25.2. Finite horizon

1. The arbitrage-free price of the *American put option* with *finite horizon* (cf. (25.1.1) above) is given by

$$V(t,x) = \sup_{0 \le \tau \le T-t} \mathsf{E}_{t,x}\big(e^{-r\tau}(K - X_{t+\tau})^+\big) \tag{25.2.1}$$

where τ is a stopping time of the geometric Brownian motion $X = (X_{t+s})_{s\ge 0}$ solving

$$dX_{t+s} = rX_{t+s}\,ds + \sigma X_{t+s}\,dB_s \tag{25.2.2}$$

with $X_t = x > 0$ under $\mathsf{P}_{t,x}$. We recall that $B = (B_s)_{s\ge 0}$ denotes a standard Brownian motion process started at zero, $T > 0$ is the expiration date (maturity), $r > 0$ is the interest rate, $K > 0$ is the strike (exercise) price, and $\sigma > 0$ is the volatility coefficient. Similarly to (25.1.2) the strong solution of (25.2.2) under $\mathsf{P}_{t,x}$ is given by

$$X_{t+s} = x \exp\big(\sigma B_s + (r - \sigma^2/2)\,s\big) \tag{25.2.3}$$

whenever $t \ge 0$ and $x > 0$ are given and fixed. The process X is strong Markov (diffusion) with the infinitesimal generator given by

$$\mathbb{L}_X = rx\frac{\partial}{\partial x} + \frac{\sigma^2}{2}\,x^2\,\frac{\partial^2}{\partial x^2}. \tag{25.2.4}$$

We refer to [197] for more information on the derivation and economic meaning of (25.2.1).

2. Let us determine the structure of the optimal stopping time in the problem (25.2.1).

(i) First note that since the gain function $G(x) = (K-x)^+$ is continuous, it is possible to apply Corollary 2.9 (Finite horizon) with Remark 2.10 and conclude that there exists an optimal stopping time in the problem (25.2.1). From our earlier considerations we may therefore conclude that the *continuation set* equals

$$C = \{\,(t,x) \in [0,T) \times (0,\infty) : V(t,x) > G(x)\,\} \tag{25.2.5}$$

and the *stopping set* equals

$$\bar{D} = \{\,(t,x) \in [0,T] \times (0,\infty) : V(t,x) = G(x)\,\}. \tag{25.2.6}$$

It means that the stopping time $\tau_{\bar{D}}$ defined by

$$\tau_{\bar{D}} = \inf\{\,0 \le s \le T-t : X_{t+s} \in \bar{D}\,\} \tag{25.2.7}$$

is optimal in (25.2.1).

(ii) We claim that all points (t,x) with $x \geq K$ for $0 \leq t < T$ belong to the continuation set C. Indeed, this is easily verified by considering $\tau_\varepsilon = \inf\{\, 0 \leq s \leq T-t : X_{t+s} \leq K-\varepsilon \,\}$ for $0 < \varepsilon < K$ and noting that $\mathsf{P}_{t,x}(0 < \tau_\varepsilon < T-t) > 0$ if $x \geq K$ with $0 \leq t < T$. The strict inequality implies that $\mathsf{E}_{t,x}(e^{-r\tau_\varepsilon}(K - X_{t+\tau_\varepsilon})^+) > 0$ so that (t,x) with $x \geq K$ for $0 \leq t < T$ cannot belong to the stopping set $\bar{D}$ as claimed.

(iii) Recalling the solution to the problem (25.2.1) in the case of infinite horizon, where the stopping time $\tau_* = \inf\{\, s > 0 : X_s \leq A_* \,\}$ is optimal and $0 < A_* < K$ is explicitly given by Theorem 25.1 above, we see that all points (t,x) with $0 < x \leq A_*$ for $0 \leq t \leq T$ belong to the stopping set $\bar{D}$. Moreover, since $x \mapsto V(t,x)$ is convex on $(0,\infty)$ for each $0 \leq t \leq T$ given and fixed (the latter is easily verified using (25.2.1) and (25.2.3) above), it follows directly from the previous two conclusions about C and $\bar{D}$ that there exists a function $b : [0,T] \to \mathbb{R}$ satisfying $0 < A_* \leq b(t) < K$ for all $0 \leq t < T$ (later we will see that $b(T) = K$ as well) such that the continuation set C equals

$$C = \{(t,x) \in [0,T) \times (0,\infty) : x > b(t)\} \tag{25.2.8}$$

and the stopping set $\bar{D}$ is the closure of the set

$$D = \{(t,x) \in [0,T] \times (0,\infty) : x < b(t)\} \tag{25.2.9}$$

joined with remaining points (T,x) for $x \geq b(T)$. (Below we will show that V is continuous so that C is open.)

(iv) Since the problem (25.2.1) is time-homogeneous, in the sense that the gain function $G(x) = (K-x)^+$ is a function of space only (i.e. does not depend on time), it follows that the map $t \mapsto V(t,x)$ is decreasing on $[0,T]$ for each $x \in (0,\infty)$. Hence if (t,x) belongs to C for some $x \in (0,\infty)$ and we take any other $0 \leq t' < t \leq T$, then $V(t',x) - G(x) \geq V(t,x) - G(x) > 0$, showing that (t',x) belongs to C as well. From this we may conclude that the boundary $t \mapsto b(t)$ in (25.2.8) and (25.2.9) is *increasing* on $[0,T]$.

3. Let us show that the value function $(t,x) \mapsto V(t,x)$ is *continuous* on $[0,T] \times (0,\infty)$.

For this, it is enough to prove that

$$x \mapsto V(t,x) \quad \text{is continuous at } x_0, \tag{25.2.10}$$

$$t \mapsto V(t,x) \quad \text{is continuous at } t_0 \text{ uniformly over } x \in [x_0-\delta, x_0+\delta] \tag{25.2.11}$$

for each $(t_0,x_0) \in [0,T] \times (0,\infty)$ with some $\delta > 0$ small enough (it may depend on x_0).

Since (25.2.10) follows from the fact that $x \mapsto V(t,x)$ is convex on $(0,\infty)$, it remains to establish (25.2.11).

For this, let us fix arbitrary $0 \le t_1 < t_2 \le T$ and $x \in (0,\infty)$, and let $\tau_1 = \tau_*(t_1,x)$ denote the optimal stopping time for $V(t_1,x)$. Set $\tau_2 = \tau_1 \wedge (T-t_2)$ and note, since $t \mapsto V(t,x)$ is decreasing on $[0,T]$, that upon denoting $S_t = \exp(\sigma B_t + \gamma t)$ with $\gamma = r - \sigma^2/2$ we have

$$
\begin{aligned}
0 &\le V(t_1,x) - V(t_2,x) \qquad\qquad (25.2.12)\\
&\le \mathsf{E}\left(e^{-r\tau_1}(K - xS_{\tau_1})^+\right) - \mathsf{E}\left(e^{-r\tau_2}(K - xS_{\tau_2})^+\right)\\
&\le \mathsf{E}\left(e^{-r\tau_2}\left[(K - xS_{\tau_1})^+ - (K - xS_{\tau_2})^+\right]\right)\\
&\le x\,\mathsf{E}\,(S_{\tau_2} - S_{\tau_1})^+
\end{aligned}
$$

where we use that $\tau_2 \le \tau_1$ and that $(K-y)^+ - (K-z)^+ \le (z-y)^+$ for $y,z \in \mathbb{R}$.

