FYSH555 ex. 13

1. a)

$$\phi = \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\overline{K^0} \\ \sqrt{2}K^- & \sqrt{2}K^0 & \frac{2}{\sqrt{3}}\eta \end{pmatrix}$$

$\pi^0 \xrightarrow{C} \pi^0$

$\eta \to \eta$

$\pi^\pm \to \pi^\mp$

$K^\pm \to K^\mp$

$K^0 \to \overline{K^0}$

$$\xrightarrow{C} \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^- & \sqrt{2}K^- \\ \sqrt{2}\pi^+ & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^+ & \sqrt{2}\overline{K^0} & \frac{2}{\sqrt{3}}\eta \end{pmatrix}$$

ex. 12.2

$$= \begin{pmatrix} \phi_3 + \frac{1}{\sqrt{3}}\phi_8 & \phi_1 + i\phi_2 & \phi_4 + i\phi_5 \\ \phi_1 - i\phi_2 & -\phi_3 + \frac{1}{\sqrt{3}}\phi_8 & \phi_6 + i\phi_7 \\ \phi_4 - i\phi_5 & \phi_6 - i\phi_7 & -\frac{2}{\sqrt{3}}\phi_8 \end{pmatrix}$$

$$= \begin{pmatrix} \phi_3 + \frac{1}{\sqrt{3}}\phi_8 & \phi_1 - i\phi_2 & \phi_4 - i\phi_5 \\ \phi_1 + i\phi_2 & -\phi_3 + \frac{1}{\sqrt{3}}\phi_8 & \phi_6 - i\phi_7 \\ \phi_4 + i\phi_5 & \phi_6 + i\phi_7 & -\frac{2}{\sqrt{3}}\phi_8 \end{pmatrix}^T$$

$$= \phi^T$$

Thus $\phi \xrightarrow{C} \phi^T$

b) Theorem: $(A^T)^n = (A^n)^T$ $\qquad$ $(AB)^T = B^T A^T$

Proof: induction. $(A^2)^T = (AA)^T \overset{\checkmark}{=} A^T A^T = (A^T)^2$

$\rightarrow$ works for $n=2$ (and $n=1$ trivially).

Assuming that the theorem holds for certain n:

$$(A^{n+1})^T = (A^n A)^T = A^T (A^n)^T = A^T (A^T)^n$$

$$= (A^T)^{n+1}$$

$\square$

Now

$$U = \exp\left(\frac{i\phi}{F_0}\right) \overset{C}{\rightarrow} \exp\left(\frac{i\phi'}{F_0}\right) \qquad \Big| \ \phi' = \phi^T$$

$$= \sum_n \left(\frac{i}{F_0}\right)^n \frac{1}{n!} (\phi^T)^n$$

$$= \left[\sum_n \frac{1}{n!}\left(\frac{i}{F_0}\right)^n \phi^n\right]^T = U^T .$$

c) $D_\mu U = \partial_\mu U - i r_\mu U + i U l_\mu$

$r_\mu = v_\mu + a_\mu$, $\quad r_\mu \xrightarrow{C} r_\mu' = -v_\mu^T + a_\mu^T = -l_\mu^T$

$l_\mu = v_\mu - a_\mu$ $\quad l_\mu \xrightarrow{C} l_\mu' = -v_\mu^T - a_\mu^T = -r_\mu^T$

d) $D_\mu U \xrightarrow{C} \partial_\mu U^T - i(-l_\mu^T) U^T + i U^T (-r_\mu^T)$

$$= \partial_\mu U^T - i U^T r_\mu^T + i l_\mu^T U^T$$

$$= [\partial_\mu U - i r_\mu U + i U l_\mu]^T$$

$$= (D_\mu U)^T$$

e) $\mathcal{L}_2 = \frac{F_0^2}{4} Tr[D_\mu U (D^\mu U)^\dagger]$

$\frac{4}{F_0^2} \mathcal{L}_2 \xrightarrow{C} Tr[(D_\mu U)^T [(D^\mu U)^T]^\dagger]$

$= Tr[(D_\mu U)^T ([D^\mu U]^\dagger)^T]$

$= Tr[\{(D^\mu U)^\dagger D_\mu U\}^T]$ $\quad | \; Tr(A^T) = Tr(A)$

$= Tr[(D^\mu U)^\dagger D_\mu U] = \underline{Tr[D_\mu U (D^\mu U)^\dagger]} = \frac{4}{F_0^2} \mathcal{L}_2$

same $\mathcal{L}$ (same physics) for particles and antiparticles.

