FYSH300 fall 2011

Exercise 8, return by Mon Nov 7th at 9.00 to box in the lobby, discussed Mon Nov 7th, at 12.15 in FYS5

1. Consider an interacting real scalar field in 1+1 dimensions (time and the x-coordinate) with the Lagrangian
$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi),$$
where $V(\phi) = \frac{\lambda}{2}(\phi^2 - a^2)^2$. In this case we take a to be a positive constant with the same dimension as the field. The coefficient $\lambda > 0$.

   a) Derive the equation of motion for the field ϕ using the Euler-Lagrange field equations. Consider a static case (set $\partial\phi/\partial t = 0$) and the so called soliton solution, which is a finite energy solution. For this solution we can require $\phi(x) \to a$ and $\phi'(x) \to 0$, when $x \to \infty$. Show that in this case the equation of motion can be written in the form
$$\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 = V(\phi).$$

   b) Solve the equation and show that $\phi(x) = a\tanh(b(x+c))$, where c is a constant. What is the constant b in terms of parameters λ and a?

2. Verify that the 2-dimensional rotation matrices
$$O(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$
form a group. This group is called $SO(2)$. Show that the rotation matrices $O(\theta)$ can be written in the form $e^{\theta\tau}$, where
$$\tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$
(Hint: Series expansion.) Determine the generators and the dimension of this group.

3. Show that unitary $N \times N$ -matrices form a group. This group is called $U(N)$. Show that the complex phase factors $e^{i\alpha}$, where α is real form the group $U(1)$.

4. Show that unitary $N \times N$ -matrices with determinant 1 also form a group. This group is called $SU(N)$ and is a subgroup of $U(N)$.

5. More on $SU(N)$:

   a) Knowing that the $SU(N)$ group matrices can be written in the form
$$U(\alpha) = \exp[i\sum_{i=1}^{d} \alpha_i T_i],$$
where T_i are the group generator matrices and α_i are real parameters, convince yourself that the generators have to be hermitian and traceless. You can use the facts $\det(e^A) = e^{\mathrm{Tr}(A)}$ and $e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\cdots}$. The purpose of this problems is that you understand why the color symmetry group $SU(3)$ generators (The Gell-Mann -matrices) satisfy these conditions.

   b) The generators are linearly independent matriced and fulfill $\mathrm{Tr}(T_i T_j) = \lambda\delta_{ij}$. Using the Lie algebra for this group and the properties of the trace, show that the structure constants can be expressed as $C_{ljk} = -\frac{i}{\lambda}\mathrm{Tr}(T_l[T_j, T_k])$ and that C_{ljk} are fully antisymmetric in any mutual exchange of indices.

6. The so-called γ-matrices are defined as
$$\gamma^0 = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix} \quad \text{and} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$
where $\mathbf{1}_2$ is the 2-dim unit matrix and σ^i are the Pauli spin matrices.

   Turn over!

Verify the following, representation-independent, results:

i) $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbf{1}_4$, known as the Clifford algebra,

ii) $\gamma^i = -\gamma^0\gamma^i\gamma^0$,

iii) $\gamma^{0\dagger} = \gamma^0$ and $\gamma^{i\dagger} = \gamma^0\gamma^i\gamma^0 = -\gamma^i$,

iv) $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$.