FYSH555 ex. 14

1. $K(L, R, U) = \sqrt{R U L^\dagger}^{-1} R \sqrt{U}$

a) $K(L_1, R_1, R_2 U L_2^\dagger) K(L_2, R_2, U)$

$$= \sqrt{R_1 R_2 U L_2^\dagger L_1^\dagger}^{-1} R_1 \sqrt{R_2 U L_2^\dagger} \sqrt{R_2 U L_2^\dagger}^{-1} R_2 \sqrt{U}$$

$$= \sqrt{R_1 R_2 U (L_1 L_2)^\dagger}^{-1} R_1 R_2 \sqrt{U}$$

$$= K(L_1 L_2, R_1 R_2, U)$$

b) Isospin transform $R = L = V$

$$K(V, V, U) = \sqrt{V U V^\dagger}^{-1} V \sqrt{U}$$

observe:

$$\left(\sqrt{V U V^\dagger}\right)^2 = V U V^\dagger = V \sqrt{U} V^\dagger V \sqrt{U} V^\dagger$$

$$\Rightarrow \sqrt{V U V^\dagger} = V \sqrt{U} V^\dagger$$

$$\Rightarrow K(V, V, U) = (V \sqrt{U} V^\dagger)^{-1} V \sqrt{U}$$

$$= (V^\dagger)^{-1} \sqrt{U}^{-1} V^{-1} V \sqrt{U} = V \sqrt{U}^{-1} \sqrt{U} = V$$

Thus $\psi \to \psi' = K(V, V, U)\psi = V\psi.$ independent of U.

2. $\mathcal{L}_{\pi N} = \bar{\Psi}\left(i\not{D} - m_N + \frac{g_A}{2}\gamma^\mu \gamma_5 u_\mu\right)\Psi$

$D_\mu = \partial_\mu + \Gamma_\mu \qquad \Gamma_\mu = \frac{1}{2}\left(u^\dagger \partial_\mu u + u\, \partial_\mu u^\dagger\right)$

a) $u = \exp\left[i\,\frac{\bar{\tau}\cdot\bar{\phi}}{2F}\right] = 1 + i\,\frac{\bar{\tau}\cdot\bar{\phi}}{2F} - \frac{\phi^2}{8F^2}, \qquad \phi = \bar{\tau}\cdot\bar{\phi}$

as $(\bar{\tau}\cdot\bar{\phi})(\bar{\tau}\cdot\bar{\phi}) = \phi^2 \;(= \bar{\phi}\cdot\bar{\phi})$ and $\tau_i^\dagger = \tau_i$

we get

$$2\Gamma_\mu = \left(1 - i\,\frac{\bar{\tau}\cdot\bar{\phi}}{2F} - \frac{\phi^2}{8F^2}\right)\left(\frac{i}{2F}\,\bar{\tau}\cdot\partial_\mu\bar{\phi} - \frac{\partial_\mu\phi\cdot\phi + \phi\cdot\partial_\mu\phi}{8F^2}\right)$$

$$+ \left(1 + i\,\frac{\bar{\tau}\cdot\bar{\phi}}{2F} - \frac{\phi^2}{8F^2}\right)\left(-\frac{i}{2F}\,\bar{\tau}\cdot\partial_\mu\bar{\phi} - \frac{\partial_\mu\phi\cdot\phi + \phi\,\partial_\mu\phi}{8F^2}\right)$$

$$= \cancel{\frac{i}{2F}\,\tau\cdot\partial_\mu\bar{\phi}} + \frac{(\bar{\tau}\cdot\bar{\phi})(\bar{\tau}\cdot\partial_\mu\bar{\phi})}{4F^2} - \frac{\partial_\mu\phi\cdot\phi + \phi\cdot\partial_\mu\phi}{8F^2}$$

$$\cancel{-\frac{i}{2F}\,\bar{\tau}\cdot\partial_\mu\bar{\phi}} + \frac{(\bar{\tau}\cdot\bar{\phi})(\bar{\tau}\cdot\partial_\mu\bar{\phi})}{4F^2} - \frac{\partial_\mu\phi\cdot\phi + \phi\cdot\partial_\mu\phi}{8F^2} + O(\phi^3)$$

$$\Gamma_\mu = \frac{(\bar{\tau}\cdot\bar{\phi})(\bar{\tau}\cdot\partial_\mu\bar{\phi})}{4F^2} - \frac{1}{8F^2}\left(\partial_\mu\phi\cdot\phi + \bar{\phi}\cdot\partial_\mu\phi\right)$$

similarly we get $\quad u_\mu = i\left[u^\dagger\partial_\mu u - u\,\partial_\mu u^\dagger\right] = -\frac{\tau\cdot\partial_\mu\bar{\phi}}{F}$

