Exercise 7 Return by 16.00 on Tue 13.11.

1. Solve the differential equation
$$i\frac{\mathrm{d}a_0(t)}{\mathrm{d}t} = H_{01}e^{-i(E_1-E_0)t} - i\frac{\Gamma}{2}a_0(t),$$
introduced on page 190 in the lecture notes, with the initial condition $a_0(0) = 0$ and show that
$$P_0 \equiv |a_0(t \gg \Gamma^{-1})|^2 = \frac{|H_{01}|^2}{(E_1-E_0)^2 + \frac{\Gamma^2}{4}}.$$

2. On page 217 in the lecture notes we stated that the color singlet condition $T_1\chi_B^C = 0$ leads to the conditions
$$\alpha_1 = -\alpha_2, \quad \alpha_3 = -\alpha_4, \quad \alpha_5 = -\alpha_6$$
for the coefficients α_i in the general color wavefunction for baryons χ_B^C. Verify this result and find the conditions resulting from the other conditions $T_i\chi_B^C = 0$, and show that these lead to the totally antisymmetric form for the baryon color wavefunction boxed on page 218 in the lecture notes.

3. What spectrum of low-lying baryon states composed of light quarks u, d and s with zero orbital angular momentum $L_{12} = L_3 = 0$ would be predicted, if we assumed that the product of the spatial and the spin parts of the wavefunction is <u>antisymmetric</u> under the exchange of like quarks. You do not have to consider the color part of the wave function here.

4. (a) Show that a general 2×2 unitary matrix U, which has $\det(U) = 1$, can be written in the form
$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}.$$
   (b) Isospin rotations are defined
$$\begin{array}{ll} \chi_{u'} = U\chi_u & \chi_{\bar{u}'} = U\chi_{\bar{u}} \\ \chi_{d'} = U\chi_d & \chi_{\bar{d}'} = U\chi_{\bar{d}} \end{array} \quad \text{with} \quad \chi_u = -\chi_{\bar{d}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \chi_d = \chi_{\bar{u}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
   and U is the above mentioned $SU(2)$-matrix. Express the states $|u'\rangle$, $|d'\rangle$, $|\bar{u}'\rangle$ and $|\bar{d}'\rangle$ using the original states $|u\rangle$, $|d\rangle$, $|\bar{u}\rangle$ and $|\bar{d}\rangle$.
   (c) Verify explicitly that an isospin singlet is invariant under isospin rotations, i.e. show that
$$\frac{1}{\sqrt{2}}(|u'\bar{u}'\rangle + |d'\bar{d}'\rangle) = \frac{1}{\sqrt{2}}(|u\bar{u}\rangle + |d\bar{d}\rangle).$$

5. Consider the process $\gamma + p \to \pi^0 + p$. If $E_\gamma = 3K$ (3 Kelvin!) (*remember that* $k_B = 1$*)*, $\sqrt{s} = m_\Delta = 1.232\,\mathrm{GeV}$ (the Δ resonance mass) and the collision is head-on (incoming p and γ have momenta in the opposite direction).

   (a) What is the energy of the proton?

   (b) Find out and explain briefly what is the "GZK cutoff".

   (c) What are the energies of the incoming and outgoing proton in the CMS?

   (d) How much does the proton energy change in the frame where $E_\gamma = 3K$ ($\approx$ Earth rest frame)? You can assume that the scattering angle in that frame is $\theta = 0$. *(Hint: once you know the proton energy in the original frame and in the CMS, you can solve the relative velocities of these frames and the Lorentz boost factor* γ*. Then you can boost the final CMS energy/momentum back to the original frame. One technical difficulty here is that as this velocity is very close to unity* ($1 - v \sim 10^{-24}$)*, your calculator will most likely give* $v = 1$*, which is of course impossible. You either have to do some series expansions or use e.g. Mathematica's* `WorkingPrecision` *option.*