Comparing binomial and Gaussian tails with an application to utility maximization

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Joint work with Walter Schachermayer

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1, The standardized simple random walk

For arbitrary $p \in (0,1)$ consider $(\alpha_n)_{n\geq 1}$ iid Bernoulli with

$$P[\alpha_n = 0] = 1 - p, \qquad P[\alpha_n = 1] = p, \tag{1}$$

Denote by $(\zeta_n)_{n\geq 1}$ the standardized variables, so that $E[\zeta_n] = 0$ and $\mathrm{Var}[\zeta_n] = 1$, and let $\xi_{n,k}$ the scaled partial sums, that is,

$$\zeta_n = \frac{\alpha_n - p}{\sqrt{p(1-p)}}, \qquad \xi_{n,k} = \frac{1}{\sqrt{n}} \sum_{j=1}^{k} \zeta_j. \tag{2}$$

For $k = 0, \ldots, n$ set

$$z_{n,k} = \frac{k - np}{\sqrt{np(1-p)}}, \qquad f_{n,k} = \binom{n}{k} p^k (1-p)^{n-k}. \tag{3}$$

Then

$$f_{n,k} = P[\xi_{n,n} = z_{n,k}]. \tag{4}$$

2, The tail comparison proposition

Denote the lower and upper tail for the standardized binomial distributions by

$$F_n(x) = P[\xi_{n,n} \leq x], \qquad \bar{F}_n(x) = P[\xi_{n,n} > x]. \tag{5}$$

If $p \in [1/2, 1)$, then there is $C > 0$ such that, for $n \geq 1$, we have

$$f_{n,k} \leq C \cdot \frac{1}{\sqrt{np(1-p)}} \phi(z_{n,k-1}), \qquad 0 \leq k \leq \lceil np \rceil, \tag{6}$$

$$f_{n,k} \leq C \cdot \frac{1}{\sqrt{np(1-p)}} \phi(z_{n,k+1}), \qquad \lfloor np \rfloor \leq k \leq n. \tag{7}$$

Furthermore

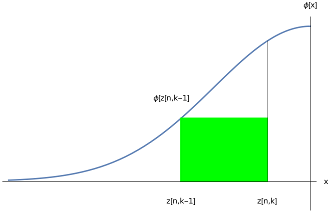
$$F_n(x) \leq C\Phi(x), \qquad x \leq 0 \tag{8}$$

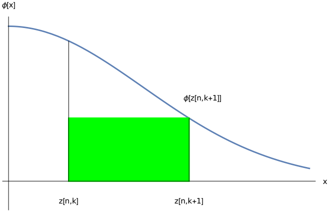
and

$$\bar{F}_n(x) \leq C\bar{\Phi}(x), \qquad x \geq 0. \tag{9}$$

Here ϕ is the density, Φ the lower tail and $\bar{\Phi}$ the upper tail of $N(0,1)$.

3, An illustration of the proposition





Remark

The terms $\frac{1}{\sqrt{np(1-p)}}\phi(z_{n,k\pm1})$ *in (6) and (7) are a lower bound for the area under the density* $\varphi(x)$ *between* $z_{n,k-1}$ *and* $z_{n,k}$ *on the left and* $z_{n,k}$ *and* $z_{n,k+1}$ *on the right tail, respectively.*

4, Classical, well-known, trivial, boring. . . ?

- The CLT is about absolute errors.
- Both the binomial and the Gaussian tails are negligibly small.
- Look at large deviation results?
- Inbetween a gap, the moderate deviation region. . .
- Please show me a precise argument or reference!

Masashi Okamoto, Some inequalities relating to the partial sum of binomial probabilities. Ann Inst Stat Math 10, 29–35 (1959) [Hoeffding, Chernoff/Rubin, . . .]

$$\Rightarrow \bar{F}_n(x) \leq C\phi(x)$$

Almost there, but miss an x^{-1} since $\bar{\Phi}(x) \sim x^{-1}\phi(x)$ as $x \to +\infty$. Recall