Plug these into the Lagrangian and get the interaction terms

$$\mathcal{L}_{\pi N} = \bar{\Psi}\left[i\partial_\mu + \frac{i}{4F^2}(\bar{\tau}\cdot\bar{\phi})(\bar{\tau}\cdot\partial_\mu\bar{\phi}) - \frac{i}{8F^2}(\partial_\mu\phi\cdot\phi + \phi\cdot\partial_\mu\phi) - m_N + \frac{g_A}{2}\gamma^\mu\gamma_5\left(-\frac{\bar{\tau}\cdot\partial_\mu\bar{\phi}}{F}\right)\right]$$

$$= \bar{\Psi}(i\partial_\mu - m_N)\Psi - \frac{1}{2}\frac{g_A}{F}\gamma^\mu\gamma_5\,\bar{\tau}\cdot\partial_\mu\phi\;\Psi$$

$$+ \frac{i}{4F^2}\bar{\Psi}\gamma^\mu\left[(\bar{\tau}\cdot\bar{\phi})(\bar{\tau}\cdot\partial_\mu\bar{\phi}) - \frac{\partial_\mu\phi\cdot\phi + \phi\cdot\partial_\mu\phi}{2}\right]$$

(typo in the problem text, missing i and wrong sign)

b) $$(\bar{\tau}\cdot\bar{A})(\bar{\tau}\cdot\bar{B}) = \bar{A}\cdot\bar{B}\;\mathbb{1} + i(\bar{A}\times\bar{B})\cdot\bar{\tau}$$

Now $$(\bar{\tau}\cdot\bar{\phi})(\bar{\tau}\cdot\partial_\mu\bar{\phi}) = \bar{\phi}\cdot\partial_\mu\bar{\phi} + i\,\varepsilon_{abc}\,\tau^a\,\bar{\phi}^b(\partial_\mu\bar{\phi})^c$$

and

$$\frac{1}{2}(\partial_\mu\phi\cdot\phi + \phi\,\partial_\mu\phi) = \frac{1}{2}\left[(\bar{\tau}\cdot\partial_\mu\bar{\phi})(\bar{\tau}\cdot\bar{\phi}) + (\bar{\tau}\cdot\bar{\phi})(\bar{\tau}\cdot\partial_\mu\bar{\phi})\right]$$

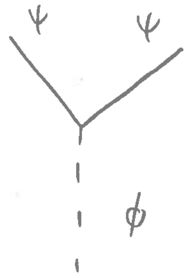
$$= \frac{1}{2}\left[\underbrace{\partial_\mu\bar{\phi}\cdot\bar{\phi} + \bar{\phi}\cdot\partial_\mu\bar{\phi}}_{2\bar{\phi}\cdot\partial_\mu\bar{\phi}} + i(\partial_\mu\bar{\phi}\times\bar{\phi}) + i(\bar{\phi}\times\partial_\mu\bar{\phi})\right]$$

$$= \bar{\phi}\cdot\partial_\mu\bar{\phi}$$

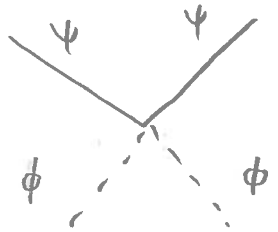
Thus the 2nd interaction term reads

$$\frac{i}{4F^2}\bar{\Psi}\gamma^\mu\left[i\,\varepsilon_{abc}\,\tau^a\bar{\phi}^b(\partial_\mu\bar{\phi})^c\right]\Psi = \underline{-\frac{1}{4F^2}\bar{\Psi}\gamma^\mu\varepsilon_{abc}\tau^a\bar{\phi}^b(\partial_\mu\bar{\phi})^c\Psi}$$

c. $-\frac{1}{2}\frac{g_A}{F}\bar{\Psi}\gamma^\mu\gamma_5\vec{\tau}\cdot\partial_\mu\vec{\phi}\,\Psi$