Set $Z_t = \sigma B_t + \gamma t$ and recall that stationary independent increments of $Z = (Z_t)_{t\ge 0}$ imply that $(Z_{\tau_2+t} - Z_{\tau_2})_{t\ge 0}$ is a version of Z, i.e. the two processes have the same law. Using that $\tau_1 - \tau_2 \le t_2 - t_1$ hence we get

$$
\begin{aligned}
\mathsf{E}\,(S_{\tau_2} - S_{\tau_1})^+ &= \mathsf{E}\Big(\mathsf{E}\left((S_{\tau_2} - S_{\tau_1})^+ \mid \mathcal{F}_{\tau_2}\right)\Big) \qquad (25.2.13)\\
&= \mathsf{E}\Big(S_{\tau_2}\,\mathsf{E}\left((1 - S_{\tau_1}/S_{\tau_2})^+ \mid \mathcal{F}_{\tau_2}\right)\Big)\\
&= \mathsf{E}\Big(S_{\tau_2}\,\mathsf{E}\left((1 - e^{Z_{\tau_1} - Z_{\tau_2}})^+ \mid \mathcal{F}_{\tau_2}\right)\Big)\\
&= \mathsf{E}\,(S_{\tau_2})\,\mathsf{E}\left(1 - e^{Z_{\tau_1} - Z_{\tau_2}}\right)^+\\
&= \mathsf{E}\,(S_{\tau_2})\,\mathsf{E}\Big(1 - \inf_{0\le t\le t_2-t_1} e^{Z_{\tau_2+t} - Z_{\tau_2}}\Big)\\
&= \mathsf{E}\,(S_{\tau_2})\,\mathsf{E}\Big(1 - \inf_{0\le t\le t_2-t_1} e^{Z_t}\Big) =: \mathsf{E}\,(S_{\tau_2})\,L(t_2 - t_1)
\end{aligned}
$$

where we also used that $Z_{\tau_1} - Z_{\tau_2}$ is independent from $\mathcal{F}_{\tau_2}$. By basic properties of Brownian motion it is easily seen that $L(t_2 - t_1) \to 0$ as $t_2 - t_1 \to 0$.

Combining (25.2.12) and (25.2.13) we find by the martingale property of $\big(\exp(\sigma B_t - (\sigma^2/2)t)\big)_{t\ge 0}$ that

$$
0 \le V(t_1,x) - V(t_2,x) \le x\,\mathsf{E}\,(S_{\tau_2})\,L(t_2 - t_1) \le x\,e^{rT} L(t_2 - t_1) \qquad (25.2.14)
$$

from where (25.2.11) becomes evident. This completes the proof.

4. In order to prove that the *smooth-fit* condition (25.2.28) holds, i.e. that $x \mapsto V(t,x)$ is C^1 at $b(t)$, let us fix a point $(t,x) \in (0,T) \times (0,\infty)$ lying on the boundary b so that $x = b(t)$. Then $x < K$ and for all $\varepsilon > 0$ such that $x + \varepsilon < K$ we have

$$
\frac{V(t,x+\varepsilon) - V(t,x)}{\varepsilon} \ge \frac{G(x+\varepsilon) - G(x)}{\varepsilon} = -1 \qquad (25.2.15)
$$

and hence, taking the limit in (25.2.15) as $\varepsilon \downarrow 0$, we get

$$
\frac{\partial^+ V}{\partial x}(t,x) \ge G'(x) = -1 \qquad (25.2.16)
$$

where the right-hand derivative exists (and is finite) by virtue of the convexity of the mapping $x \mapsto V(t,x)$ on $(0,\infty)$. (Note that the latter will also be proved independently below.)

To prove the converse inequality, let us fix $\varepsilon > 0$ such that $x+\varepsilon < K$, and consider the stopping time $\tau_\varepsilon = \tau_*(t, x+\varepsilon)$ being optimal for $V(t, x+\varepsilon)$. Then we have

$$\begin{aligned} &V(t,x+\varepsilon) - V(t,x) &(25.2.17)\\ &\quad\le \mathsf{E}\left(e^{-r\tau_\varepsilon}(K-(x+\varepsilon)S_{\tau_\varepsilon})^+\right) - \mathsf{E}\left(e^{-r\tau_\varepsilon}(K-xS_{\tau_\varepsilon})^+\right)\\ &\quad\le \mathsf{E}\Big(e^{-r\tau_\varepsilon}\big[(K-(x+\varepsilon)S_{\tau_\varepsilon})^+ - (K-xS_{\tau_\varepsilon})^+\big]I\big((x+\varepsilon)S_{\tau_\varepsilon} < K\big)\Big)\\ &\quad= -\varepsilon\,\mathsf{E}\Big(e^{-r\tau_\varepsilon}S_{\tau_\varepsilon}I\big((x+\varepsilon)S_{\tau_\varepsilon} < K\big)\Big). \end{aligned}$$

Using that $s \mapsto -\frac{\gamma}{\sigma}s$ is a lower function of B at zero and the fact that the optimal boundary $s \mapsto b(s)$ is increasing on $[t,T]$, it is not difficult to verify that $\tau_\varepsilon \to 0$ P-a.s. as $\varepsilon \downarrow 0$. In particular, this implies that

$$\mathsf{E}\left(e^{-r\tau_\varepsilon}S_{\tau_\varepsilon}\,I((x+\varepsilon)S_{\tau_\varepsilon} < K)\right) \to 1 \tag{25.2.18}$$

as $\varepsilon \downarrow 0$ by the dominated convergence theorem.

Combining (25.2.17) and (25.2.18) we see that

$$\frac{\partial^+ V}{\partial x}(t,x) \le G'(x) = -1 \tag{25.2.19}$$

which together with (25.2.16) completes the proof.

5. We proceed to prove that the boundary b is *continuous* on $[0,T]$ and that $b(T) = K$.

(i) Let us first show that the boundary b is right-continuous on $[0,T]$. For this, fix $t \in (0,T]$ and consider a sequence $t_n \downarrow t$ as $n \to \infty$. Since b is increasing, the right-hand limit $b(t+)$ exists. Because $(t_n, b(t_n)) \in \bar{D}$ for all $n \ge 1$, and $\bar{D}$ is closed, we get that $(t, b(t+)) \in \bar{D}$. Hence by (25.2.9) we see that $b(t+) \le b(t)$. The reverse inequality follows obviously from the fact that b is increasing on $[0,T]$, thus proving the claim.