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

5, From a Feller paper

W. Feller, On the Normal Approximation to the Binomial Distribution
Ann. Math. Statist. 16 (1945), 319–329.

1. Although the problem of an efficient estimation of the error in the normal approximation to the binomial distribution is classical, the many papers which are still being written on the subject show that not all pertinent questions have found a satisfactory solution. Let for a fixed n and $0 < p < 1$, $q = 1 - p$,

so that any estimate of the form $O(\sigma$ $)$ leaves us without guidance. With some modifications this remains true also for more refined estimates like Uspensky's remarkable result[2]

$$P_{\lambda,\nu} = \Pi_{\lambda,\nu} + \frac{q-p}{6\sigma(2\pi)^{1/2}}\left[(1-z^2)e^{-z^2/2}\right]\Big|_{z_\lambda-1/2\sigma}^{z_\nu+1/2\sigma} + \omega \qquad (4)$$

with

$$|\,\omega\,| < \{.13 + .18\,|\,p-q\,|\}\sigma^{-2} + e^{-3\sigma/2}$$

provided $\sigma \geq 5$. What is really needed in many applications is an estimate of the relative error, but this seems difficult to obtain.

It should also be noticed that the accuracy of the normal approximation to the binomial is by no means quite as good as many texts would make appear. Exam-

Precisely: Needed for the utility maximization application below!

6, Kraft on Kambo and Kotz

Olaf Krafft, A note on exponential bounds for binomial probabilities Annals of the Institute of Statistical Mathematics volume 21, pages 219–220 (1969)

In [1] Kambo and Kotz gave bounds on deviations from the mean of binomial probabilities in order to strengthen results of Okamoto [2]. The bounds are based on the following inequality:

$$(1)\qquad L(p, c)=(p+c)\log\left(1+\frac{c}{p}\right)+(1-p-c)\log\left(1-\frac{c}{1-p}\right)\geqq 2c^2+\frac{4}{3}c^4\,,$$

where $0<p<1$ and $c\geqq 0$. Putting $c=\frac{1}{3}$, $p=\frac{1}{3}$ one sees immediately that (1) does not hold in the stated generality. An inspection of the proof of that inequality shows that the argument of symmetry does not go through. On the other hand, for most of the values (p, c) for which (1) holds, the inequality can be sharpened considerably. It may be remarked that values $c>1-p$ are of no interest for the considered problem

7, McKay on Littlewood

Brendan McKay, On Littlewood's Estimate for the Binomial Distribution, Advances in Applied Probability 1989, Vol. 21, No. 2, 475–478.

We begin with Littlewood's Theorem 2. With the errors corrected, we can state that theorem as follows.

Theorem 1. *Let p, $0<p<1$, be fixed. Let $t=t(n)$ be such that $pn+t$ is an integer and $0 \leqq t \leqq \frac{3}{4}qn$. Define $x=t/\sigma$ and $\rho = q - t/n$. Then*

$$B(pn+t;n,p) = Q(x)\exp\left(A_1 + A_2/\sqrt{\rho(1-\rho)n} + O(n^{-1})\right),$$

where

$$A_1 = \frac{t^2}{2pqn} - (pn+t-\tfrac{1}{2})\log\left(1+\frac{t}{pn}\right) - (qn-t+\tfrac{1}{2})\log\left(1-\frac{t}{qn}\right)$$

and

$$A_2 = \tfrac{1}{6}(1-2\rho)((1-x^2)/Y(x)+x^3) + \tfrac{1}{2}(1/Y(x)-x).$$

Proof. Littlewood's statement of this theorem has three errors:

(i) The coefficient $\frac{1}{2}$ should be $\frac{1}{6}$ (it comes from c_3).
(ii) The definition of ρ should be as in (his) Theorem 1.
(iii) The sign of the $O(n^{-1/2})$ term is wrong.

Errors (i) and (ii) are merely typos. Error (iii) can be traced to an incorrect sign change in passing from (22.5) to (22.6). Apart from this, Littlewood's proof appears valid. To guard against other gross errors, we have succesfully checked the theorem numerically for a wide range of values of the parameters.

I wish to thank Nick Wormald for some useful ideas. Also, the referee deserves thanks for detecting an error in the first draft of this note, thereby (one hopes) obviating the need for a third iteration!

8, Motivation

The Kreps book

David M. Kreps
The Black–Scholes–Merton Model as an Idealization of Discrete-Time Economies
Cambridge University Press, Cambridge, 2019.

The Kreps and Schachermayer paper

David M. Kreps and Walter Schachermayer
Convergence of Optimal Expected Utility for a Sequence of Discrete-Time Markets
Mathematical Finance 30(4):1205–1228, 2020.

