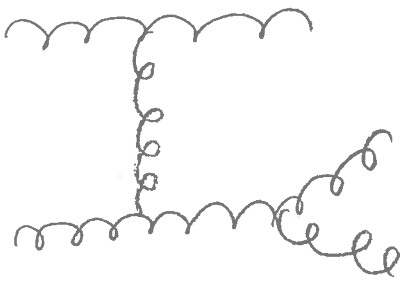
qqg vertex: 1

quark propagator GeV^{-1}

Dimension of A: $(\text{GeV}^{1/2})^2 \, \text{GeV}^{-2} \, (\text{GeV}^{-1})^2 = \text{GeV}^{-2}$

1b cont.

$gg \to ggg$



ggg vertex: GeV

Gluon propagator GeV^{-2}

External legs: 1

Dimension of A: $(GeV)^3(GeV^{-2})^2 = GeV^{-1}$

ok!

2. $\mathrm{Tr}\; t_a t_b = \frac{1}{2}\delta_{ab}$ $\qquad [t_a, t_b] = i f_{abc} t_c$

$t_a t_a = C_F \, \mathbb{1}_{N_c \times N_c}$ $\qquad | \mathrm{Tr}$ $\qquad$ (note: sum over a)

$\mathrm{Tr}(t_a t_a) = C_F \, \mathrm{Tr}(\mathbb{1}_{N_c \times N_c}) = N_c C_F$

$\frac{1}{2}(N_c^2 - 1) = N_c C_F$

$C_F = \frac{N_c^2 - 1}{2N_c}$

$(T_a)_{bc} = -i f_{abc}$

$f_{abc} f_{abd} = C_A \delta_{cd}$ $\qquad | \cdot t_c t_d$

$\underbrace{f_{abc} t_c}_{-i[t_a, t_b]} \underbrace{f_{abd} t_d}_{-i[t_a, t_b]} = C_A t_c t_d \delta_{cd}$ $\qquad | \mathrm{Tr}$

$- \mathrm{Tr}\left[(t_a t_b - t_b t_a)(t_a t_b - t_b t_a) \right] = C_A \underbrace{\mathrm{Tr}(t_c t_d)\delta_{cd}}_{\mathrm{Tr}(t_c t_c) = N_c C_F}$

$= -\mathrm{Tr}(t_a t_b t_a t_b) + \mathrm{Tr}(t_a \overbrace{t_b t_b}^{C_F \mathbb{1}_{N_c \times N_c}} t_a) + \mathrm{Tr}(t_b t_a t_a t_b)$

$- \underbrace{\mathrm{Tr}(t_b t_a t_b t_a)}_{\mathrm{Tr}(t_a t_b t_a t_b)} = N_c C_F C_A$

$-2\,\mathrm{Tr}(t_a t_b t_a t_b) + 2 C_F^2 \,\mathrm{Tr}(\mathbb{1}_{N_c \times N_c}) = N_c C_F C_A$

$$\mathrm{Tr}\,(t_a t_b t_a t_b) = N_c\, C_F^{\,2} - \frac{1}{2} N_c\, C_F\, C_A$$

$$= N_c \left(\frac{N_c^2-1}{2N_c}\right)^2 - \frac{1}{2} N_c^2\, \frac{N_c^2-1}{2N_c}$$

$$= \frac{(N_c^2-1)^2}{4N_c} - \frac{N_c}{4}(N_c^2-1)$$

$$= -\frac{2}{3} \quad \text{for } N_c = 3$$

How to get C_A:

- Group theory proof: Peskin p. 501
- Direct calculation using the Fierz identity:

