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Optimal Stopping and Free-Boundary Problems

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Preface

The present monograph, based mainly on studies of the authors and their co-authors, and also on lectures given by the authors in the past few years, has the following particular aims:

> To present basic results (with proofs) of optimal stopping theory in both *discrete* and *continuous time* using both *martingale* and *Markovian approaches*;
>
> To select a series of *concrete* problems of *general interest* from the theory of probability, mathematical statistics, and mathematical finance that can be reformulated as *problems of optimal stopping* of stochastic processes and solved by reduction to *free-boundary problems* of real analysis (*Stefan problems*).

The table of contents found below gives a clearer idea of the material included in the monograph. Credits and historical comments are given at the end of each chapter or section. The bibliography contains a material for further reading.

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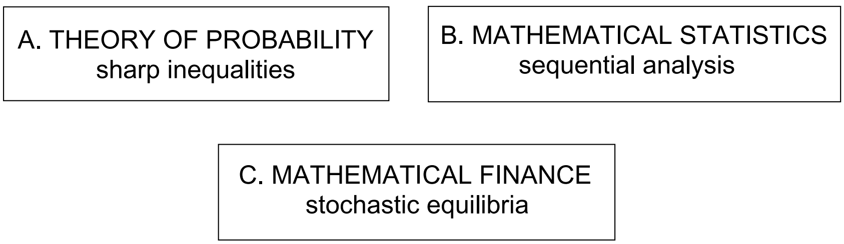
7th March 2006 G.P. & A.Sh.

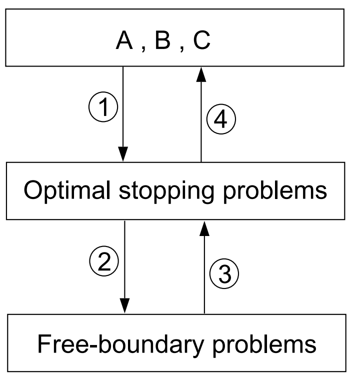
Contents

Introduction

1. The following scheme illustrates the kind of *concrete* problems of *general interest* that will be studied in the monograph:



The solution method for problems A, B, C consists of *reformulation* to an optimal stopping problem and *reduction* to a free-boundary problem as follows:



Steps 1 and 2 indicate the way of reformulation and reduction. Steps 3 and 4 indicate the way of finding a solution to the initial problem.

2. To get some idea of the character of problems A, B, C that will be studied, let us briefly consider the following simple examples.

(A) If $B = (B_t)_{t \geq 0}$ is a standard Brownian motion, then it is well known that the following *maximal equality* holds:

$$\mathsf{E}\Big(\max_{0 \leq t \leq T} |B_t| \Big) = \sqrt{\frac{\pi}{2}\, T} \tag{1}$$

for every deterministic time T. Suppose now that instead of the deterministic time T we are given some (random) stopping time τ of B. The question then arises naturally of how to determine $\mathsf{E}\,(\max_{0\le t\le\tau}|B_t|)$. On closer inspection, however, it becomes clear that it is virtually impossible to compute this expectation for every stopping time τ of B. Thus, as the second best thing, one can try to bound the expectation with a quantity which is easier to compute. A natural candidate for the latter is $\mathsf{E}\,\tau$ at least when finite. In this way a problem A has appeared. This problem then leads to the following *maximal inequality*:

$$\mathsf{E}\Big(\max_{0\le t\le\tau}|B_t|\Big)\le C\sqrt{\mathsf{E}\,\tau} \tag{2}$$

which is valid for all stopping times τ of B with the best constant C equal to $\sqrt{2}$.

We will see in Chapter V that the problem A just formulated can be solved in the form (2) by reformulation to the following optimal stopping problem:

$$V_*=\sup_\tau \mathsf{E}\Big(\max_{0\le t\le\tau}|B_t|-c\tau\Big) \tag{3}$$

where the supremum is taken over all stopping times τ of B satisfying $\mathsf{E}\,\tau<\infty$, and the constant $c>0$ is given and fixed. It constitutes Step 1 in the diagram above.

If $V_*=V_*(c)$ can be computed, then from (3) we get

$$\mathsf{E}\Big(\max_{0\le t\le\tau}|B_t|\Big)\le V_*(c)+c\,\mathsf{E}\,\tau \tag{4}$$

for all stopping times τ of B and all $c>0$. Hence we find

$$\mathsf{E}\Big(\max_{0\le t\le\tau}|B_t|\Big)\le \inf_{c>0}\big(V_*(c)+c\,\mathsf{E}\,\tau\big) \tag{5}$$

for all stopping times τ of B. The right-hand side in (5) defines a function of $\mathsf{E}\,\tau$ that, in view of (3), provides a sharp bound of the left-hand side.

We will see in Chapter IV that the optimal stopping problem (3) can be reduced to a free-boundary problem. This constitutes Step 2 in the diagram above. Solving the free-boundary problem one finds that $V_*(c)=1/2c$. Inserting this into (5) yields

$$\inf_{c>0}\mathsf{E}\big(V_*(c)+c\,\mathsf{E}\,\tau\big)=\sqrt{2\,\mathsf{E}\,\tau} \tag{6}$$

so that the inequality (5) reads as follows:

$$\mathsf{E}\Big(\max_{0\le t\le\tau}|B_t|\Big)\le\sqrt{2\,\mathsf{E}\,\tau} \tag{7}$$

for all stopping times τ of B. This is exactly the inequality (2) above with $C=\sqrt{2}$. From the formulation of the optimal stopping problem (3) it is not surprising that equality in (7) is attained at a stopping time for which both sides in (7) are non-zero. This shows that the constant $\sqrt{2}$ is best possible in (7) as claimed in (2) above. The solution of (3) and its use in (2) just explained constitute Steps 3 and 4 in the diagram above and complete the solution to the initial problem.