Our note

Friedrich Hubalek and Walter Schachermayer
Convergence of Optimal Expected Utility for a Sequence of Binomial Models
arXiv:2009.09751 [math.PR]

9, Two central topics in mathematical finance

- Pricing and hedging of derivatives
- Dynamic utility maximization

If we have a sequence of market models S^n approximating a limit model S do

- Option prices, trading strategies, gains processes, hedging errors etc. from the sequence converge [...] to the corresponding [...] quantities in the limit model?
- And do utilities and optimal investments etc. converge?

More precise: Consider also measures P^n and P and possibly equivalent martingale measures Q^n and Q in this discussion!

10, On the convergence of option prices

Friedrich Hubalek and Walter Schachermayer, When Does Convergence of Asset Price Processes Imply Convergence of Option Prices? Volume 8, Issue 4, October 1998, Pages 385-403.

- The usual (homogeneous) approximation is fine.
- Minor variations (e.g. odd-even model) behave wildly. Connected to asymptotic arbitrage. [. . .]

Many other results, e.g.,

Jean-Luc Prigent, Weak Convergence of Financial Markets, Springer, 2003.

11, Contiguity and entire separation

1. Contiguity $(Q^n) \lhd (P^n)$: If

$$P^n(A^n) \to 0 \quad \Rightarrow \quad Q^n(A^n) \to 0$$

   as $n \to \infty$ for all $A^n \in \mathcal{F}^n$.

2. Entire separation $(Q^n) \bigtriangleup (P^n)$: If there is a subsequence $n_k \to \infty$ and for each k a set A^{n_k}, such that

$$P^{n_k}(A^{n_k}) \to 1, \quad \text{and} \quad Q^{n_k}(A^{n_k}) \to 0$$

   as $k \to \infty$.

Useful criteria in terms of

$$h^n(\alpha) = \sum_{k=1}^{n} \left[1 - H(\alpha; p_k^n, q_k^n)\right], \tag{10}$$

where $H(\alpha; p_k^n, q_k^n)$ is the Hellinger integral of order $\alpha \in (0, 1)$.

12, The sequence of discrete-time markets for utility maximization

Let $(\zeta_n)_{n\geq 1}$ an iid sequence of standardized random variables with bounded support [. . . !], i.e.,

$$E[\zeta] = 0, \quad \mathrm{Var}[\zeta] = 1, \quad \exists M > 0 : P[|\zeta| < M] = 1$$

For $n \geq 1$ define a sequence of continuous time processes $(\xi^n(t) : t \in [0,1])$ on the grid

$$\xi^n\left(\frac{k}{n}\right) = \frac{1}{\sqrt{n}} \sum_{j=1}^{k} \zeta_j, \quad k = 0, \ldots, n$$

and extend for all $t \in [0,1]$ by linear interpolation. [. . .]

The n-th discrete time market consists of a bond $(B(t) : t \in [0,1])$ and a stock $(S^n(t) : t \in [0,1])$ with

$$B(t) = 1, \quad S^n(t) = e^{\xi^n(t)}, \quad t \in [0,1].$$

13, The continuous-time limit

Let $(W(t) : t \in [0,1])$ a standard Brownian motion.

The continuous-time limit market consists of a bond $(B(t) : t \in [0,1])$ and a stock $(S(t) : t \in [0,1])$ with

$$B(t) = 1, \quad S(t) = e^{W(t)}, \quad t \in [0,1].$$

This is the theorist's version of the Black-Scholes-Merton model, i.e.,

$$T = 1, \quad S(0) = 1, \quad r = 0, \quad q = 0, \quad \mu = \frac{1}{2}, \quad \sigma = 1.$$

Donsker's Theorem implies $\xi^n \stackrel{d}{\to} W$ as $n \to \infty$, and thus

$$S^n \stackrel{d}{\to} S.$$

14, Utility maximization

Suppose $U : (0, \infty) \to \mathbb{R}$ is a utility function [...].

For $x > 0$ let $\mathcal{X}_n(x)$ resp. $\mathcal{X}(x)$ the set of all outcomes (terminal wealth)

- from trading in the n-th discrete-time market
- resp. the continuous-time limit market
- starting with capital $x > 0$,
- following an admissible, self-financing strategy. [...]

