Chapter VII.

Optimal stopping in mathematical finance

25. The American option

25.1. Infinite horizon

1. According to theory of modern finance (see e.g. [197]) the arbitrage-free price of the *American put option* with *infinite horizon* (perpetual option) is given by

$$V(x) = \sup_{\tau} \mathsf{E}_x \big(e^{-r\tau} (K - X_\tau)^+ \big) \tag{25.1.1}$$

where the supremum is taken over all stopping times τ of the geometric Brownian motion $X = (X_t)_{t \ge 0}$ solving

$$dX_t = rX_t \, dt + \sigma X_t \, dB_t \tag{25.1.2}$$

with $X_0 = x > 0$ under P_x. We recall that $B = (B_t)_{t \ge 0}$ is a standard Brownian motion process started at zero, $r > 0$ is the interest rate, $K > 0$ is the strike (exercise) price, and $\sigma > 0$ is the volatility coefficient.

The equation (25.1.2) under P_x has a unique (strong) solution given by

$$X_t = x \exp\big(\sigma B_t + (r - \sigma^2/2)\, t \big) \tag{25.1.3}$$

for $t \ge 0$ and $x > 0$. The process X is strong Markov (diffusion) with the infinitesimal generator given by

$$\mathbb{L}_X = r\, x \, \frac{\partial}{\partial x} + \frac{\sigma^2}{2}\, x^2 \, \frac{\partial^2}{\partial x^2}. \tag{25.1.4}$$

The aim of this subsection is to compute the arbitrage-free price V from (25.1.1) and to determine the optimal exercise time τ_* (at which the supremum in (25.1.1) is attained).

2. The optimal stopping problem (25.1.1) will be solved in two steps. In the first step we will make a guess for the solution. In the second step we will verify that the guessed solution is correct (Theorem 25.1).

From (25.1.1) and (25.1.3) we see that the closer X gets to 0 the less likely that the gain will increase upon continuation. This suggests that there exists a point $b \in (0, K)$ such that the stopping time

$$\tau_b = \inf\{\, t \geq 0 : X_t \leq b \,\} \tag{25.1.5}$$

is optimal in the problem (25.1.1). [In (25.1.5) we use the standard convention that $\inf(\varnothing) = \infty$ (see Remark 25.2 below).]

Standard arguments based on the strong Markov property (cf. Chapter III) lead to the following *free-boundary problem* for the *unknown value function* V and the *unknown point* b:

$$\mathbb{L}_X V = rV \quad \text{for } x > b, \tag{25.1.6}$$
$$V(x) = (K-x)^+ \quad \text{for } x = b, \tag{25.1.7}$$
$$V'(x) = -1 \quad \text{for } x = b \quad \text{(smooth fit)}, \tag{25.1.8}$$
$$V(x) > (K-x)^+ \quad \text{for } x > b, \tag{25.1.9}$$
$$V(x) = (K-x)^+ \quad \text{for } 0 < x < b. \tag{25.1.10}$$

3. To solve the free-boundary problem note that the equation (25.1.6) using (25.1.4) reads

$$Dx^2 V'' + rxV' - rV = 0 \tag{25.1.11}$$

where we set $D = \sigma^2/2$. One may now recognize (25.1.11) as the Cauchy–Euler equation. Let us thus seek a solution in the form

$$V(x) = x^p. \tag{25.1.12}$$

Inserting (25.1.12) into (25.1.11) we get

$$p^2 - \Big(1 - \frac{r}{D}\Big)p - \frac{r}{D} = 0. \tag{25.1.13}$$

The quadratic equation (25.1.13) has two roots, $p_1 = 1$ and $p_2 = -r/D$. Thus the general solution of (25.1.11) can be written as

$$V(x) = C_1\, x + C_2\, x^{-r/D} \tag{25.1.14}$$

where C_1 and C_2 are undetermined constants. From the fact that $V(x) \leq K$ for all $x > 0$, we see that C_1 must be zero. Thus (25.1.7) and (25.1.8) become two algebraic equations in two unknowns C_2 and b (free-boundary). Solving this system one gets

$$C_2 = \frac{D}{r}\Big(\frac{K}{1+D/r}\Big)^{1+r/D}, \tag{25.1.15}$$

$$b = \frac{K}{1+D/r}. \tag{25.1.16}$$

Inserting (25.1.15) into (25.1.14) upon using that $C_1 = 0$ we conclude that

$$V(x) = \begin{cases} \frac{D}{r}\Big(\frac{K}{1+D/r}\Big)^{1+r/D} x^{-r/D} & \text{if } x \in [b,\infty), \\ K - x & \text{if } x \in (0,b]. \end{cases} \tag{25.1.17}$$

Note that V is C^2 on $(0,b) \cup (b,\infty)$ but only C^1 at b. Note also that V is convex on $(0,\infty)$.

