Portfolio optimization for a large trader

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Literature review

- Frey, Platen & Schweizer,...
- Bank & Baum: universal martingale measure; Cetin, Jarrow, Protter:
- Schied *et al.* : optimal execution

Typical observations

- Trading strategies should be continuous and of finite variation, otherwise transaction costs will get incurred.
- Even if the primitive financial market is complete, the large trader will only be able to hedge approximately.
- Despite having a smaller set of admissible trading strategies, the large trader has an information advantage.
- Even if the market is arbitrage free, it can get destabilized in the presence of a large trader.

Setting

- The financial market consists of a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ modelling the evolution of information for the large trader. We assume that $\mathbb{F}$ satisfies the usual conditions with $\mathcal{F}_0$ being trivial apart from zero sets, and denote by T some finite time horizon. An *equivalent martingale measure* for a continuous semimartingale S is a probability measure Q equivalent to P such that S is a local Q-martingale.
- At the core of our model we have a family $(S(\vartheta, \cdot))$, $\vartheta \in \mathbb{R}$, of adapted stochastic processes with the following properties:

1. (i) $S(\vartheta, \cdot)$ is a continuous semimartingale for all $\vartheta \in \mathbb{R}$;
2. (ii) S is continuous in the space parameter ϑ.

- $(S(\vartheta, \cdot))_{\vartheta\in\mathbb{R}}$ serves as a model of the discounted price process of some risky asset given the large trader has a constant position of ϑ shares in that asset.

Stochastic Differential Equations

- We assume that the primal price processes $S(\vartheta, \cdot)$ are given as strong solutions of the SDE

$$dS(\vartheta, t) = b^{\vartheta}(S(\vartheta, t))\, dt + \sigma(S(\vartheta, t))\, dW_t.$$

Here W is a Brownian motion, the function

$$\begin{aligned} b : \mathbb{R} \times \mathbb{R} &\longrightarrow \mathbb{R} \\ (\vartheta, x) &\longmapsto b^{\vartheta}(x) \end{aligned}$$

is assumed to be continuous and non-decreasing in the first argument and Lipschitz continuous in the second argument. Furthermore we assume that σ is a function that is bounded from below by some $\varepsilon > 0$ and satisfying $|\sigma(x) - \sigma(y)|^2 \leq \rho(|x - y|)$ for some $\rho > 0$. Note that in general there is no universal martingale measure simultaneously for all $S(\vartheta, \cdot)$.

Reaction-Diffusion Setting

- Assume that $\psi(t,x,\vartheta)$ is a $\mathcal{C}^{1,2,1}$–function. Define similarly as e.g. in Frey (1998) the price process $S = (S(\vartheta,\cdot))_{\vartheta\in\mathbb{R}}$ via $S(\vartheta,t) = \psi(t,W_t,\vartheta)$, where the Brownian motion W models some fundamental state variable. Due to Ito's formula, the dynamics of the primitive price processes have the form

$$d\psi_t = \left(\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right)\psi_t\,dt + \frac{\partial}{\partial x}\psi_t\,dW_t.$$

We assume that $\partial\psi/\partial x$ is strictly positive and that S satisfies **(MM)**. Again, there is in general no universal martingale measure. An explicit example for a reaction function is given by

$$\psi(t,W_t,\vartheta) = \exp(\sigma W_t + \kappa\vartheta t),\ \text{where } \sigma,\kappa > 0.$$

Simple strategies

- A *simple strategy* θ for the large trader is a simple predictable process with representation

$$\theta(t) = \theta_{-1}\mathbf{1}_{\{0\}} + \sum_{i=0}^{n} \theta_i \mathbf{1}_{(\tau_i, \tau_{i+1}]}(t),$$

where $0 = \tau_0 \leq \tau_1 \leq ... \leq \tau_{n+1} = T$ is a finite sequence of $(\mathcal{F}_t)$-stopping times, θ_{-1} is some constant, and θ_i is for each $i = 0, ..., n$ a bounded $\mathcal{F}_{\tau_i}$-measurable random variable. Moreover, the stopping times τ_i and the random variables θ_i take only finitely many values. We denote the space of all simple strategies by $\mathbf{S}$. The closure of $\mathbf{S}$ wrt. the *ucp*-convergence is the space $\mathbb{L}$ of càglàd processes. By $b\mathbb{L}$ we denote the subspace of bounded càglàd processes.