(ii) Suppose that at some point $t_* \in (0,T)$ the function b makes a jump, i.e. let $b(t_*) > b(t_*-)$. Let us fix a point $t' < t_*$ close to t_* and consider the half-open set $R \subseteq C$ being a curved trapezoid formed by the vertices $(t', b(t'))$, $(t_*, b(t_*-))$, (t_*, x') and (t', x') with any x' fixed arbitrarily in the interval $(b(t_*-), b(t_*))$.

Recall that the strong Markov property (cf. Chapter III) implies that the value function V is $C^{1,2}$ in C. Note also that the gain function G is C^2 in

R so that by the Newton–Leibniz formula using (25.2.27) and (25.2.28) it follows that

$$V(t,x) - G(x) = \int_{b(t)}^{x} \int_{b(t)}^{u} \big(V_{xx}(t,v) - G_{xx}(v)\big)\, dv\, du \tag{25.2.20}$$

for all $(t,x) \in R$. Moreover, the strong Markov property (cf. Chapter III) implies that the value function V solves the equation (25.2.26) from where using that $t \mapsto V(t,x)$ and $x \mapsto V(t,x)$ are decreasing so that $V_t \le 0$ and $V_x \le 0$ in C, we obtain

$$\begin{aligned} V_{xx}(t,x) &= \frac{2}{\sigma^2 x^2}\big(rV(t,x) - V_t(t,x) - rxV_x(t,x)\big) \\ &= \frac{2}{\sigma^2 x^2}\, r(K-x)^+ \ge c > 0 \end{aligned} \tag{25.2.21}$$

for each $(t,x) \in R$ where $c > 0$ is small enough.

Hence by (25.2.20) using that $G_{xx} = 0$ in R we get

$$V(t',x') - G(x') \ge c\,\frac{(x' - b(t'))^2}{2} \longrightarrow c\,\frac{(x' - b(t_*))^2}{2} > 0 \tag{25.2.22}$$

as $t' \uparrow t_*$. This implies that $V(t_*,x') > G(x')$ which contradicts the fact that (t_*,x') belong to the stopping set $\bar{D}$. Thus $b(t_*+) = b(t_*)$ showing that b is continuous at t_* and thus on $[0,T]$ as well.

(iii) We finally note that the method of proof from the previous part (ii) also implies that $b(T) = K$. To see this, we may let $t_* = T$ and likewise suppose that $b(T) < K$. Then repeating the arguments presented above word by word we arrive at a contradiction with the fact that $V(t,x) = G(x)$ for all $x \in [b(T),K]$.

6. Summarizing the facts proved in paragraphs 1–5 above we may conclude that the following hitting time is optimal in the problem (25.2.1):

$$\tau_b = \inf\{\, 0 \le s \le T-t : X_{t+s} \le b(t+s) \,\} \tag{25.2.23}$$

(the infimum of an empty set being equal to $T-t$) where the boundary b satisfies the properties

$$b : [0,T] \to (0,K] \quad \text{is continuous and increasing,} \tag{25.2.24}$$

$$b(T) = K. \tag{25.2.25}$$

(see Figure VII.1).

Standard arguments based on the strong Markov property (cf. Chapter III) lead to the following *free-boundary problem* for the *unknown value function* V

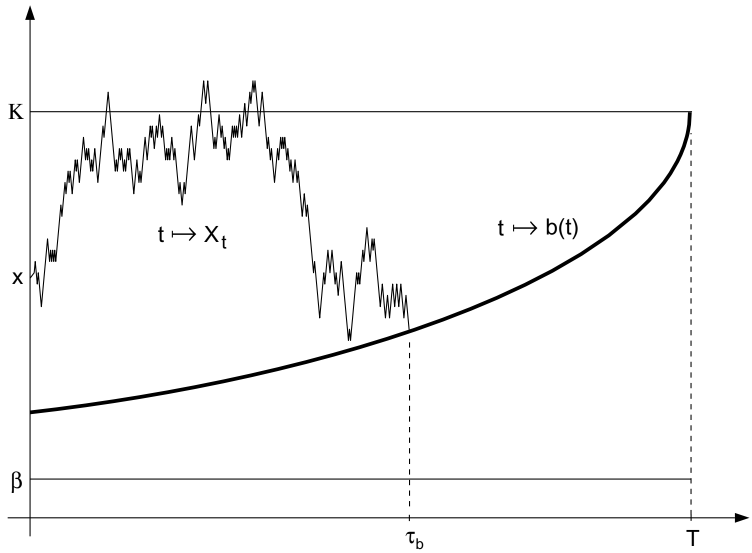

Figure VII.1: A computer drawing of the optimal stopping boundary b from Theorem 25.3. The number β is the optimal stopping point in the case of infinite horizon (Theorem 25.1).

and the *unknown boundary* b:

$$V_t + \mathbb{L}_X V = rV \quad \text{in } C, \tag{25.2.26}$$

$$V(t,x) = (K-x)^+ \quad \text{for } x = b(t), \tag{25.2.27}$$

$$V_x(t,x) = -1 \quad \text{for } x = b(t) \qquad (\textit{smooth fit}), \tag{25.2.28}$$

$$V(t,x) > (K-x)^+ \quad \text{in } C, \tag{25.2.29}$$

$$V(t,x) = (K-x)^+ \quad \text{in } D \tag{25.2.30}$$

where the continuation set C is defined in (25.2.8) above and the stopping set $\bar{D}$ is the closure of the set D in (25.2.9) above.

7. The following properties of V and b were verified above:

$$V \text{ is continuous on } [0,T]\times\mathbb{R}_+, \tag{25.2.31}$$

$$V \text{ is } C^{1,2} \text{ on } C \text{ (and } C^{1,2} \text{ on } \bar{D}), \tag{25.2.32}$$

$$x \mapsto V(t,x) \text{ is decreasing and convex with } V_x(t,x) \in [-1,0], \tag{25.2.33}$$

$$t \mapsto V(t,x) \text{ is decreasing with } V(T,x) = (K-x)^+, \tag{25.2.34}$$

$$t \mapsto b(t) \text{ is increasing and continuous with } 0 < b(0+) < K \tag{25.2.35}$$

and $b(T-) = K$.

Note also that (25.2.28) means that $x \mapsto V(t,x)$ is C^1 at $b(t)$.