Expected utility from terminal wealth

$$u_n(x) = \sup_{X \in \mathcal{X}_n(x)} E[U(X)], \qquad u(x) = \sup_{X \in \mathcal{X}(x)} E[U(X)]$$

Is this utility maximization *continuous* ? I.e.,

$$\lim_{n \to \infty} u_n(x) \stackrel{?}{=} u(x), \quad x > 0.$$

15, Assumptions on the utility function

The utility function $U : (0, \infty) \to \mathbb{R}$ is

- strictly increasing, strictly concave, continuously differentiable,
- satisfies the Inada conditions

$$\lim_{x \to 0} U'(x) = +\infty, \quad \lim_{x \to \infty} U'(x) = 0$$

- and (w.l.o.g.) has $\lim_{x \to \infty} U(x) > 0$.

Note we allow

- $\lim_{x \to 0} U(x) = -\infty$ or finite, and
- $\lim_{x \to \infty} U(x) = +\infty$ or finite positive!

16, The Kreps-Schachermayer convergence result

The asymptotic elasticity of U is

$$\text{Æ}(U) = \limsup_{x\to\infty} \frac{xU'(x)}{U(x)}.$$

Theorem (Kreps and Schachermayer 2020, Theorem 8.1)

Given

- *a sequence of discrete-time markets as above, and*
- *a utility function U satisfying the conditions above,*
- *which has $\text{Æ}(U) < 1$*

then

$$u(x) < \infty \quad \forall x > 0$$

and

$$\lim_{n\to\infty} u_n(x) = u(x) \quad \forall x > 0.$$

□

17, The Kreps-Schachermayer counterexample

There is

- a sequence of discrete-time markets as above, and
- a utility function U satisfying the conditions above,
- which has $AE(U) = 1$

such that

$$u(x) < \infty \quad \forall x > 0$$

and

$$\lim_{n\to\infty} u_n(x) = +\infty \quad \forall x > 0.$$

- The sequence of markets is easy, need only $E[\zeta^3] > 0$.
- Construction of the appropriate utility function U is more tricky! [...]

18, The binomial counterexample and the binomial question

A standardized Ber(p) has

$$E[\zeta^3] = \frac{1-2p}{\sqrt{p(1-p)}}$$

and thus $E[\zeta^3] > 0$ for $0 < p < 1/2$.

In other words: In a sequence of Cox-Ross-Rubinstein binomial models with $p \in (0, 1/2)$ utility explodes!

Urgent question

What happens for $p = \frac{1}{2}$ or more generally $\frac{1}{2} \leq p < 1$?

19, A binomial convergence result

Theorem (Hubalek and Schachermayer 2020, Theorem 1)

Given

- *a sequence of binomial markets with* $p \in [1/2, 1)$*, and*
- *a utility function* U *satisfying the conditions above,*

then

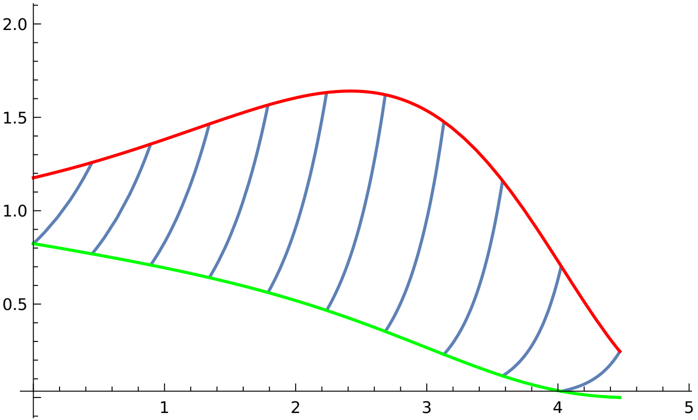
$$u(x) < \infty \quad \forall x > 0$$

and

$$\lim_{n \to \infty} u_n(x) = u(x) \quad \forall x > 0.$$

□

20, Back to the tail comparison: An illustration



Illustration

$$C_n(x) = \frac{\bar{F}_n(x)}{\bar{\Phi}(x)}$$

for $n = 20$ and $p = 1/2$ on $0 \leq x \leq \sqrt{n}$.

21, Binomial tail and Beta functions

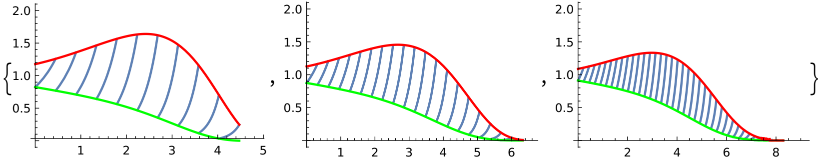
Well-known and old: Binomial CDF by Regularized Incomplete Beta-Function

$$\bar{F}_n(x) = 1 - I_{\frac{1}{2}}(n - \lfloor x \rfloor, 1 + \lfloor x \rfloor),$$

$$I_z(a,b) = \frac{B_z(a,b)}{B(a,b)}, \quad B_z(a,b) = \int_0^z t^{a-1}(1-t)^{b-1} dt$$

Well-known asymptotics of $I_z(a,b)$? [...]