2. Previously we found

$$-2\,\mathrm{tr}(t^a t^b t^a t^b) = N_c C_F C_A - 2 C_F^2 N_c \qquad (*)$$

using the Fierz identity (derivation: Peskin p. 608)

$$t^a_{ij} t^a_{kl} = \frac{1}{2}(\delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl})$$

we get

$$\mathrm{tr}(t^a t^b t^a t^b) = t^a_{ij} t^b_{jk} t^a_{kl} t^b_{li}$$

$$= \frac{1}{2}\left(\delta_{il}\delta_{kj} - \frac{1}{N_c}\delta_{ij}\delta_{kl}\right)\frac{1}{2}\left(\delta_{ji}\delta_{kl} - \frac{1}{N_c}\delta_{jk}\delta_{li}\right)$$

$$= \frac{1}{4}\Big[\delta_{il}\delta_{kj}\delta_{ji}\delta_{kl} - \frac{1}{N_c}\delta_{ij}\delta_{kl}\delta_{ji}\delta_{kl} - \frac{1}{N_c}\delta_{il}\delta_{kj}\delta_{jk}\delta_{li}$$

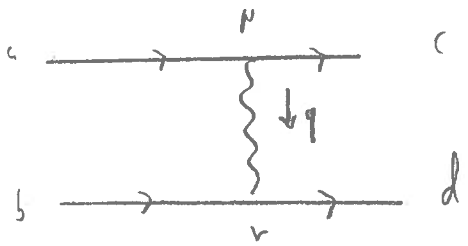
$$+ \frac{1}{N_c^2}\delta_{ij}\delta_{kl}\delta_{jk}\delta_{li}\Big]$$

$$= \frac{1}{4}\left[\delta_{ll} - \frac{2}{N_c}\delta_{ii}\delta_{kk} + \frac{N_c}{N_c^2}\right]$$

$$= \frac{1}{4}\left[N_c - \frac{2}{N_c}N_c^2 + \frac{N_c}{N_c^2}\right]$$

substituting this to $(*)$ and solving C_A gives $\underline{C_A = N_c}$

3.



Denote: $\bar{u}_a = \bar{u}_{s_a}(p_a)$

spin of particle $i = s_i$

$$-iA = \bar{u}_c(-ie\gamma^\mu)u_a \; \bar{u}_d(-ie\gamma^\nu)u_b \; \frac{-ig_{\mu\nu}}{q^2}$$

Sum over final state spis
Average over initial state spins

$$|A|^2 = \frac{e^4}{q^4}\left(\bar{u}_c\gamma^\mu u_a \; \bar{u}_d\gamma_\mu u_b\right)^*\left(\bar{u}_c\gamma^\nu u_a \bar{u}_d\gamma_\nu u_b\right)$$

Use: $(\bar{u}_c\gamma^\mu u_a)^* = (\bar{u}_c\gamma^\mu u_a)^\dagger = u_a^\dagger \underbrace{(\gamma^\mu)^\dagger}_{\gamma^0\gamma^\mu\gamma^0} \underbrace{\bar{u}_c^{\,\dagger}}_{(u_c^\dagger\gamma^0)^\dagger = \gamma^0 u_c}$

$$= \underbrace{u^\dagger_a\gamma^0}_{\bar{u}_a}\gamma^\mu\underbrace{\gamma^0\gamma^0}_{\mathbb{1}}u_c = \bar{u}_a\gamma^\mu u_c$$

Thus $$|A|^2 = \frac{e^4}{q^4}\bar{u}_b\gamma_\mu u_d \; \bar{u}_a\gamma^\mu u_c \bar{u}_c\gamma^\nu u_a \underbrace{\bar{u}_d\gamma_\nu u_b}_{\in\mathbb{C}}$$

And

$$\overline{|A|} = \frac{1}{4}\sum_{\substack{s_a s_b\\ s_c s_d}}|\overline{A}| = \frac{1}{4}\frac{e^4}{q^4}\sum_{s_b s_d}\bar{u}_b\gamma_\mu u_d \bar{u}_d\gamma_\nu u_b$$

$$\times\sum_{s_a s_c}\bar{u}_a\gamma^\mu u_c \bar{u}_c\gamma^\nu u_a$$

Use: $\sum_s u_s\bar{u}_s = \not{p} + m \simeq \not{p}$

$$|\overline{A}|^2 = \frac{e^4}{4q^4} \sum_{s_b} \bar{u}_b \gamma_\mu \not{p}_d \gamma_\nu u_b$$