Once we know that V satisfying (25.2.28) is "sufficiently regular" (cf. footnote 14 in [27] when $t \mapsto V(t,x)$ is known to be C^1 for all x), we can apply Itô's formula (page 67) to $e^{-rs}V(t+s, X_{t+s})$ in its standard form and take the $\mathsf{P}_{t,x}$-expectation on both sides in the resulting identity. The martingale term then vanishes by the optional sampling theorem (page 60) using the final part of (25.2.33) above, so that by (25.2.26) and (25.2.27)+(25.2.30) upon setting $s = T-t$ (being the key advantage of the finite horizon) one obtains the *early exercise premium representation* of the value function

$$\begin{aligned} V(t,x) &= e^{-r(T-t)}\mathsf{E}_{t,x}G(X_T) \qquad\qquad (25.2.36)\\ &\quad - \int_0^{T-t} e^{-ru}\mathsf{E}_{t,x}\Big(H(t-u, X_{t+u})\, I\big(X_{t+u} \le b(t+u)\big)\Big)\, du \\ &= e^{-r(T-t)}\mathsf{E}_{t,x}G(X_T) + rK\int_0^{T-t} e^{-ru}\mathsf{P}_{t,x}\big(X_{t+u} \le b(t+u)\big)\, du \end{aligned}$$

for all $(t,x) \in [0,T]\times\mathbb{R}_+$ where we set $G(x) = (K-x)^+$ and $H = G_t + \mathbb{L}_X G - rG$ so that $H = -rK$ for $x < b(t)$.

A detail worth mentioning in this derivation (see (25.2.47) below) is that (25.2.36) follows from (3.5.9) with $F(t+s, X_{t+s}) = e^{-rs}V(t+s, X_{t+s})$ without knowing a priori that $t \mapsto V(t,x)$ is C^1 at $b(t)$ as required under the condition of "sufficiently regular" recalled prior to (25.2.36) above. This approach is more direct since the sufficient conditions (3.5.10)–(3.5.13) for (3.5.9) are easier verified than sufficient conditions [such as b is C^1 or (locally) Lipschitz] for $t \mapsto V(t,x)$ to be C^1 at $b(t)$. This is also more in the spirit of the free-boundary equation (25.2.39) to be derived below where neither differentiability nor a Lipschitz property of b plays a role in the formulation.

Since $V(t,x) = G(x) = (K-x)^+$ in $\bar{D}$ by (25.2.27)+(25.2.30), we see that (25.2.36) reads

$$\begin{aligned} K - x &= e^{-r(T-t)}\, \mathsf{E}_{t,x}(K - X_T)^+ \qquad\qquad (25.2.37)\\ &\quad + rK\int_0^{T-t} e^{-ru}\mathsf{P}_{t,x}\big(X_{t+u} \le b(t+u)\big)\, du \end{aligned}$$

for all $x \in (0, b(t)]$ and all $t \in [0,T]$.

8. A natural candidate equation is obtained by inserting $x = b(t)$ in (25.2.37). This leads to the *free-boundary equation* (cf. Subsection 14.1)

$$\begin{aligned} K - b(t) &= e^{-r(T-t)}\, \mathsf{E}_{t,b(t)}(K - X_T)^+ \qquad\qquad (25.2.38)\\ &\quad + rK\int_0^{T-t} e^{-ru}\, \mathsf{P}_{t,b(t)}\big(X_{t+u} \le b(t+u)\big)\, du \end{aligned}$$

which upon using (25.2.3) more explicitly reads as follows:

$$
\begin{aligned}
&K - b(t) &&(25.2.39)\\
&= e^{-r(T-t)} \int_0^K \Phi\Big(\frac{1}{\sigma\sqrt{T-t}}\Big(\log\frac{K-z}{b(t)} - \Big(r-\frac{\sigma^2}{2}\Big)(T-t)\Big)\Big)\, dz\\
&\quad + rK \int_0^{T-t} e^{-ru}\, \Phi\Big(\frac{1}{\sigma\sqrt{u}}\Big(\log\frac{b(t+u)}{b(t)} - \Big(r-\frac{\sigma^2}{2}\Big)u\Big)\Big)\, du
\end{aligned}
$$

for all $t \in [0,T]$ where $\Phi(x) = (1/\sqrt{2\pi})\int_{-\infty}^{x} e^{-z^2/2}dz$ for $x \in \mathbb{R}$. It is a *nonlinear Volterra integral equation of the second kind* (see [212]).

9. The main result of the present subsection may now be stated as follows (see also Remark 25.5 below).

Theorem 25.3. *The optimal stopping boundary in the American put problem* (25.2.1) *can be characterized as the unique solution of the free-boundary equation* (25.2.39) *in the class of continuous increasing functions* $c : [0,T] \to \mathbb{R}$ *satisfying* $0 < c(t) < K$ *for all* $0 < t < T$.

Proof. The fact that the optimal stopping boundary b solves (25.2.38) i.e. (25.2.39) was derived above. The main emphasis of the theorem is thus on the claim of *uniqueness*. Let us therefore assume that a continuous increasing $c : [0,T] \to \mathbb{R}$ solving (25.2.39) is given such that $0 < c(t) < K$ for all $0 < t < T$, and let us show that this c must then coincide with the optimal stopping boundary b. The proof of this implication will be presented in the nine steps as follows.

1°. In view of (25.2.36) and with the aid of calculations similar to those leading from (25.2.38) to (25.2.39), let us introduce the function

$$
\begin{aligned}
&U^c(t,x) &&(25.2.40)\\
&= e^{-r(T-t)}\, \mathsf{E}_{t,x} G(X_T) + rK \int_0^{T-t} e^{-ru}\, \mathsf{P}_{t,x}\big(X_{t+u} \le c(t+u)\big)\, du\\
&= e^{-r(T-t)}\, U_1^c(t,x) + rK\, U_2^c(t,x)
\end{aligned}
$$

where U_1^c and U_2^c are defined as follows:

$$
U_1^c(t,x) = \int_0^K \Phi\Big(\frac{1}{\sigma\sqrt{T-t}}\Big(\log\frac{K-z}{x} - \gamma\,(T-t)\Big)\Big)\, dz, \tag{25.2.41}
$$

$$
U_2^c(t,x) = \int_t^T e^{-r(v-t)}\, \Phi\Big(\frac{1}{\sigma\sqrt{v-t}}\Big(\log\frac{c(v)}{x} - \gamma\,(v-t)\Big)\Big)\, dv \tag{25.2.42}
$$

for all $(t,x) \in [0,T)\times(0,\infty)$ upon setting $\gamma = r-\sigma^2/2$ and substituting $v = t+u$.

Denoting $\varphi = \Phi'$ we then have

$$\frac{\partial U_1^c}{\partial x}(t,x) = -\frac{1}{\sigma x\sqrt{T-t}} \int_0^K \varphi\Big(\frac{1}{\sigma\sqrt{T-t}}\Big(\log\frac{K-z}{x} - \gamma\,(T-t)\Big)\Big)\,dz, \quad (25.2.43)$$

$$\frac{\partial U_2^c}{\partial x}(t,x) = -\frac{1}{\sigma x} \int_t^T \frac{e^{-r(v-t)}}{\sqrt{v-t}}\,\varphi\Big(\frac{1}{\sigma\sqrt{v-t}}\Big(\log\frac{c(v)}{x} - \gamma\,(v-t)\Big)\Big)\,dv \quad (25.2.44)$$

for all $(t,x) \in [0,T) \times (0,\infty)$ where the interchange of differentiation and integration is justified by standard means. From (25.2.43) and (25.2.44) we see that $\partial U_1^c/\partial x$ and $\partial U_2^c/\partial x$ are continuous on $[0,T)\times(0,\infty)$, which in view of (25.2.40) implies that U_x^c is continuous on $[0,T) \times (0,\infty)$.