22, More tail comparison illustrations



Illustration

$$C_n(x) = \frac{\bar{F}_n(x)}{\bar{\Phi}(x)}$$

for $n = 20, 40, 80$ and $p = 1/2$ on $1/2 \leq x \leq \sqrt{n}$.

23, Approximating factorials

Recall: For $k = 0, \ldots, n$ set

$$z_{n,k} = \frac{k - np}{\sqrt{np(1-p)}}, \qquad f_{n,k} = \binom{n}{k} p^k (1-p)^{n-k}. \tag{11}$$

Most cumbersome

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}. \tag{12}$$

- Central area: Exploit interaction of k and n when both growing more or less 'together'.
- Far tails: Have k or $n-k$ much smaller than n.
- But I do not want to split and paste!

Need an accurate version of Stirling's formula for all arguments!

24, A fine version of Stirling's formula

Milton Abramowitz and Irene A. Stegun, editors. Handbook of mathematical functions with formulas, graphs, and mathematical tables. Dover Publications, Inc., New York, 1992. Reprint of the 1972 edition. [6.1.38, p.257]

$$x! = \sqrt{2\pi} x^{x+\frac{1}{2}} \exp\left(-x + \frac{\theta(x)}{12x}\right), \quad x > 0, \tag{13}$$

with $0 < \theta(x) < 1$ for all $x > 0$. Notation: $x! = \Gamma(x+1)$.

[DLMF, 5.6.1] NIST Digital Library of Mathematical Functions. `http://dlmf.nist.gov/,Release1.0.28of2020-09-15`. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.

NIST Handbook of Mathematical Functions Hardback and CD-ROM Edited by Frank W. J. Olver, Daniel W. Lozier, Ronald F. Boisvert Charles W. Clark

25, The key auxiliary function, $p = 1/2$

From

$$z_{n,k} = \frac{2k-n}{\sqrt{n}}, \qquad f_{n,k} = \binom{n}{k} 2^{-n}. \tag{14}$$

with

$$x! = \sqrt{2\pi} x^{x+\frac{1}{2}} \exp\left(-x + \frac{\theta(x)}{12x}\right), \quad x > 0, \tag{15}$$

and patience and/or computer help

$$\log\left(\frac{\sqrt{n} f_{n,k}}{2\phi(z_{n,k+1})}\right) \leq g_n\left(\frac{k}{n}\right), \tag{16}$$

with $g_n(w) = \alpha(w)n + \beta_n(w)$, where $w \in [\frac{1}{2}, 1]$, and

$$\alpha(w) = -w\log w - (1-w)\log(1-w) - 2w(1-w) + \frac{1}{2} - \log 2 \tag{17}$$

and

$$\beta_n(w) = -\frac{1}{2}\log w - \frac{1}{2}\log(1-w) + 4w - 2 + \frac{25}{12n}. \tag{18}$$

26, An elementary discussion

It remains to show that $g_n(w) = \alpha(w)n + \beta_n(w)$ is bounded from above uniformly in $n \in \mathbb{N}$ and $w \in [1/2, 1 - 1/n]$.

$$\alpha(w) = -w \log w - (1-w)\log(1-w) - 2w(1-w) + \frac{1}{2} - \log 2 \quad (19)$$

and

$$\beta_n(w) = -\frac{1}{2}\log w - \frac{1}{2}\log(1-w) + 4w - 2 + \frac{25}{12n}. \quad (20)$$

We have

$$\alpha\left(\frac{1}{2}\right) = \alpha'\left(\frac{1}{2}\right) = \alpha''\left(\frac{1}{2}\right) = \alpha'''\left(\frac{1}{2}\right) = 0$$

and

$$\alpha^{iv}(w) = -\frac{2}{(1-w)^3} - \frac{2}{w^3} < 0$$

for $w \in (\frac{1}{2}, 1)$. Thus $\alpha'''(w)$, $\alpha''(w)$, $\alpha'(w)$, and $\alpha(w)$ is strictly negative and decreasing for $w \in (\frac{1}{2}, 1)$. [...]