$$\sum_{s_a} \bar{u}_a \gamma^\mu \not{p}_c \gamma^\nu u_a$$

Note: $\sum_{s_a} \bar{u}_a \gamma^\mu \not{p}_c \gamma^\nu u_a = \sum_{s_a} (\bar{u}_a)_i (\gamma^\mu)_{ij} (\not{p}_c)_{jk} (\gamma^\nu)_{kl} (u_a)_l$

$$= \underbrace{\sum_{s_a} (u_a)_l (\bar{u}_a)_i}_{(\not{p}_a + m)_{li}} (\gamma^\mu)_{ij} (\not{p}_c)_{jk} (\gamma^\nu)_{kl}$$

$$\approx \mathrm{Tr}(\not{p}_a \gamma^\mu \not{p}_c \gamma^\nu)$$

$$= p_{a,\alpha} \, p_{c,\beta} \underbrace{\mathrm{Tr}(\gamma^\alpha \gamma^\mu \gamma^\beta \gamma^\nu)}_{4(g^{\alpha\mu} g^{\beta\nu} - g^{\alpha\beta} g^{\mu\nu} + g^{\alpha\nu} g^{\beta\mu})}$$

$$= 4(p_a^\mu p_c^\nu - p_a \cdot p_b \, g^{\mu\nu} + p_a^\nu p_b^\mu)$$

Similarly $\sum_{s_b} \bar{u}_b \gamma_\mu \not{p}_d \gamma_\nu u_b$

$$= 4(p_{b,\mu} p_{d,\nu} + p_{b,\nu} p_{d,\mu} - p_d \cdot p_b \, g_{\mu\nu})$$

Thus

$$\overline{|A|}^2 = 4\frac{e^4}{q^4}\left(p_a^{\mu} p_c^{\nu} + p_a^{\nu} p_c^{\mu} - p_a\cdot p_c\, g^{\mu\nu}\right)$$
$$\left(p_{b,\mu} p_{d,\nu} + p_{b,\nu} p_{d,\mu} - p_b\cdot p_d\, g_{\mu\nu}\right)$$

$$= 4\frac{e^4}{q^4}\Big[\, p_a\cdot p_b\ p_c\cdot p_d + p_a\cdot p_d\ p_c\cdot p_b$$
$$- p_a\cdot p_c\ p_b\cdot p_d$$

$$+ p_a\cdot p_d\ p_c\cdot p_b + p_a\cdot p_b\ p_c\cdot p_d - p_a\cdot p_c\ p_b\cdot p_d$$

$$- p_a\cdot p_c\ p_b\cdot p_d - p_a\cdot p_c\ p_b\cdot p_d + p_a\cdot p_c\ p_b\cdot p_d\, \underbrace{g^{\mu\nu} g_{\mu\nu}}_{4}\Big]$$

$$= 4\frac{e^4}{q^4}\, 2\left(p_a\cdot p_b\ p_c\cdot p_d + p_a\cdot p_d\ p_c\cdot p_b\right)$$

In massless limit
$$s = (p_a + p_b) = 2\, p_a\cdot p_b = 2\, p_c\cdot p_d$$
$$t = (p_a - p_c) = -2\, p_a\cdot p_c = -2\, p_b\cdot p_d$$
$$u = (p_a - p_d) = -2\, p_a\cdot p_d = -2\, p_b\cdot p_c$$

$$\overline{|A|}^2 = \frac{8e^4}{q^4}\left(\frac{s}{2}\,\frac{s}{2} + \frac{-u}{2}\,\frac{-u}{2}\right)$$

using $q^2 = (p_c - p_a)^2 = t$ we get

$$|\overline{A}|^2 = \frac{2e^4}{t^2}(s^2+u^2)$$

High energy limit $s \sim -u$

$$|\overline{A}|^2 = \frac{2e^4}{t^2} 2s^2 = 4e^4 \frac{s^2}{t^2}$$

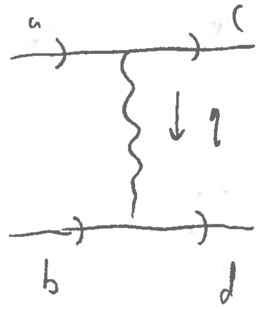
$$\frac{d\sigma}{dt} = \frac{1}{16\pi s^2} |A|^2 = \frac{e^4}{8\pi s^2 t^2}(s^2+u^2)$$

$$\approx \frac{e^4}{4\pi t^2}$$ in the high-energy limit.