4. In this way we have arrived at the two conclusions of the following theorem.

Theorem 25.1. *The arbitrage-free price V from* (25.1.1) *is given explicitly by* (25.1.17) *above. The stopping time τ_b from* (25.1.5) *with b given by* (25.1.16) *above is optimal in the problem* (25.1.1).

Proof. To distinguish the two functions let us denote the value function from (25.1.1) by $V_*(x)$ for $x > 0$. We need to prove that $V_*(x) = V(x)$ for all $x > 0$ where $V(x)$ is given by (25.1.17) above.

1°. The properties of V stated following (25.1.17) above show that Itô's formula (page 67) can be applied to $e^{-rt}V(X_t)$ in its standard form (cf. Subsection 3.5). This gives

$$e^{-rt}V(X_t) = V(x) + \int_0^t e^{-rs}(\mathbb{L}_X V - rV)(X_s) I(X_s \neq b)\, ds \tag{25.1.18}$$
$$+ \int_0^t e^{-rs}\sigma X_s V'(X_s)\, dB_s.$$

Setting $G(x) = (K-x)^+$ we see that $(\mathbb{L}_X G - rG)(x) = -rK < 0$ so that together with (25.1.6) we have

$$(\mathbb{L}_X V - rV) \le 0 \tag{25.1.19}$$

everywhere on $(0,\infty)$ but b. Since $\mathsf{P}_x(X_s = b) = 0$ for all s and all x, we see that (25.1.7), (25.1.9)–(25.1.10) and (25.1.18)–(25.1.19) imply that

$$e^{-rt}(K - X_t)^+ \le e^{-rt}V(X_t) \le V(x) + M_t \tag{25.1.20}$$

where $M = (M_t)_{t\ge 0}$ is a continuous local martingale given by

$$M_t = \int_0^t e^{-rs}\sigma X_s V'(X_s)\, dB_s. \tag{25.1.21}$$

(Using that $|V'(x)| \le 1$ for all $x > 0$ it is easily verified by standard means that M is a martingale.)

Let $(\tau_n)_{n \ge 1}$ be a localization sequence of (bounded) stopping times for M (for example $\tau_n \equiv n$ will do). Then for every stopping time τ of X we have by (25.1.20) above

$$e^{-r(\tau\wedge\tau_n)}(K - X_{\tau\wedge\tau_n})^+ \le V(x) + M_{\tau\wedge\tau_n} \tag{25.1.22}$$

for all $n \ge 1$. Taking the P_x-expectation, using the optional sampling theorem (page 60) to conclude that $\mathsf{E}_x M_{\tau\wedge\tau_n} = 0$ for all n, and letting $n \to \infty$, we find by Fatou's lemma that

$$\mathsf{E}_x\big(e^{-r\tau}(K - X_\tau)^+\big) \le V(x). \tag{25.1.23}$$

Taking the supremum over all stopping times τ of X we find that $V_*(x) \le V(x)$ for all $x > 0$.

2°. To prove the reverse inequality (equality) we observe from (25.1.18) upon using (25.1.6) (and the optional sampling theorem as above) that

$$\mathsf{E}_x\Big(e^{-r(\tau_b\wedge\tau_n)}V(X_{\tau_b\wedge\tau_n})\Big) = V(x) \tag{25.1.24}$$

for all $n \ge 1$. Letting $n \to \infty$ and using that $e^{-r\tau_b}V(X_{\tau_b}) = e^{-r\tau_b}(K - X_{\tau_b})^+$ (both expressions being 0 when $\tau_b = \infty$), we find by the dominated convergence theorem that

$$\mathsf{E}_x\big(e^{-r\tau_b}(K - X_{\tau_b})^+\big) = V(x). \tag{25.1.25}$$

This shows that τ_b is optimal in (25.1.1). Thus $V_*(x) = V(x)$ for all $x > 0$ and the proof is complete. □

Remark 25.2. It is evident from the definition of τ_b in (25.1.5) and the explicit representation of X in (25.1.3) that τ_b is not always finite. Using the well-known Doob formula (see e.g. [197, Chap. VIII, § 2a, (51)])

$$\mathsf{P}\Big(\sup_{t\ge 0}(B_t - \alpha t) \ge \beta\Big) = e^{-2\alpha\beta} \tag{25.1.26}$$

for $\alpha > 0$ and $\beta > 0$, it is straightforwardly verified that

$$\mathsf{P}_x(\tau_b < \infty) = \begin{cases} 1 & \text{if } r \le D \text{ or } x \in (0, b], \\ \left(\frac{b}{x}\right)^{(r/D)-1} & \text{if } r > D \text{ and } x \in (b, \infty) \end{cases} \tag{25.1.27}$$

for $x > 0$.