Similar, more asymmetric, for $p \in (1/2, 1)$.

27, Variants of the comparison

Always $p \in [1/2, 1)$ fixed.

Our proposition, as required for the application:

$$\exists C > 0, \quad \forall n \geq 1 \quad \forall x > 0 \qquad \bar{F}_n(x) \leq C\bar{\Phi}(x)$$

Sharpening from the proof:

$$\forall C > 1 \quad \exists n_0(C) \geq 1 \quad \forall n \geq n_0(C) \quad \forall x > 0 \qquad \bar{F}_n(x) \leq C\bar{\Phi}(x)$$

Similar for the left tail.

Remark: We cannot have $C = 1$.
Scaled binomial and Gaussian interlace. [. . .]

Also interesting: Asymptotics [. . . ?]

$$\sup_{x>0} \frac{\bar{F}_n(x)}{\bar{\Phi}(x)} \qquad n \to \infty$$

28, Sketch of utility convergence proof

Recall

- Utility from trading in the n-th binomial market is $u_n(x)$,
- Utility from trading in the Black-Scholes-Merton market is $u(x)$.

We want to show

$$\lim_{n\to\infty} u_n(x) = u(x).$$

Recall the solutions from [...]

- Dimitrij Kramkov and Walter Schachermayer, The asymptotic elasticity of utility functions and optimal investment in incomplete markets. The Annals of Applied Probability, 9(3):904–950, 1999. [KS99]
- David M. Kreps and Walter Schachermayer, Convergence of Optimal Expected Utility for a Sequence of Discrete-Time Markets, Mathematical Finance 30(4):1205–1228, 2020. [KS20]

All our markets are complete! Easy!

29, Solutions for utility maximization

Conjugate function of U, namely

$$V(y) = \sup_{x>0} \{U(x) - xy\}.$$

Unique equivalent martingale measures Q^n and Q with

$$\frac{dQ^n}{dP^n} = Z_n, \qquad \frac{dQ}{dP} = Z$$

where

$$Z_n = \exp\left(-a_n \xi^n(1) - b_n\right), \qquad Z = \exp\left(-\frac{1}{2} W(1) - \frac{1}{8}\right)$$

with $a_n = 1/2$ and $b_n = n \log\cosh\left(\frac{1}{2\sqrt{n}}\right)$ when $p = 1/2$. [...]

30, Solutions for utility maximization (continued)

Set

$$v_n(y) = E[V(yZ_n)], \qquad v(y) = E[V(yZ)]$$

then

$$u_n(x) = \inf_{y>0} \{v_n(y) + xy\}$$

and

$$u(x) = \inf_{y>0} \{v(y) + xy\}.$$

To show $u_n(x) \to u(x)$ it is sufficient [KS20] to have

$$\lim_{n\to\infty} v_n(y) = v(y), \quad y > 0.$$

From [KS20, Prop.2] we know

$$\liminf_{n\to\infty} v_n(y) \geq v(y), \quad y > 0.$$

31, The liminf

We only have to show

$$\liminf_{n\to\infty} v_n(y) \le v(y), \quad y > 0.$$

For $x \in \mathbb{R}$ let

$$H_n(x) = V\left(y \exp\left(-a_n x - b_n\right)\right), \qquad H(x) = V\left(y \exp\left(-\frac{x}{2} - \frac{1}{8}\right)\right),$$

with $a_n = 1/2$ when $p = 1/2$ [...] and then $H_n(x) \le H(x)$ for all $x \in \mathbb{R}$ and

$$v_n(y) = E[H_n(\xi^n(1)] \le E[H(\xi^n(1)], \qquad v(y) = E[H(W(1))].$$

We know H is continuous. If H was bounded, we are done by weak convergence. Typically it is not. We need uniform integrability.

$$\mathbb{E}\left[|H(\xi^n(1))|\mathbb{1}_{\{|H(\xi^n(1))|>M\}}\right] = \sum_{k=0}^{n} |H(z_{n,k})|\mathbb{1}_{\{|H(z_{n,k})|>M\}} f_{n,k} < \varepsilon,$$

This is provided by our tail comparison proposition. [...] □

Asymmetric $p \in (1/2, 1)$ slightly more technical, but similar.

32, Interlacing tails, $n = 40$, $p = 1/2$