using $\alpha = \frac{e^2}{4\pi}$ we get

$$\frac{d\sigma}{dt} \simeq \frac{4\pi \alpha^2}{t^2}$$

4. Massless scalar $\begin{cases}\text{propagator} \\ \text{"photon"}\end{cases}$ $\frac{-i}{q^2}$



vertex: $-ie$

$$-iA = \bar{u}_c (-ie) u_a \frac{-i}{q^2} \bar{u}_d (-ie) u_b$$

$$= \frac{e^2}{q^2} \bar{u}_c u_a \bar{u}_d u_b$$

$$\overline{|A|}^2 = \frac{1}{4} \sum_{\substack{s_a s_b \\ s_c s_d}} (\bar{u}_c u_a \bar{u}_d u_b)^* (\bar{u}_c u_a \bar{u}_d u_b) \frac{e^4}{q^4}$$

using $(\bar{u}_c u_a)^* = (\bar{u}_c u_a)^\dagger = u_a^\dagger (u_c^\dagger \gamma^0)^\dagger$
$= \bar{u}_a u_c$

we get

$$\overline{|A|}^2 = \frac{1}{4} \frac{e^4}{q^4} \sum_{s_a s_c} \bar{u}_a u_c \bar{u}_c u_a \sum_{s_b s_d} \bar{u}_b u_d \bar{u}_d u_b$$

and

$$\sum_{s_a s_c} \bar{u}_a u_c \bar{u}_c u_a = \sum_{s_a} \bar{u}_a \not{p}_c u_a = \sum_{s_a} \bar{u}_{a,i} (\not{p}_c)_{ij} (u_a)_j$$

$$= \sum_{s_a} (u_a)_j (\bar{u}_a)_i (\not{p}_c)_{ij} = (\not{p}_a)_{ji} (\not{p}_c)_{ij}$$

$$= \mathrm{Tr}(\not{p}_a \not{p}_c) = p_a^\mu p_c^\nu \,\mathrm{Tr}(\gamma^\mu \gamma^\nu) = 4\, p_a \cdot p_c$$

$$\overline{|A|}^2 = 4\frac{e^4}{q^4}\underbrace{p_a \cdot p_c}_{-\frac{t}{2}} \underbrace{p_b \cdot p_d}_{-\frac{t}{2}}$$

$$\frac{d\sigma}{dt} = \frac{1}{16\pi s^2} \frac{e^4}{t^2} \frac{t^2}{4} = \frac{e^4}{16\pi s^2} \to 0$$ in the high-energy limit.

5.

$$(\omega^2+\nabla^2)\, G(x,y) = -\delta^3(x-y) \qquad (\nabla = \nabla_x)$$

$$G(x,y) = \int \frac{d^3p}{(2\pi)^3}\, e^{ip\cdot(x-y)}\, G(p)$$

$$(\omega^2+\nabla^2)\, G(x,y) = \int \frac{d^3p}{(2\pi)^3}\, (\omega^2-p^2)\, e^{ip\cdot(x-y)}\, G(p)$$

$$= -\delta^3(x-y) = \int \frac{d^3p}{(2\pi)^3}\, e^{-ip\cdot(x-y)}$$

Thus

$$G(p) = \frac{1}{p^2-\omega^2} \rightarrow \frac{1}{p^2-(\omega+i\epsilon)^2}$$

Now $G(x,y) = \int \frac{d^3p}{(2\pi)^3} \frac{e^{ip\cdot(x-y)}}{p^2-(\omega+i\epsilon)^2}$

$$= \frac{1}{(2\pi)^2}\int dp\, p^2 \int_{-1}^{1} d(\cos\theta)\, \frac{e^{ip|x-y|\cos\theta}}{p^2-(\omega+i\epsilon)^2}$$

$$= \frac{-i}{(2\pi)^2}\frac{1}{|x-y|}\int_0^\infty dp\, p\left[\frac{e^{ip|x-y|}}{p^2-(\omega+i\epsilon)} - \frac{e^{-ip|x-y|}}{p^2-(\omega+i\epsilon)}\right]$$