Price process as non-linear stochastic integral

- To motivate the definition let $\theta \in \mathbf{S}$ and denote the sequence of stopping times appearing in its decomposition by $(\tau_i)_{i \le n+1}$. We interpret $(\tau_i)_{i \le n+1}$ as those points in the future when the large trader changes her strategy and causes a change of the price dynamics. We model the discounted price process caused by those changes by the elementary non-linear stochastic integral

$$\int_0^t S(\theta_s, ds) := S(\theta_{-1}, 0) + \sum_{i=0}^{n} \{S(\theta_{\tau_i \wedge t+}, \tau_{i+1} \wedge t) - S(\theta_{\tau_i \wedge t+}, \tau_i \wedge t)$$

Note that if $S(\vartheta, .) = S(0, .)$ for all $\vartheta \in \mathbb{R}$, the definition coincides with the classical one for small traders. In this case the trading activities of the large trader have no impact on the price evolution.

No arbitrage for small traders

- Our main assumption is that there are no arbitrage opportunities for the small trader in periods where the large investor only employs simple strategies. A mathematical condition ensuring this (as we shall see below) is formulated in the following standing assumption:
- **Assumption (MM)** For each $\vartheta \in \mathbb{R}$ there exist equivalent martingale measures Q^{ϑ} for $S(\vartheta, \cdot)$.

Theorem

Let $\theta \in \mathbf{S}$ be a simple strategy of the large trader. Under **(MM)**, *there exists an equivalent martingale measure Q^{θ} for $\int S(\theta, ds)$.*

Carmona-Nualart nonlinear stochastic integral

- Let $(\theta_n)_{n\geq 1} \subset \mathbf{S}$ denotes a sequence of simple strategies such that
$$\theta_n \xrightarrow[ucp]{} \theta,$$
where *ucp* indicates the uniform convergence on compacts in probability. Since $\theta \in \mathbb{L}$ is locally bounded, we may and do assume that $(\theta_n)_{n\geq 1} \subset \mathbf{S}$ is locally uniformly bounded.
- The discounted price process $\int S(\theta, ds)$ affected by a large trader strategy $\theta \in \mathbb{L}$ is the limit of $\left(\int S(\theta_n, ds)\right)_{n\geq 1}$ in the semimartingale topology $\mathcal{SM}$, i.e.
$$\int S(\theta_n, ds) \xrightarrow[\mathcal{SM}]{} \int S(\theta, ds).$$
- The existence of the limit follows from assumption **(SI)**, see Carmona & Nualart (1990).

Structure condition and no arbitrage

- A continuous semimartingale $S = M + A$ satisfies the *weak structure condition* **(SC′)**, if $dA \ll d\langle M\rangle$ holds. S satisfies the *structure condition* **(SC)**, if $A = \int \lambda \, d\langle M\rangle$, where $\lambda \in L^2_{loc}(M)$.
- The next result is due to Delbaen & Schachermayer (1994) and Hulley & Schweizer (2010).