2°. In accordance with (25.2.36) define a function $V^c : [0,T) \times (0,\infty) \to \mathbb{R}$ by setting $V^c(t,x) = U^c(t,x)$ for $x > c(t)$ and $V^c(t,x) = G(x)$ for $x \le c(t)$ when $0 \le t < T$. Note that since c solves (25.2.39) we have that V^c is continuous on $[0,T) \times (0,\infty)$, i.e. $V^c(t,x) = U^c(t,x) = G(x)$ for $x = c(t)$ when $0 \le t < T$. Let C_1 and C_2 be defined by means of c as in (3.5.3) and (3.5.4) with $[0,T)$ instead of $\mathbb{R}_+$, respectively.

Standard arguments based on the Markov property (or a direct verification) show that V^c i.e. U^c is $C^{1,2}$ on C_1 and that

$$V_t^c + \mathbb{L}_X V^c = rV^c \quad \text{in} \quad C_1. \quad (25.2.45)$$

Moreover, since U_x^c is continuous on $[0,T)\times(0,\infty)$ we see that V_x^c is continuous on $\bar{C}_1$. Finally, since $0 < c(t) < K$ for $0 < t < T$ we see that V^c i.e. G is $C^{1,2}$ on $\bar{C}_2$.

3°. Summarizing the preceding conclusions one can easily verify that with $(t,x) \in [0,T) \times (0,\infty)$ given and fixed, the function $F : [0,T-t) \times (0,\infty) \to \mathbb{R}$ defined by

$$F(s,y) = e^{-rs} V^c(t+s, xy) \quad (25.2.46)$$

satisfies (3.5.10)–(3.5.13) (in the relaxed form) so that (3.5.9) can be applied. In this way we get

$$\begin{aligned} e^{-rs}V^c(t+s, X_{t+s}) &= V^c(t,x) \quad (25.2.47)\\ &+ \int_0^s e^{-ru}\Big(V_t^c + \mathbb{L}_X V^c - rV^c\Big)(t+u, X_{t+u})\, I(X_{t+u} \ne c(t+u))\,du \\ &+ M_s^c + \frac{1}{2}\int_0^s e^{-ru}\Delta_x V_x^c(t+u, c(t+u))\,d\ell_u^c(X) \end{aligned}$$

where $M_s^c = \int_0^s e^{-ru} V_x^c(t+u, X_{t+u})\,\sigma X_{t+u}\, I(X_{t+u} \ne c(t+u))\,dB_u$ and we set $\Delta_x V_x^c(v, c(v)) = V_x^c(v, c(v)+) - V_x^c(v, c(v)-)$ for $t \le v \le T$. Moreover, it is easily seen from (25.2.43) and (25.2.44) that $(M_s^c)_{0\le s\le T-t}$ is a martingale under $\mathsf{P}_{t,x}$ so that $\mathsf{E}_{t,x} M_s^c = 0$ for each $0 \le s \le T-t$.

4°. Setting $s = T - t$ in (25.2.47) and then taking the $\mathsf{P}_{t,x}$-expectation, using that $V^c(T,x) = G(x)$ for all $x > 0$ and that V^c satisfies (25.2.45) in C_1, we get

$$e^{-r(T-t)}\mathsf{E}_{t,x}G(X_T) = V^c(t,x) + \int_0^{T-t} e^{-ru}\mathsf{E}_{t,x}\big(H(t+u, X_{t+u})\, I(X_{t+u} \le c(t+u))\big)\, du + \frac{1}{2}\int_0^{T-t} e^{-ru}\Delta_x V_x^c(t+u, c(t+u))\, d_u\mathsf{E}_{t,x}(\ell_u^c(X)) \tag{25.2.48}$$

for all $(t,x) \in [0,T) \times (0,\infty)$ where $H = G_t + \mathbb{L}_X G - rG = -rK$ for $x \le c(t)$. From (25.2.48) we thus see that

$$V^c(t,x) = e^{-r(T-t)}\mathsf{E}_{t,x}G(X_T) + rK\int_0^{T-t} e^{-ru}\mathsf{P}_{t,x}(X_{t+u} \le c(t+u))\, du - \frac{1}{2}\int_0^{T-t} e^{-ru}\Delta_x V_x^c(t+u, c(t+u))\, d_u\mathsf{E}_{t,x}(\ell_u^c(X)) \tag{25.2.49}$$

for all $(t,x) \in [0,T) \times (0,\infty)$. Comparing (25.2.49) with (25.2.40), and recalling the definition of V^c in terms of U^c and G, we get

$$\int_0^{T-t} e^{-ru}\Delta_x V_x^c(t+u, c(t+u))\, d_u\mathsf{E}_{t,x}(\ell_u^c(X)) = 2\,\big(U^c(t,x) - G(x)\big)\, I(x \le c(t)) \tag{25.2.50}$$

for all $0 \le t < T$ and $x > 0$, where $I(x \le c(t))$ equals 1 if $x \le c(t)$ and 0 if $x > c(t)$.

5°. From (25.2.50) we see that if we are to prove that

$$x \mapsto V^c(t,x) \quad \text{is} \quad C^1 \quad \text{at} \quad c(t) \tag{25.2.51}$$

for each $0 \le t < T$ given and fixed, then it will follow that

$$U^c(t,x) = G(x) \quad \text{for all} \quad 0 < x \le c(t). \tag{25.2.52}$$

On the other hand, if we know that (25.2.52) holds, then using the general fact

$$\frac{\partial}{\partial x}\Big(U^c(t,x) - G(x)\Big)\Big|_{x=c(t)} = V_x^c(t, c(t)+) - V_x^c(t, c(t)-) = \Delta_x V_x^c(t, c(t)) \tag{25.2.53}$$

for all $0 \le t < T$, we see that (25.2.51) holds too (since U_x^c is continuous). The equivalence of (25.2.51) and (25.2.52) just explained then suggests that instead of dealing with the equation (25.2.50) in order to derive (25.2.51) above (which was the content of an earlier proof) we may rather concentrate on establishing (25.2.52) directly. [To appreciate the simplicity and power of the probabilistic argument to be given shortly below one may differentiate (25.2.50) with respect to x, compute the left-hand side explicitly (taking care of a jump relation), and then try to prove the uniqueness of the zero solution to the resulting (weakly singular) Volterra integral equation using any of the known analytic methods (see e.g. [212]).]