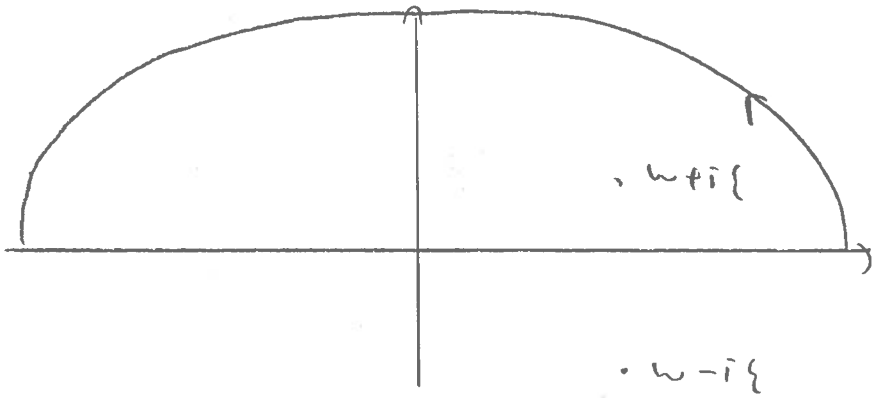
change $p \rightarrow -p$

int. limits $\int_0^{-\infty}$

$$= \frac{-i}{(2\pi)^2}\frac{1}{|x-y|}\int_{-\infty}^{\infty} dp\, p\, \frac{e^{ip|x-y|}}{p^2-(\omega+i\epsilon)}$$

Integrand has poles at $p = \omega + i\epsilon$ and
$p = -\omega - i\epsilon$, as

$$p^2 - (\omega + i\epsilon)^2 = [p + (\omega + i\epsilon)][p - (\omega + i\epsilon)]$$



Integrand vanishes on the upper half plane at infinity. Thus using the residue theorem we obtain

$$\underbrace{\int dp \; p \frac{e^{ip|x-y|}}{p^2 - (\omega + i\epsilon)^2}}_{f(p)} = 2\pi i \operatorname*{Res}_{p \to \omega + i\epsilon} f(p)$$

$$= 2\pi i \frac{(\omega + i\epsilon)\, e^{i(\omega + i\epsilon)|x-y|}}{2(\omega + i\epsilon)}$$

$$\overset{\epsilon \to 0}{=} i\pi \, e^{i\omega |x-y|}$$

Thus $\quad G(x,y) = \dfrac{-i}{(2\pi)^2} \dfrac{1}{|x-y|} \, i\pi \, e^{i\omega|x-y|}$

$$= \frac{e^{i\omega|x-y|}}{4\pi |x-y|}$$

5. Cont.

using regularization $\omega \to \omega - i\varepsilon$ the poles would be $\omega - i\varepsilon$ and $-\omega + i\varepsilon$, closing contour on the upper half plane again gives $-\omega$ instead of ω

$\sim e^{-i\omega|x-y|}$ result, corresponding to a wave propagating in an opposite direction.

6.

a) $$\frac{d\sigma_{el}}{d^2 q_T} = \left| \frac{i}{2\pi} \int d^2 b_T \, e^{-i q_T \cdot b_T} \, \Gamma(b_T) \right|^2$$

$$\sigma_{el} = \int d^2 q_T \, \frac{1}{4\pi^2} \int d^2 b_T \, d^2 b_T' \, e^{-i q_T \cdot b_T} \, \Gamma(b_T)$$