Theorem

1. *S satisfies* **(SC)** *if and only if $\mathcal{E}\left(-\int \lambda \, dM\right)$ is a strictly positive local martingale density for S.*
2. *There exists an equivalent martingale measure for S if and only if S satisfies* **(SC)** *and the classical* **(NA)***-condition.*

- Due to this theorem, it is natural to use the convergence in $\mathcal{SM}$ to find conditions that ensure the existence of an equivalent martingale measure. Indeed, Assumption **(MM)** and the last theorem allow us to write the canonical decompositions of $(S(\vartheta,\cdot))_{\vartheta\in\mathbb{R}}$ as

$$S(\vartheta,\cdot) = S_0 + M^\vartheta + \int \lambda^\vartheta \, d[M^\vartheta],$$

where $\int_0^T (\lambda^\vartheta)^2 \, d[M^\vartheta] < \infty$ a.s. Moreover, the canonical decompositions of $\left(\int S(\theta_n, ds)\right)_{n\geq 1}$ can be written as

$$\int S(\theta_n, ds) = M^n + \int \lambda^{(n)} \, d[M^n],$$

where for $\theta_n(t) = \sum_{i=0}^m \theta_i \mathbf{1}_{(\tau_i,\tau_{i+1}]}(t)$ we have

$$M_t^n = \sum_{i=0}^m \left(M^{\theta_i}_{\tau_{i+1}\wedge t} - M^{\theta_i}_{\tau_i\wedge t} \right), \; \lambda_t^{(n)} = \sum_{i=0}^m \lambda_t^{\theta_i} \mathbf{1}_{(\tau_i,\tau_{i+1}]}(t).$$

No arbitrage with dynamic strategies

Theorem

With the above assumptions and notation, if

$$\liminf_{n\to\infty} \int_0^T (\lambda^{(n)})^2 \, d\left[M^n\right] < \infty, \qquad P-\text{a.s.},$$

then $\int S(\theta, ds)$ *satisfies the structure condition.*

- This condition is satisfied under natural assumptions in both the stochastic differential equation as well as the reaction-diffusion settings.
- If $\mathcal{E}\left(-\int \lambda \, dM\right)$ is a true martingale, under the theorem asssumptions it therefore is an equivalent martingale measure for $\int S(\theta, ds)$.

Admissible strategies

- A large trader strategy $\theta \in \mathbb{L}$ is called *admissible* if the following conditions hold:

1. The set $\mathcal{M}_e\left(\int S(\theta, ds)\right)$ of all equivalent martingale measures for $\int S(\theta, ds)$ is not empty;
2. The large trader wealth process $V(\theta) = V_0 + \left(\theta \cdot \int S(\theta, ds)\right)$ is a supermartingale for all $Q \in \mathcal{M}_e\left(\int S(\theta, ds)\right)$.

- We denote the set of all admissible trading strategies by Θ.

- We analyse the utility maximization problem of a large trader in a basic setting. Despite its rather simple structure it highlights new phenomena that are not present in the classical utility maximization theory for small traders. At this point we want to emphasise that these phenomena appear even though the market is arbitrage free. These new phenomena are a consequence of the non-linear structure of the value process involved in the problem.
- Let M be a continuous and square-integrable martingale starting at zero a.s., such that $[M]$ is a deterministic process. Further let $\mu, \sigma \in C^2(\mathbb{R}, \mathbb{R})$. For $\vartheta \in \mathbb{R}$ and $S_0 > 0$ we set

$$S(\vartheta, t) := S_0 + \mu(\vartheta)\,[M]_t + \sigma(\vartheta) M_t, \qquad t \in [0, T],$$

and we assume that **(MM)** holds.

Portfolio optimization for a large trader

- For $\theta \in \mathbb{L}$ the price process and the large trader value process are given by

$$\int S(\theta_s, ds) = S_0 + \int \mu(\theta_s)\, d[M]_s + \int \sigma(\theta_s)\, dM_s$$

and

$$V_t(\theta) = V_0 + \int_0^t \theta_s \mu(\theta_s)\, d[M]_s + \int_0^t \theta_s \sigma(\theta_s)\, dM_s,$$

respectively.

- For constant and fixed initial value V_0 we consider the exponential utility maximization problem

$$\sup_{\theta \in \Phi} E\left[u\left(V_T(\theta)\right)\right],$$

where $u(x) = 1 - e^{-\alpha x}$ for $\alpha > 0$ and $\Phi := b\mathbb{L} \subset \mathbb{L}$ is the set of all bounded processes having càglàd paths.