6°. To derive (25.2.52) first note that standard arguments based on the Markov property (or a direct verification) show that U^c is $C^{1,2}$ on C_2 and that

$$U_t^c + \mathbb{L}_X U^c - rU^c = -rK \quad \text{in} \quad C_2 . \tag{25.2.54}$$

Since F in (25.2.46) with U^c instead of V^c is continuous and satisfies (3.5.10)–(3.5.13) (in the relaxed form), we see that (3.5.9) can be applied just as in (25.2.47), and this yields

$$\begin{aligned} & e^{-rs} U^c(t+s, X_{t+s}) \\ & = U^c(t,x) - rK \int_0^s e^{-ru} I(X_{t+u} \le c(t+u))\, du + \widetilde{M}_s^c \end{aligned} \tag{25.2.55}$$

upon using (25.2.45) and (25.2.54) as well as that $\Delta_x U_x^c(t+u, c(t+u)) = 0$ for all $0 \le u \le s$ since U_x^c is continuous. In (25.2.55) we have $\widetilde{M}_s^c = \int_0^s e^{-ru} U_x^c(t+u, X_{t+u})\, \sigma X_{t+u}\, I(X_{t+u} \ne c(t+u))\, dB_u$ and $(\widetilde{M}_s^c)_{0 \le s \le T-t}$ is a martingale under $\mathsf{P}_{t,x}$.

Next note that (3.5.9) applied to F in (25.2.46) with G instead of V^c yields

$$\begin{aligned} e^{-rs} G(X_{t+s}) = G(x) - rK \int_0^s e^{-ru} I(X_{t+u} < K)\, du & \\ + M_s^K + \frac{1}{2} \int_0^s e^{-ru}\, d\ell_u^K(X) & \end{aligned} \tag{25.2.56}$$

upon using that $G_t + \mathbb{L}_X G - rG$ equals $-rK$ on $(0,K)$ and 0 on (K,∞) as well as that $\Delta_x G_x(t+u, K) = 1$ for $0 \le u \le s$. In (25.2.56) we have $M_s^K = \int_0^s e^{-ru}\, G'(X_{t+u})\, \sigma X_{t+u}\, I(X_{t+u} \ne K)\, dB_u = -\int_0^s e^{-ru}\, \sigma X_{t+u}\, I(X_{t+u} < K)\, dB_u$ and $(M_s^K)_{0 \le s \le T-t}$ is a martingale under $\mathsf{P}_{t,x}$.

For $0 < x \le c(t)$ consider the stopping time

$$\sigma_c = \inf \{\, 0 \le s \le T-t : X_{t+s} \ge c(t+s) \,\}. \tag{25.2.57}$$

Then using that $U^c(t, c(t)) = G(c(t))$ for all $0 \le t < T$ since c solves (25.2.9), and that $U^c(T,x) = G(x)$ for all $x > 0$ by (25.2.40), we see that $U^c(t+\sigma_c, X_{t+\sigma_c}) = G(X_{t+\sigma_c})$. Hence from (25.2.55) and (25.2.56) using the optional sampling theorem (page 60) we find

$$
\begin{aligned}
U^c(t,x) &= \mathsf{E}_{t,x}\Big(e^{-r\sigma_c}U^c(t+\sigma_c, X_{t+\sigma_c})\Big) \qquad (25.2.58)\\
&\qquad + rK\,\mathsf{E}_{t,x}\bigg(\int_0^{\sigma_c} e^{-ru} I(X_{t+u} \le c(t+u))\,du\bigg)\\
&= \mathsf{E}_{t,x}\big(e^{-r\sigma_c}G(X_{t+\sigma_c})\big) + rK\,\mathsf{E}_{t,x}\bigg(\int_0^{\sigma_c} e^{-ru} I(X_{t+u} \le c(t+u))\,du\bigg)\\
&= G(x) - rK\,\mathsf{E}_{t,x}\bigg(\int_0^{\sigma_c} e^{-ru} I(X_{t+u} < K)\,du\bigg)\\
&\quad + rK\,\mathsf{E}_{t,x}\bigg(\int_0^{\sigma_c} e^{-ru} I(X_{t+u} \le c(t+u))\,du\bigg) = G(x)
\end{aligned}
$$

since $X_{t+u} < K$ and $X_{t+u} \le c(t+u)$ for all $0 \le u < \sigma_c$. This establishes (25.2.52) and thus (25.2.51) holds as well as explained above.

7°. Consider the stopping time

$$\tau_c = \inf\{\, 0 \le s \le T-t : X_{t+s} \le c(t+s)\,\}. \qquad (25.2.59)$$

Note that (25.2.47) using (25.2.45) and (25.2.51) reads

$$
\begin{aligned}
e^{-rs}V^c(t+s, X_{t+s}) &= V^c(t,x) \qquad (25.2.60)\\
&\quad + \int_0^s e^{-ru} H(t+u, X_{t+u})\, I(X_{t+u} \le c(t+u))\,du + M_s^c
\end{aligned}
$$

where $H = G_t + \mathbb{L}_X G - rG = -rK$ for $x \le c(t)$ and $(M_s^c)_{0\le s\le T-t}$ is a martingale under $\mathsf{P}_{t,x}$. Thus $\mathsf{E}_{t,x} M_{\tau_c}^c = 0$, so that after inserting τ_c in place of s in (25.2.60), it follows upon taking the $\mathsf{P}_{t,x}$-expectation that

$$V^c(t,x) = \mathsf{E}_{t,x}\Big(e^{-r\tau_c}(K - X_{t+\tau_c})^+\Big) \qquad (25.2.61)$$

for all $(t,x) \in [0,T) \times (0,\infty)$ where we use that $V^c(t,x) = G(x) = (K-x)^+$ for $x \le c(t)$ or $t = T$. Comparing (25.2.61) with (25.2.1) we see that

$$V^c(t,x) \le V(t,x) \qquad (25.2.62)$$

for all $(t,x) \in [0,T) \times (0,\infty)$.

8°. Let us now show that $c \ge b$ on $[0,T]$. For this, recall that by the same arguments as for V^c we also have

$$
\begin{aligned}
e^{-rs}V(t+s, X_{t+s}) &= V(t,x) \qquad (25.2.63)\\
&\quad + \int_0^s e^{-ru} H(t+u, X_{t+u})\, I(X_{t+u} \le b(t+u))\,du + M_s^b
\end{aligned}
$$

where $H = G_t + \mathbb{L}_X G - rG = -rK$ for $x \le b(t)$ and $(M_s^b)_{0 \le s \le T-t}$ is a martingale under $\mathsf{P}_{t,x}$. Fix $(t,x) \in (0,T) \times (0,\infty)$ such that $x < b(t) \wedge c(t)$ and consider the stopping time

$$\sigma_b = \inf \{ \, 0 \le s \le T-t : X_{t+s} \ge b(t+s) \, \}. \tag{25.2.64}$$

