FYSH300 fall 2011

Exercise 12, return by Mon Dec 5th at 9.00 to box in the lobby, discussed Mon Dec 5th, at 12.15 in FYS5

1. In the lectures, we defined the helicity operator $\hat{\lambda}$ in its matrix form for a spin-$\frac{1}{2}$-particle.

   a) Show that for a spin-$\frac{1}{2}$-particle the possible values of helicity are are $\pm\frac{1}{2}$.

   b) Using the Dirac-Pauli-representation results for the spinors $u^{(s)}$ in the case $E = E_p > 0$, show that at the ultrarelativistic limit $|\bar{p}| \gg m$ the chirality operator corresponds to the helicity operator i.e. that $\gamma^5 u^{(s)} \cong \hat{\lambda} u^{(s)}$.

2. Show how the helicity of the photon is either +1 or -1. In the lectures we concluded that the two transverse-polarization 3-vectors of a free photon, whose momentum $\mathbf{p}$ is in the z-direction, can be chosen as $\vec{\epsilon}_1 = (1, 0, 0)^T$ and $\vec{\epsilon}_2 = (0, 1, 0)^T$. For the helicity discussion, let's however choose these a little differently, namely

$$\vec{\epsilon}_R = -\frac{1}{\sqrt{2}}[\vec{\epsilon}_1 + i\vec{\epsilon}_2] \quad , \quad \vec{\epsilon}_L = \frac{1}{\sqrt{2}}[\vec{\epsilon}_1 - i\vec{\epsilon}_2].$$

   In addition, from the QMI lectures we recall that the wave function of a vector particle (such as the photon) transforms under the rotations as follows

   $\vec{\psi}'(\mathbf{x}) = \exp[i\theta\mathbf{n}\cdot\hat{\mathbf{J}}]\vec{\psi}(\mathbf{x})$, where θ is the rotation angle, $\mathbf{n}$ is the rotation axis and $\hat{\mathbf{J}} = \mathbf{1}_3\hat{\mathbf{L}} + \mathbf{\Sigma}$ is the total angular momentum operator, $\mathbf{1}_3$ is the 3-dimensional unit matrix, $\hat{\mathbf{L}}$ is the angular momentum operator and $\mathbf{\Sigma}$ is the spin operator (3×3 matrix). In other words, the spin of the particle generates the rotations of the wavefunction. Now the spatial part of the photon wavefunction is (p. 305)

$$\vec{A}_i(\mathbf{x}) = \vec{\psi}_i(\mathbf{x}) = \vec{\epsilon}_i e^{-ip\cdot x},$$

   where $i = R, L$ are the two different polarizations. Given that the photon moves into the positive z direction, and that

$$\Sigma_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

   show that $\vec{A}_{R,L}(\mathbf{x})$ are eigenvectors of the helicity operator $\hat{\Lambda} \equiv \hat{\mathbf{e}}_{\mathbf{p}} \cdot \mathbf{\Sigma}$ with an eigenvalue ± 1, correspondingly.

3. In the lectures defined the projection operators P_L and P_R. Using the properties of γ^5 show that

   a) $P_L^2 = P_L$

   b) $P_R^2 = P_R$

   c) $P_R P_L = P_L P_R = 0$

   d) $\overline{\psi}_{L,R}\gamma^\mu\psi_{L,R} = \frac{1}{2}\overline{\psi}\gamma^\mu(1 \mp \gamma^5)\psi$

4. Consider pion decays to the channels $\pi^- \to \mu^-\bar{\nu}_\mu$ and $\pi^- \to e^-\bar{\nu}_e$.

   a) Given that

$$\Gamma(\pi^- \to l\bar{\nu}_l) = \frac{G^2}{8\pi} f_\pi^2 m_\pi m_l^2 \left(1 - \frac{m_l^2}{m_\pi^2}\right)^2$$

   estimate the ratio

$$\frac{\Gamma(\pi^- \to e^-\bar{\nu_e})}{\Gamma(\pi^- \to \mu^-\bar{\nu_\mu})}.$$

   Compare your result with PDG.

   b) The physical explanation for this small ratio is that at the extreme ultrarelativistic limit only lefthanded spin-$\frac{1}{2}$-fermions couple to the weak current. Draw the (first order) Feynman diagrams for such decays and figure out the helicities of the final state particles. Show that e^- is now ultrarelativistic and righthanded.