Exercise 8 Return by 16.00 on Tue 20.11.

1. Show that the Lagrangian

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi - V_I(\phi^* \phi)$$

gives the equations of motion boxed on page 233 in the lecture notes for the complex field $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$ and it's conjugate field ϕ^*. In addition derive the corresponding equations of motion for the component fields ϕ_1 and ϕ_2 also boxed on page 233.

2. Starting from the definition of Maxwell's field tensor $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$, convince yourself that the matrix representation of $F^{\mu\nu}$ given on page 234 in the lecture notes is correct. Show also that Maxwell's equations (i) and (iv) can be written as $\partial_\mu F^{\mu\nu} = j^\nu$. For the four-current j^ν show that $\partial_\nu j^\nu = 0$.

3. Show that the equations of motion given by the Lagrangian

$$\mathcal{L}_{ED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu$$

are $\partial_\mu F^{\mu\nu} = j^\nu$ i.e. the Maxwell equations (i) and (iv).

4. Verify that the 2-dimensional rotation matrices

$$O(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

form a group. This group is called $SO(2)$. Show that the rotation matrices $O(\theta)$ can be written in the form $e^{\theta\tau}$, where

$$\tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(Hint: Series expansion.) Determine the generators and the dimension of this group.

5. (a) Show that unitary $N \times N$ -matrices form a group i.e. they satisfy the conditions (i-iv) on page 240 in the lecture notes. This group is called $U(N)$. Show that the complex phase factors $e^{i\alpha}$, where α is real form the group $U(1)$.

   (b) Show that unitary $N \times N$ -matrices with determinant 1 also form a group. This group is called $SU(N)$ and is a subgroup of $U(N)$.