Inserting σ_b in place of s in (25.2.60) and (25.2.63) and taking the $\mathsf{P}_{t,x}$-expectation, we get

$$\begin{aligned} \mathsf{E}_{t,x}\big(e^{-r\sigma_b} V^c(t+\sigma_b, X_{t+\sigma_b})\big) &= G(x) \\ &\quad - rK \, \mathsf{E}_{t,x}\bigg(\int_0^{\sigma_b} e^{-ru} I(X_{t+u} \le c(t+u)) \, du \bigg), \end{aligned} \tag{25.2.65}$$

$$\mathsf{E}_{t,x}\big(e^{-r\sigma_b} V(t+\sigma_b, X_{t+\sigma_b})\big) = G(x) - rK \, \mathsf{E}_{t,x}\bigg(\int_0^{\sigma_b} e^{-ru} \, du \bigg). \tag{25.2.66}$$

Hence by (25.2.62) we see that

$$\mathsf{E}_{t,x}\bigg(\int_0^{\sigma_b} e^{-ru} I(X_{t+u} \le c(t+u)) \, du \bigg) \ge \mathsf{E}_{t,x}\bigg(\int_0^{\sigma_b} e^{-ru} \, du \bigg) \tag{25.2.67}$$

from where it follows by the continuity of c and b that $c(t) \ge b(t)$ for all $0 \le t \le T$.

9°. Finally, let us show that c must be equal to b. For this, assume that there is $t \in (0,T)$ such that $c(t) > b(t)$, and pick $x \in (b(t), c(t))$. Under $\mathsf{P}_{t,x}$ consider the stopping time τ_b from (25.2.23). Inserting τ_b in place of s in (25.2.60) and (25.2.63) and taking the $\mathsf{P}_{t,x}$-expectation, we get

$$\begin{aligned} \mathsf{E}_{t,x}\big(e^{-r\tau_b} G(X_{t+\tau_b})\big) &= V^c(t,x) \\ &\quad - rK \, \mathsf{E}_{t,x}\bigg(\int_0^{\tau_b} e^{-ru} I(X_{t+u} \le c(t+u)) \, du \bigg), \end{aligned} \tag{25.2.68}$$

$$\mathsf{E}_{t,x}\big(e^{-r\tau_b} G(X_{t+\tau_b})\big) = V(t,x). \tag{25.2.69}$$

Hence by (25.2.62) we see that

$$\mathsf{E}_{t,x}\bigg(\int_0^{\tau_b} e^{-ru} I(X_{t+u} \le c(t+u)) \, du \bigg) \le 0 \tag{25.2.70}$$

from where it follows by the continuity of c and b that such a point x cannot exist. Thus c must be equal to b, and the proof is complete. □

Remark 25.4. The fact that U^c defined in (25.2.40) must be equal to G below c when c solves (25.2.39) is truly remarkable. The proof of this fact given above (paragraphs $2°$–$6°$) follows the way which led to its discovery. A shorter but somewhat less revealing proof can also be obtained by introducing U^c as in (25.2.40) and then verifying directly (using the Markov property only) that

$$e^{-rs}U^c(t+s, X_{t+s}) + rK \int_0^s e^{-ru} I(X_{t+u} \le c(t+u))\, du \tag{25.2.71}$$

is a martingale under $\mathsf{P}_{t,x}$ for $0 \le s \le T-t$. In this way it is possible to circumvent the material from paragraphs 2°–4° and carry out the rest of the proof starting with (25.2.56) onward. Moreover, it may be noted that the martingale property of (25.2.71) does not require that c is increasing (but only measurable). This shows that the claim of uniqueness in Theorem 25.3 holds in the class of continuous (or left-continuous) functions $c : [0,T] \to \mathbb{R}$ such that $0 < c(t) < K$ for all $0 < t < T$. It also identifies some limitations of the approach based on the local time-space formula (cf. Subsection 3.5) as initially undertaken (where c needs to be of bounded variation).

Remark 25.5. Note that in Theorem 25.3 above we do not assume that the solution starts (ends) at a particular point. The equation (25.2.39) is highly nonlinear and seems to be out of the scope of any existing theory on nonlinear integral equations (the kernel having four arguments). Similar equations arise in the first-passage problem for Brownian motion (cf. Subsection 14.2).

Notes. According to theory of modern finance (see e.g. [197]) the arbitrage-free price of the American put option with a strike price K coincides with the value function V of the optimal stopping problem with the gain function $G = (K-x)^+$. The optimal stopping time in this problem is the first time when the price process (geometric Brownian motion) falls below the value of a time-dependent boundary b. When the option's expiration date T is finite, the mathematical problem of finding V and b is inherently two-dimensional and therefore analytically more difficult (for infinite T the problem is one-dimensional and b is constant).

The first mathematical analysis of the problem is due to McKean [133] who considered a "discounted" American call with the gain function $G = e^{-\beta t}(x-K)^+$ and derived a free-boundary problem for V and b. He further expressed V in terms of b so that b itself solves a countable system of nonlinear integral equations (p. 39 in [133]). The approach of expressing V in terms of b was in line with the ideas coming from earlier work of Kolodner [114] on free-boundary problems in mathematical physics (such as Stefan's ice melting problem). The existence and uniqueness of a solution to the system for b derived by McKean was left open in [133].

McKean's work was taken further by van Moerbeke [215] who derived a single nonlinear integral equation for b (pp. 145–146 in [215]). The connection to the physical problem is obtained by introducing the auxiliary function $\widetilde{V} = \partial(V-G)/\partial t$ so that the "smooth-fit condition" from the optimal stopping problem translates into the "condition of heat balance" (i.e. the law of conservation of energy) in the physical problem. A motivation for the latter may be seen from the fact that in the mathematical physics literature at the time it was realized that the existence and local uniqueness of a solution to such nonlinear integral equations can be proved by applying the contraction principle (fixed point theorem), first for a small time interval and then extending it to any interval of time by induction (see [137] and [70]). Applying this method, van Moerbeke has proved the existence and local uniqueness of a solution to the integral equations of a general optimal stopping problem (see Sections 3.1 and 3.2 in [215]) while the proof of the same claim in the context of the discounted American call [133] is merely indicated (see Section 4.4 in [215]). One of the technical difficulties in this context is that the derivative b' of the optimal boundary b is not bounded at the initial point T as used in the general proof (cf. Sections 3.1 and 3.2 in [215]).

The fixed point method usually results in a long and technical proof with an indecisive end where the details are often sketched or omitted. Another consequence of the approach is the fact that the integral equations in [133] and [215] involve both b and its derivative b', so that either the fixed point method results in proving that b is differentiable, or this needs to be proved a priori if the existence is claimed simply by identifying b with the boundary of the set where $V = G$. The latter proof, however, appears difficult to give directly, so that if one is only interested in the actual values of b which indicate when to stop, it seems that the differentiability of b plays a minor role. Finally, since it is not obvious to see (and it was never explicitly addressed) how the "condition of heat balance" relates to the economic mechanism of "no-arbitrage" behind the American option, one is led to the conclusion that the integral equations derived by McKean and van Moerbeke, being motivated purely by the mathematical tractability arising from the work in mathematical physics, are perhaps more complicated then needed from the standpoint of optimal stopping.