Changing measures

- Due to Novikov's condition, we can define probability measures $Q^\theta \sim P$ via

$$\frac{dQ^\theta}{dP} := \exp\left(-\alpha \int_0^T \theta_s \sigma(\theta_s)\, dM_s - \frac{\alpha^2}{2} \int_0^T \theta_s^2 \sigma^2(\theta_s)\, d\,[M]_s\right), \qquad \forall \theta \in$$

Since

$$E\left[u(V_T(\theta))\right] = 1 - \exp\left(-\alpha V_0\right) E_{Q^\theta}\left[\exp\left(-\alpha \int_0^T p(\theta_s)\, d\,[M]_s\right)\right],$$

with

$$p(x) := x\mu(x) - \frac{\alpha}{2} x^2 \sigma^2(x), \qquad x \in \mathbb{R},$$

the utility maximization problem is well posed.

Different regimes

- As we will see below two different scenarios might happen. In the first scenario, the so-called *stable regime*, we can find at least one optimal strategy. These optimal strategies are constant.
- In the second scenario, the *unstable regime*, the presence of the large trader completely destabilises the market. This is caused by the fact that in an unstable regime it is optimal for the large trader to buy/sell as many shares as possible to maximize her expected utility from terminal wealth.
- It will turn out that the existence of an optimal strategy boils down to the existence of a maximum of the function p. The following observations will show that the market is stable, if and only if the function p with

$$p(x) := x\mu(x) - \frac{\alpha}{2}x^2\sigma^2(x), \qquad x \in \mathbb{R},$$

attains at least one maximum.

Stable regime

- Stable regime: Let us suppose that p has at least one maximum $\vartheta^* \in \mathbb{R}$. We get for $\theta \in \Phi$

$$E\left[u(V_T(\theta))\right]$$

$$= 1 - \exp\left(-\alpha(V_0 + p(\vartheta^*)\left[M\right]_T)\right) E_{Q^\theta}\left[\exp\left(-\alpha\int_0^T (p(\theta_s) - p(\vartheta^*))\right.\right.$$

Since

$$P\left(p(\theta_s) - p(\vartheta^*) \leq 0,\ \forall\theta \in \Phi\right) = 1,$$

it follows that ϑ^* is the optimal strategy and

$$\sup_{\theta\in\Phi} E\left[u\left(V_T\left(\theta\right)\right)\right] = 1 - \exp\left(-\alpha(V_0 + p(\vartheta^*)\left[M\right]_T)\right).$$

Unstable regime

- Unstable regime: Let us suppose that p has no maximum. Due to the continuity of p we can find a sequence $(\vartheta_n)_{n\in\mathbb{N}} \subset \Phi$ of constant strategies such that

$$\sup_{\vartheta\in\mathbb{R}} p(\vartheta) = \begin{cases} \lim_{n\to\infty} p(\vartheta_n) =: p^* \in \mathbb{R}, & \text{if } p \text{ is bounded from above,} \\ +\infty, & \text{else.} \end{cases}$$

Keeping this in mind, it follows that

$$\sup_{\theta\in\Phi} E\left[u\left(V_T\left(\theta\right)\right)\right] = \begin{cases} 1 - \exp\left(-\alpha\left(V_0 + p^*\left[M\right]_T\right)\right), & \text{if } p \text{ is bounded f} \\ 1, & \text{else.} \end{cases}$$

Since continuous functions attain their extreme points on compact intervals, it is clear that $\vartheta_n \to \pm\infty$. Obviously $\vartheta_n \to +\infty$ means that the large trader tries to buy as many shares as possible in order to reach her maximal expected utility of terminal wealth. $\vartheta_n \to -\infty$ means that she achieves her goal by short selling. Therefore there is no optimal strategy $\theta \in \Phi$. Such trading strategies lead to exploding or collapsing prices and therefore destabilize the market.