FYSH300 Particle Physics Autumn 2012

Exercise 11 Return by 16.00 on Tuesday 11.12.

1. **a)** The Maxwell field strength tensor of QED is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. In the lecture notes (p. 310) we defined the covariant derivative as $D_\mu = \partial_\mu + ieQ_f A_\mu$. Show that

$$F_{\mu\nu} = -\frac{i}{eQ_f}[D_\mu, D_\nu].$$

   b) For QCD, we defined the covariant derivative as $D_\mu = \partial_\mu \mathbf{1}_3 + igA_\mu$, where $\mathbf{1}_3$ is the 3×3 unit matrix and $A_\mu = \sum_a A_\mu^a t^a$ is the Yang-Mills gauge field, which is a 3×3 matrix and expressed in terms of the generator matrices t^a of the group $SU(3)$ and the coefficient fields A_μ^a. Using the Lie algebra of $SU(3)$, $[t^a, t^b] = i\sum_c f^{abc}t^c$, show that when we define the non-Abelian field strength tensor as the commutator $F_{\mu\nu} \equiv -\frac{i}{g_s}[D_\mu, D_\nu] \equiv \sum_a F_{\mu\nu}^a t^a$, we get

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g_s f^{abc} A_\mu^b A_\nu^c.$$

   c) As suggested at the bottom of p. 312, show that in the $SU(3)$ gauge transformation U, the tensor $F_{\mu\nu}$ transforms as $F_{\mu\nu} \to F'_{\mu\nu} = UF_{\mu\nu}U^{-1}$.

2. On p. 347' we listed the differential cross sections for various partonic QCD processes. Let's verify the third one, i.e. compute $\mathrm{d}\hat{\sigma}/\mathrm{d}\hat{t}$ for the quark-antiquark scattering $q_i + \hat{q}_i \to q_j + \bar{q}_j$, where $i \neq j$. Write down the invariant amplitude $\mathcal{M}$ starting from the QCD Feynman rules, then square it, sum(average) over final(initial) state spins and colours. Note that in obtaining the trace of the Dirac gamma matrices, you can and may directly use the results we derived in the previous exercises. Remember to compute the colour factor, too. The quarks can be considered massless here.

3. In the lecture notes (p. 342), it is mentioned that the parton distribution functions (PDFs) of the neutron are obtained from those of the proton through isospin symmetry, meaning that $f_{d/n}(x, Q^2) = f_{u/p}(x, Q^2)$, $f_{u/n}(x, Q^2) = f_{d/p}(x, Q^2)$ and $f_{\bar{d}/n}(x, Q^2) = f_{\bar{u}/p}(x, Q^2)$, $f_{\bar{u}/n}(x, Q^2) = f_{\bar{d}/p}(x, Q^2)$, while the distributions of the other quark flavors (and gluons) in the neutron are the same as those in the proton. The standard convention is that when one writes f_i (without the specification p or n), one refers to the distribution of the parton type i in the proton.

   (a) Starting from the parton model form of the structure function F_2 given on p. 338, write down F_2^p for the protons and F_2^n for the neutron in terms of the total quark and antiquark distributions of the proton. Divide the total quark distributions into valence and sea quark components. Note that the sea quark content of a hadron corresponds to the $q\bar{q}$ quantum fluctuations, so that for each quark flavor the sea quark distributions are identical to those of the sea antiquark distributions. Let's assume here for simplicity that the c, b and t quark distributions can be neglected.

(b) Then, let's try to understand the experimental result shown on p. 342 for the ratio F_2^n/F_2^p. Approximate the PDFs as follows (see p.340)

$$u_V(x) \gg d_V(x), \bar{u}(x), \bar{d}(x), \bar{s}(x), \quad \text{when} \quad x \to 1$$

$$\bar{u}(x) \approx \bar{d}(x) \approx \bar{s}(x) \gg u_V(x), d_V(x), \quad \text{when} \quad x \to 0.$$

Here we have denoted the PDFs as $u_V \equiv f_{u_V}$ etc for brevity. Show that

$$\frac{F_2^n}{F_2^p} \to \frac{1}{4} \quad \text{when} \quad x \to 1$$

$$\frac{F_2^n}{F_2^p} \to 1 \quad \text{when} \quad x \to 0.$$

4. QCD exhibits singularities for soft and collinear gluon emission. To see this, consider the process $\gamma^* \to q\bar{q}$ (virtual photon splitting to quark-antiquark pair) with

$$-i\mathcal{M}^{\mu}_{q\bar{q}} = \overline{u}(p_1)(-ie_q)\gamma^{\mu}v(p_2)$$

Here μ is the Lorentz index of the $\gamma^* q\bar{q}$ vertex.

(a) Add a gluon with color a to the final state and show that the resulting diagrams yield

$$\begin{aligned} -i\mathcal{M}^{\mu}_{q\bar{q}g} &= \overline{u}(p_1)(-ig_s)\not{\epsilon}^* t^a_{ij}\frac{i(\not{p}_1 + \not{k})}{(p_1 + k)^2}(-ie_q)\gamma^{\mu}v(p_2) \\ &- \overline{u}(p_1)(-ie_q)\gamma^{\mu}\frac{i(\not{p}_2 + \not{k})}{(p_2 + k)^2}(-ig_s)\not{\epsilon}^* t^a_{ij}v(p_2), \end{aligned}$$

where ϵ^*_ν is the polarization vector of the outgoing gluon, the colors of the final state quarks are i and j and k is the momentum of the outgoing gluon.

(b) Simplify this expression using the relations $\not{a}\not{b} = 2a{\cdot}b - \not{b}\not{a}$, $\overline{u}(p_1)\not{p}_1 = \not{p}_2 v(p_2) = 0$ and $k \ll p_1, p_2$, that is, assume massless quarks and consider an emission of a soft gluon.

Show that the result can be brought into the form

$$-i\mathcal{M}^{\mu}_{q\bar{q}g} \simeq -\overline{u}(p_1)ie_q\gamma^{\mu}t^a_{ij}v(p_2)g_s\left(\frac{p_1 \cdot \epsilon^*}{p_1 \cdot k} - \frac{p_2 \cdot \epsilon^*}{p_2 \cdot k}\right)$$

(c) From this expression, show that

$$|\overline{\mathcal{M}_{q\bar{q}g}}|^2 = |\overline{\mathcal{M}_{q\bar{q}}}|^2 C_F g_s^2 \frac{2p_1p_2}{(p_1 \cdot k)(p_2 \cdot k)},$$

where $C_F = (N_c^2 - 1)/(2N_c)$.

Adding the phase space $\mathrm{d}\Phi_{q\bar{q}}$ for the quark-antiquark part and $\frac{\mathrm{d}^3k}{2E_k(2\pi)^3}$ for the gluon, show that the integral of $\overline{\mathcal{M}_{q\bar{q}g}}$ diverges when $E_k \to 0$ and when $\vec{k} \uparrow\uparrow \vec{p_1}$ or $\vec{k} \uparrow\uparrow \vec{p_2}$.