Exercise 9 Return by 16.00 on Tuesday 27.11.

1. As on p. 247-8 in the lecture notes, let's study the group $SU(N)$.
   (a) Knowing that the $SU(N)$ group matrices can be written in the form
   $$U(\alpha) = \exp\left[i\sum_{i=1}^{d}\alpha_i T_i\right],$$
   where T_i are the group generator matrices and α_i are real parameters, convince yourself that the generators have to be hermitian and traceless. You can use the facts $\det(e^A) = e^{\mathrm{Tr}(A)}$ and $e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\cdots}$. The purpose of this problems is that you understand why the color symmetry group $SU(3)$ generators (The Gell-Mann -matrices) satisfy these conditions.
   (b) The generators are linearly independent matrices and fulfil $\mathrm{Tr}(T_i T_j) = \lambda\delta_{ij}$. Using the Lie algebra for this group and the properties of the trace, show that the structure constants can be expressed as $C_{ljk} = -\frac{i}{\lambda}\mathrm{Tr}(T_l[T_j, T_k])$ and that C_{ljk} are fully antisymmetric in any mutual exchange of indices.

2. As suggested on p.249 in the lecture notes,
   (a) Generalize the Noether theorem for the globally $U(1)$ symmetric Lagrangian
   $$\mathcal{L}(\phi, \phi^*, \partial_\mu\phi, \partial_\mu\phi^*) = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi - V_I(\phi^*\phi),$$
   describing a charged scalar field (see p. 249), i.e. find the form of the conserved 4-current, analogous to that on p. 236.
   (b) Using the $U(1)$ transformation $U = e^{i\theta}$ (expand), find the variations $\delta\phi$ and $\delta\phi^*$ and based on your result above, show that you arrive at the same conserved 4-current as given on p. 249.

3. Using the general properties of the Dirac matrices α^1, α^2, α^3 and β (see p. 255 in the lecture notes), and the definitions of the Dirac gamma matrices γ^μ (p. 256), verify the following, representation-independent, results:
   (a) $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbf{1}_4$, known as the Clifford algebra,
   (b) $\gamma^i = -\gamma^0\gamma^i\gamma^0$,
   (c) $\gamma^{0\dagger} = \gamma^0$ ja $\gamma^{i\dagger} = \gamma^0\gamma^i\gamma^0 = -\gamma^i$,
   (d) $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$.

4. Using the explicit 2-block forms for the free spin-$\frac{1}{2}$ particle & antiparticle spinors $u^{(r)}(p)$ and $v^{(r)}(p)$ ($r, s = 1, 2$), which we derived in the lecture notes (p.259-263), show that with the chosen normalization $\int_V \mathrm{d}^3x \ \rho = 2E_p$ (p. 264), we have the following

   (a) normalization constant: $\quad N(\mathbf{p}) = \sqrt{E_p + m}$
   (Hints: Do the spinor and matrix multiplications using the 2-block forms. Recall the Pauli spin-matrix property $(\vec{\sigma} \cdot \vec{p})^2 = \vec{p}^2$)

   (b) orthogonality relations: $\quad u^{(r)\dagger}(p)u^{(s)}(p) = v^{(r)\dagger}(p)v^{(s)}(p) = 2E_p\delta_{rs}$

   (c) projection operators:
   (Hint: do the multiplication by γ^0 only after performing the spin sums)

$$\sum_{s=1,2} u^{(s)}(p)\overline{u}^{(s)}(p) = \not{p} + m$$

$$\sum_{s=1,2} v^{(s)}(p)\overline{v}^{(s)}(p) = \not{p} - m$$

5. Let's consider an interacting real scalar field in 1+1 dimensions (time and the x coordinate) with the lagrangian

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi),$$

where $V(\phi) = \frac{\lambda}{2}(\phi^2 - a^2)^2$. In this case we take a to be a positive constant with the same dimension as the field ϕ. The coefficient $\lambda > 0$. (Notice that the mass term has the "wrong" sign, we will return to this later when dealing with the Higgs mechanism)

   (a) Derive the equation of motion for the field ϕ using the Euler-Lagrange field equations. Consider a static case ($\partial\phi/\partial t = 0$) and the so called soliton solution, which is a finite energy solution. For this solution we can require $\phi(x) \to a$ and $\phi'(x) \to 0$, when $x \to \infty$. Show that in this case the equation of motion can be written as

$$\frac{1}{2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^2 = V(\phi).$$

   (b) Solve this equation and show that $\phi(x) = a \tanh[b(x + c)]$, where c is a constant. What is the constant b in terms of parameters λ and a?