2. $\mathcal{L}_2 = \frac{F_0^2}{4} Tr[D_\mu U (D^\mu U)^\dagger]$, $D_\mu U = \partial_\mu U - i r_\mu U + i U l_\mu$

$r_\mu = l_\mu = -e Q A_\mu$

a) $D_\mu U = \partial_\mu U + ie A_\mu (QU - UQ)$

$= \partial_\mu U + ie A_\mu [Q, U]$

$(D_\mu U)^\dagger = \partial_\mu U^\dagger - ie A_\mu \underbrace{(QU - UQ)^\dagger}_{U^\dagger Q - QU^\dagger}$

$= \partial_\mu U^\dagger + ie A_\mu [Q, U^\dagger]$

b) As $0 = \partial_\mu (U U^\dagger) = (\partial_\mu U) U^\dagger + U \partial_\mu U^\dagger$

$\Rightarrow U \partial_\mu U^\dagger = -(\partial_\mu U) U^\dagger$

$0 = \partial_\mu(U^\dagger U) \Rightarrow (\partial^\mu U^\dagger) U = -U^\dagger \partial^\mu U$

Insert $D_\mu U$ and $(D_\mu U)^\dagger$ in $\mathcal{L}_2$:

$$\mathcal{L}_2 = \frac{F_0^2}{4} Tr[(D_\mu U)(D^\mu U^\dagger)]$$

$$= \frac{F_0^2}{4} Tr[(\partial_\mu U + ieA_\mu [Q,U])(\partial^\mu U^\dagger + ieA^\mu [Q,U^\dagger])$$

$$= \frac{F_0^2}{4} Tr[(\partial_\mu U)(\partial^\mu U^\dagger)]$$

$$+ \frac{F_0^2}{4} Tr[(\partial_\mu U)\, ieA^\mu [Q,U^\dagger] + ieA_\mu [Q,U]\, \partial^\mu U^\dagger]$$

$$- e^2 \frac{F_0^2}{4} A_\mu A^\mu\, Tr([Q,U][Q,U^\dagger])$$

$$= \frac{F_0^2}{4} Tr[(\partial_\mu U)(\partial^\mu U^\dagger)]$$

$$+ \frac{F_0^2}{4} ieA_\mu\, Tr[(\partial^\mu U) Q U^\dagger - (\partial^\mu U) U^\dagger Q$$

$$+ Q \underbrace{U \partial^\mu U^\dagger}_{-(\partial^\mu U)U^\dagger} - \underbrace{U Q \partial^\mu U^\dagger}_{-Q\, U^\dagger \partial^\mu U}]$$

$$- e^2 \frac{F_0^2}{4} A_\mu A^\mu\, Tr([Q,U][Q,U^\dagger])$$

$$= \frac{F_0^2}{4} Tr\left[(\partial_\mu U)(\partial^\mu U^\dagger)\right]$$

int. terms

$$+ \frac{F_0^2}{2} ie A_\mu \, Tr\left[Q \left(-(\partial^\mu U) U^\dagger + U^\dagger \partial^\mu U \right)\right] \Big\} \mathcal{L}_1^i$$

$$- e^2 \frac{F_0^2}{4} A_\mu A^\mu \, Tr\left([Q, U][Q, U^\dagger] \right) \Big\} \mathcal{L}_2^i$$

Interactions: $\mathcal{L}_1^i$: A_μ and any # of scalars

$\mathcal{L}_2^i$: $2A_\mu$ and — " —

c) $U = 1 + \frac{i\phi}{F_0} - \frac{\phi^2}{2F_0^2}$

Substitute to $\mathcal{L}_1^i$ and $\mathcal{L}_2^i$ and drop terms $O(\phi^3)$

$$\mathcal{L}_1^i = \frac{F_0^2}{2} ie A_\mu \, Tr\Big[Q \Big\{ \left(-\partial^\mu \frac{i\phi}{F_0}\right)\left(\frac{-i\phi}{F_0}\right)$$

$$+ \left(-\partial^\mu \cancel{\left(-\frac{\phi^2}{2F_0^2}\right)} \right) 1 + 1\, \partial^\mu \cancel{\left(-\frac{\phi^2}{2F_0}\right)}$$

$$+ \left(\frac{-i\phi}{F_0}\right)\left(\partial^\mu \frac{i\phi}{F_0}\right) \Big\} \Big]$$