$-\frac{1}{4F^2}\bar{\Psi}\gamma^\mu\varepsilon_{abc}\tau^a\vec{\phi}^b(\partial_\mu\phi)^c\,\Psi$



3. Hadronic states $\langle B|$, $|A\rangle$

Momentum eigenvalues p_A^μ , p_B^μ

a)

$$\langle B|\partial_\mu A_i^\mu(x)|A\rangle \overset{\text{transl. inv.}}{=} \partial_\mu \underbrace{\langle B| e^{i\hat{p}\cdot x}}_{} A_i^\mu(0)\, e^{-i\hat{p}\cdot x}|A\rangle \quad (\to p_A^\mu)$$

$$\left(e^{-i\hat{p}\cdot x}|B\rangle\right)^\dagger = \left(e^{-ip_B\cdot x}|B\rangle\right)^\dagger$$

$$= \langle B| e^{ip_B\cdot x}$$

$$= \partial_\mu\left[e^{i\overbrace{(p_B-p_A)}^{q}\cdot x}\langle B|A_i^\mu(0)|A\rangle\right]$$

$$= e^{iq\cdot x}\, i q_\mu \langle B|A_i^\mu(0)|A\rangle$$

$$= e^{iq\cdot x}\, i\langle B|q_\mu A_i^\mu(0)|A\rangle$$

Thus

$$\langle B|\underbrace{\partial_\mu A_i^\mu(x)}_{m_q P_i(x)}|A\rangle = e^{iq\cdot x}\, i\langle B|q_\mu A_i^\mu(0)|A\rangle$$

$$m_q\langle B|P_i(x)|A\rangle = i e^{iq\cdot x}\langle B|q_\mu A_i^\mu(0)|A\rangle$$

similarly using the rotational invariance for $P_i(x) = e^{i\hat{p}\cdot x}P_i(0)\,e^{-i\hat{p}\cdot x}$ we get

$$m_q\, \cancel{e^{iq\cdot x}}\langle B|P_i(0)|A\rangle = \cancel{e^{iq\cdot x}}\, i\langle B|q_\mu A_i^\mu(0)|A\rangle$$

h) $\langle N(p') | A_i^\mu(0) | N(p)\rangle = \bar{u}(p')\left[\gamma^\mu G_A(t) + \frac{(p'-p)^\mu}{2m_N} G_p(t)\right]\gamma_5 \frac{\tau_i}{2} u(p)$

Now

$$\frac{M_\pi^2 F_\pi}{M_\pi^2 - t} G_{\pi N}(t)\, i\bar{u}(p')\gamma_5 \tau_i u(p)$$

$$= m_q \langle N(p') | P_i(0) | N(p)\rangle$$

$$= iq_\mu \langle N(p') | A_i^\mu(0) | N(p)\rangle$$

$$= i\bar{u}(p')\left[(\not{p}' - \not{p}) G_A(t) + \frac{(p'-p)^2}{2m_N} G_p(t)\right]\gamma_5 \frac{\tau_i}{2} u(p)$$

use DE: $\bar{u}(p')\not{p}' = m_N u(p)$

$\not{p} u(p) = m_N u(p)$

and $\not{p}\gamma_5 = -\gamma_5 \not{p}$

$$= i\bar{u}\left[(m_N + m_N) G_A(t) + \frac{t}{2m_N} G_p(t)\right]\gamma_5 \frac{\tau_i}{2} u(p)$$

$$= i\bar{u}(p')\frac{1}{2}\left[2m_N G_A(t) + \frac{t}{2m_N} G_p(t)\right]\gamma_5 \tau_i u(p)$$

Thus

$$\frac{M_\pi^2 F_\pi}{M_\pi^2 - t} G_{\pi N}(t) = m_N G_A(t) + \frac{t}{4m_N} G_p(t)$$