$$\times \, e^{i q_T \cdot b_T'} \, \Gamma^*(b_T')$$

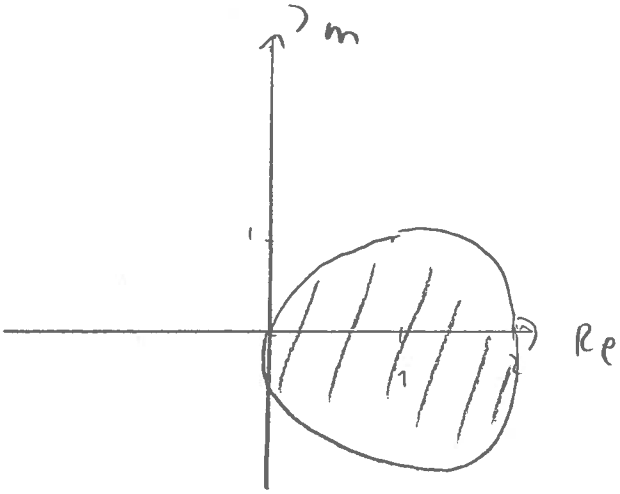
$$= \frac{1}{4\pi^2} \int d^2 b_T \, d^2 b_T' \, \underbrace{d^2 q_T \, e^{i q_T \cdot (b_T - b_T')}}_{(2\pi)^2 \delta(b_T - b_T')} \, \Gamma(b_T) \Gamma^*(b_T')$$

$$= \int d^2 b_T \, |\Gamma(b_T)|^2$$

b) Unitarity: $|\Gamma(b_T)|^2 \leq 2\,\mathrm{Re}\,\Gamma(b_T)$

$$\mathrm{Re}\left[\Gamma(b_T)\right]^2 + \mathrm{Im}\left[\Gamma(b_T)\right]^2 \leq 2\,\mathrm{Re}\left[\Gamma(b_T)\right]$$

$$\left(\mathrm{Re}\left[\Gamma(b_T)\right] - 1\right)^2 + \mathrm{Im}\left[\Gamma(b_T)\right]^2 \leq 1$$



(that should be a circle...)

c) $\mathrm{Re}\left[\Gamma(b_T)\right] = \Gamma_0\, e^{-b_T^2/2B}$

$\mathrm{Im}\left[\Gamma(b_T)\right] = 0.141\ \mathrm{Re}\left[\Gamma(b_T)\right]$

$\sigma_{el} = 25.4$ mb $\qquad \sigma_{tot} = 98.6$ mb

$$\sigma_{tot} = 2\int d^2b_T\ \Gamma_0\, e^{-b_T^2/2B} = 4\pi B\Gamma_0 = 98.6 \text{ mb}$$

$$\sigma_{el} = \int d^2b_T\ |\Gamma(b_T)|^2 = (1+0.141^2)\int d^2b_T\ \Gamma_0^2\, e^{-b_T^2/B}$$

$$= \Gamma_0^2(1+0.141^2)\,\pi B = 25.4 \text{ mb}$$

which gives $\quad B = 7.766$ mb

$\qquad\qquad\quad \Gamma_0 = 1.01$ mb

6. cont

Gaussian b_T form gives

$$\frac{d\sigma_{el}}{d^2q_T} = \frac{1}{4\pi^2} \int d^2b_T\, e^{-iq_T\cdot b_T} e^{-b_T^2/2B} \int d^2b_T'\, e^{iq_T\cdot b_T'} e^{-b_T'^2/2B}$$

$$\times\, \Gamma_0^2 (1+r^2) \qquad r = 0.141$$

$$= \frac{\Gamma_0^2(1+r^2)}{4\pi^2} \int d^2b_T\, e^{-\frac{1}{2B}(b_T - Biq_T)^2 - Bq_T^2/2}$$

$$\times \underbrace{\int d^2b_T\, e^{-\frac{1}{2B}(b_T + Biq_T)^2 - Bq_T^2/2}}_{e^{-Bq_T^2/2}\, B}$$

$$= \frac{\Gamma_0^2(1+r^2)}{4\pi^2} B^2 e^{-Bq_T^2} \sim e^{-Bt}$$

Note: $7.766\ \text{mb} = 7.766 \cdot 0.1\, \underbrace{\text{fm}^2}_{(5\,\text{GeV}^{-1})^2} = 19.4\ \text{GeV}^{-2}$

TOTEM slope $19.9\ \text{GeV}^{-2}$ consistent.