This was to be confirmed in the beginning of the 1990's when Kim [110], Jacka [102] and Carr, Jarrow, Myneni [27] independently arrived at a nonlinear integral equation for b that is closely linked to the *early exercise premium representation* of V having a clear economic meaning (see Section 1 in [27] and Corollary 3.1 in [142]). In fact, the equation is obtained by inserting $x = b(t)$ in this representation, and for this reason it is called the *free-boundary equation* (see (25.2.39) above). The early exercise premium representation for V follows transparently from the free-boundary formulation (given that the smooth-fit condition holds) and moreover corresponds to the decomposition of the superharmonic function V into its harmonic and its potential part (the latter being the basic principle of optimal stopping established in the works of Snell [206] and Dynkin [52]).

The superharmonic characterization of the value function V (cf. Chapter I) implies that $e^{-rs}V(t-s, X_{t+s})$ is the smallest supermartingale dominating $e^{-rs}(K - X_{t+s})^+$ on $[0, T-t]$, i.e. that $V(t, x)$ is the smallest superharmonic function (relative to $\partial/\partial t + \mathbb{L}_X - rI$) dominating $(K-x)^+$ on $[0, T] \times \mathbb{R}_+$. The two requirements (i.e. smallest and superharmonic) manifest themselves in the single analytic condition of smooth fit (25.2.28).

The derivation of the smooth-fit condition given in Myneni [142] upon integrating the second formula on p. 15 and obtaining the third one seems to violate the Newton–Leibniz formula unless $x \mapsto V(t,x)$ is smooth at $b(t)$ so that there is nothing to prove. Myneni writes that this proof is essentially from McKean [133]. A closer inspection of his argument on p. 38 in [133] reveals the same difficulty. Alternative derivations of the smooth-fit principle for Brownian motion and diffusions are given in Grigelionis & Shiryaev [88] and Chernoff [30] by a Taylor expansion of V at $(t, b(t))$ and in Bather [11] and van Moerbeke [215] by a Taylor expansion of G at $(t, b(t))$. The latter approach seems more satisfactory generally since V is not known a priori. Jacka [104] (Corollary 7) develops a different approach which he applies in [102] (Proposition 2.8) to verify (25.2.28).

It follows from the preceding that the optimal stopping boundary b satisfies the free-boundary equation, however, as pointed out by Myneni [142] (p. 17) "the uniqueness and regularity of the stopping boundary from this integral equation remain open". This attempt is in line with McKean [133] (p. 33) who wrote that "another inviting unsolved problem is to discuss the integral equation for the free-boundary of section 6", concluding the paper (p. 39) with the words "even the existence and uniqueness of solutions is still unproved". McKean's integral equations [133] (p. 39) are more complicated (involving b' as well) than the equation (25.2.37). Thus the simplification of his equations to the equations (25.2.37) and finally the equation (25.2.39) may be viewed as a step to the solution of the problem. Theorem 4.3 of Jacka [102] states that if $c : [0,T] \to \mathbb{R}$ is a "left-continuous" solution of (25.2.37) for all $x \in (0, c(t)]$ satisfying $0 < c(t) < K$ for all $t \in (0,T)$, then c is the optimal stopping boundary b. Since (25.2.37) is a different equation for each new $x \in (0, c(t)]$, we see that this assumption in effect corresponds to c solving a countable system of nonlinear integral equations (obtained by letting x in $(0, c(t)]$ run through rationals for instance). From the standpoint of numerical calculation it is therefore of interest to reduce the number of these equations.

The main purpose of the present section (following [164]) is to show that the question of Myneni can be answered affirmatively and that the *free-boundary equation alone does indeed characterize the optimal stopping boundary* b. The key argument in the proof is based on the local time-space formula [163] (see Subsection 3.5). The same method of proof can be applied to more general continuous Markov processes (diffusions) in problems of optimal stopping with finite horizon. For example, in this way it is also possible to settle the somewhat more complicated problem of the Russian option with finite horizon [165] (see Section 26 below).

With reference to [133] and [215] it is claimed in [142] (and used in some other papers too) that b is C^1 but we could not find a complete/transparent proof in either of these papers (nor anywhere else). If it is known that b is C^1, then the proof above shows that C in (25.2.32) can be replaced by $\bar{C}$, implying also that $s \mapsto V(s, b(t))$ is C^1 at t. For both, in fact, it is sufficient to know that b is (locally) Lipschitz, but it seems that this fact is no easier to establish directly, and we do not know of any transparent proof.

For more information on the American option problem we refer to the survey paper [142], the book [197] and Sections 2.5–2.8 in the book [107] where further references can also be found. For a numerical discussion of the free-boundary equation and possible improvements along these lines see e.g. [93]. For asymptotics of the optimal stopping boundary see [121], and for a proof that it is convex see [58]. For random walks and Lévy processes see [33], [140] and [2].

26. The Russian option

26.1. Infinite horizon

1. The arbitrage-free price of the *Russian option* with *infinite horizon* (perpetual option) is given by

$$V = \sup_{\tau} \mathsf{E}\left(e^{-(r+\lambda)\tau} M_\tau\right) \tag{26.1.1}$$

where the supremum is taken over all stopping times τ of the geometric Brownian motion $S = (S_t)_{t\geq 0}$ solving

$$dS_t = rS_t\,dt + \sigma S_t\,dB_t \quad (S_0 = s) \tag{26.1.2}$$

and $M = (M_t)_{t\geq 0}$ is the *maximum* process given by

$$M_t = \left(\max_{0\leq u\leq t} S_u\right) \vee m \tag{26.1.3}$$

where $m \geq s > 0$ are given and fixed. We recall that $B = (B_t)_{t\geq 0}$ is a standard Brownian motion process started at zero, $r > 0$ is the interest rate, $\lambda > 0$ is the discounting rate, and $\sigma > 0$ is the volatility coefficient.

2. The problem (26.1.1) is two-dimensional since the underlying Markov process may be identified with (S, M). Using the same method as in Section 13 it is possible to solve the problem (26.1.1) explicitly. Instead we will follow a different route to the explicit solution using a *change of measure* (cf. Subsection 15.3) which reduces the initial *two-dimensional* problem to an equivalent *one-dimensional* problem (cf. Subsection 6.2). This reduction becomes especially handy in the case when the horizon in (26.1.1) is finite (cf. Subsection 26.2 below).

Recalling that the strong solution of (26.1.2) is given by (26.1.9) below and writing M_τ in (26.1.1) as $S_\tau(M_\tau/S_\tau)$, we see by change of measure that

$$V = s \sup_{\tau} \widetilde{\mathsf{E}}(e^{-\lambda\tau} X_\tau) \tag{26.1.4}$$

where we set

$$X_t = \frac{M_t}{S_t} \tag{26.1.5}$$