$$= \frac{1}{2} ie A_\mu \, Tr\left[-Q (\partial^\mu \phi)\phi + Q \phi \partial^\mu \phi \right]$$

$$= -\frac{i}{2} e A_\mu \, Tr\left[Q \left[\partial^\mu \phi, \phi \right] \right]$$

and similarly

$$\mathcal{L}_2^i \simeq -\frac{e^2 F_0^2}{4} A_\mu A^\mu \, \mathrm{Tr}\Big(\Big[Q, 1 + \frac{i\phi}{F_0} - \frac{\phi^2}{2F_0^2}\Big]$$

$$\Big[Q, 1 - \frac{i\phi}{F_0} - \frac{\phi^2}{2F_0}\Big] \Big)$$

$$= -\frac{e^2}{4} A_\mu A^\mu \, \mathrm{Tr}\Big([Q, i\phi][Q, -i\phi]$$

$$+ \Big[Q, -\frac{\phi^2}{2F_0^2}\Big] \underbrace{[Q, 1]}_{0} + \underbrace{[Q, 1]}_{0} \Big[Q, -\frac{\phi^2}{2F_0}\Big] \Big)$$

$$= -\frac{e^2}{4} A_\mu A^\mu \, \mathrm{Tr}\big([Q, \phi][Q, \phi] \big)$$

d)

$$\phi = \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\bar{K}^0 \\ \sqrt{2}K^- & \sqrt{2}K^0 & \frac{2}{\sqrt{3}}\eta \end{pmatrix}$$

$$[\partial^\mu \phi, \phi]_{11} = \partial^\mu\left(\pi^0 + \frac{1}{\sqrt{3}}\eta\right)\left(\pi^0 + \frac{1}{\sqrt{3}}\eta\right) + 2(\partial^\mu \pi^+)\pi^-$$

$$+ 2(\partial^\mu K^+)K^-$$

$$- \left(\pi^0 + \frac{1}{\sqrt{3}}\eta\right)\partial^\mu\left(\pi^0 + \frac{1}{\sqrt{3}}\eta\right) - 2\pi^+\partial^\mu\pi^-$$

$$- 2K^+\partial^\mu K^-$$

$$= 2\left((\partial^\mu \pi^+)\pi^- - \pi^+\partial^\mu\pi^- + (\partial^\mu K^+)K^- - K^+\partial^\mu K^-\right.$$

$$[Q, \phi] = \begin{pmatrix} 0 & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ -\sqrt{2}\pi^- & 0 & 0 \\ -\sqrt{2}K^- & 0 & 0 \end{pmatrix}$$

$$[Q, \phi][Q, \phi] = -2\begin{pmatrix} \pi^+\pi^- + K^+K^- & 0 & 0 \\ 0 & \pi^-\pi^+ & \pi^-K^+ \\ 0 & K^-\pi^+ & K^-K^+ \end{pmatrix}$$

e) $Tr([Q,\phi][Q,\phi]) = 4(\pi^+\pi^- + K^+K^-)$

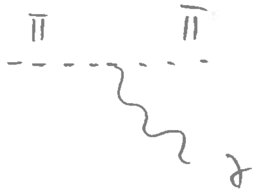
$\to$ interaction terms from $\mathcal{L}_2^i$:

$$e^2 A_\mu A^\mu (\pi^+\pi^- + K^+K^-)$$

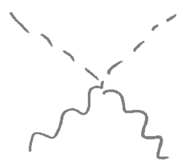
From $\mathcal{L}_1^i$:

$$-eA_\mu \frac{i}{2} Tr(Q, [\partial^\mu \phi, \phi]) = -\frac{ieA_\mu}{2}\Big(\frac{2}{3}[\partial^\mu\phi,\phi]_{11}$$
$$-\frac{1}{3}[\partial^\mu\phi,\phi]_{22} - \frac{1}{3}[\partial^\mu\phi,\phi]_{33}\Big)$$

$$= \dots = -ieA_\mu\left[(\partial^\mu\pi^+)\pi^- - \pi^+\partial^\mu\pi^- + (\partial^\mu K^+)K^- - K^+\partial^\mu K^-\right]$$

$\mathcal{L}_1^i$ corresponds to  vertex,

proportional to e and $\partial^\mu \pi \sim$ momentum
external photon $\to$ polarisation vector

$\mathcal{L}_2^i$ corresponds to  $\sim e^2$

Two external photons $\to$ two pol. vectors.

Internal pion / kaon: Scalar propagator $\frac{i}{p^2-m^2}$