Chapter V studies similar sharp inequalities for other stochastic processes using ramifications of the method just exposed. Apart from being able to derive sharp versions of known inequalities the method can also be used to derive new inequalities.

(B) The classic example of a problem in *sequential analysis* is the problem of sequential testing of two statistical hypotheses

$$H_0\colon \mu = \mu_0 \quad \text{and} \quad H_1\colon \mu = \mu_1 \tag{8}$$

about the drift parameter $\mu \in \mathbb{R}$ of the observed process

$$X_t = \mu t + B_t \tag{9}$$

for $t \geq 0$ where $B = (B_t)_{t \geq 0}$ is a standard Brownian motion.

Another classic example of a problem in *sequential analysis* is the problem of sequential testing of two statistical hypotheses

$$H_0\colon \lambda = \lambda_0 \quad \text{and} \quad H_1\colon \lambda = \lambda_1 \tag{10}$$

about the intensity parameter $\lambda > 0$ of the observed process

$$X_t = N_t^\lambda \tag{11}$$

for $t \geq 0$ where $N = (N_t)_{t \geq 0}$ is a standard Poisson process.

The basic problem in both cases seeks to find the optimal decision rule (τ_*, d_*) in the class $\Delta(\alpha, \beta)$ consisting of decision rules (d, τ), where τ is the time of stopping and accepting H_1 if $d = d_1$ or accepting H_0 if $d = d_0$, such that the probability errors of the first and second kind satisfy:

$$\mathsf{P}(\text{accept } H_1 \mid \text{true } H_0) \leq \alpha, \tag{12}$$

$$\mathsf{P}(\text{accept } H_0 \mid \text{true } H_1) \leq \beta \tag{13}$$

and the mean times of observation $\mathsf{E}_0\tau$ and $\mathsf{E}_1\tau$ are as small as possible. It is assumed above that $\alpha > 0$ and $\beta > 0$ with $\alpha + \beta < 1$.

It turns out that with this (*variational*) problem one may associate an optimal stopping (*Bayesian*) problem which in turn can be reduced to a free-boundary problem. This constitutes Steps 1 and 2 in the diagram above. Solving the free-boundary problem leads to an optimal decision rule (τ_*, d_*) in the class $\Delta(\alpha, \beta)$

satisfying (12) and (13) as well as the following two identities:

$$\mathsf{E}_0\tau = \inf_{(\tau,d)} \mathsf{E}_0\tau, \tag{14}$$

$$\mathsf{E}_1\tau = \inf_{(\tau,d)} \mathsf{E}_1\tau \tag{15}$$

where the infimum is taken over all decision rules (τ, d) in $\Delta(\alpha, \beta)$. This constitutes Steps 3 and 4 in the diagram above.

While the methodology just described is the same for both problems (8) and (10), it needs to be pointed out that the solution of the Bayesian problem in the Poisson case is more difficult than in the Brownian case. This is primarily due to the fact that, unlike in the Brownian case, the sample paths of the observed process are discontinuous in the Poisson case.

Chapter VI studies these as well as closely related problems of *quickest detection.* Two of the prime findings of this chapter, which also reflect the historical development of these ideas, are the *principles of smooth and continuous fit*, respectively.

(C) One of the best-known specific problems of *mathematical finance*, that has a direct connection with optimal stopping problems, is the problem of determining the arbitrage-free price of the *American put option.*

Consider the Black–Scholes model where the stock price $X = (X_t)_{t\geq 0}$ is assumed to follow a geometric Brownian motion

$$X_t = x\ \exp\Big(\sigma B_t + (r - \sigma^2/2)\, t\Big) \tag{16}$$

where $x > 0$, $\sigma > 0$, $r > 0$ and $B = (B_t)_{t\geq 0}$ is a standard Brownian motion. By Itô's formula one finds that the process X solves

$$dX_t = r X_t\, dt + \sigma X_t\, dB_t \tag{17}$$

with $X_0 = x$. General theory of financial mathematics makes it clear that the initial problem of determining the arbitrage-free price of the American put option can be reformulated as the following optimal stopping problem:

$$V_* = \sup_\tau \mathsf{E}\, e^{-r\tau}(K - X_\tau)^+ \tag{18}$$

where the supremum is taken over all stopping times τ of X . This constitutes Step 1 in the diagram above. The constant $K > 0$ is called the 'strike price'. It has a certain financial meaning which we set aside for now.

It turns out that the optimal stopping problem (18) can be reduced to a free-boundary problem which can be solved explicitly. It yields the existence of a constant b_* such that the stopping time

$$\tau_* = \inf\,\{\, t \geq 0 : X_t \leq b_* \,\} \tag{19}$$

is optimal in (18). This constitutes Steps 2 and 3 in the diagram above. Both the optimal stopping point b_* and the arbitrage-free price V_* can be expressed explicitly in terms of the other parameters in the problem. A financial interpretation of these expressions constitutes Step 4 in the diagram above.

In the formulation of the problem (18) above no restriction was imposed on the class of admissible stopping times, i.e. for certain reasons of simplicity it was assumed there that τ belongs to the class of stopping times

$$\mathfrak{M} = \{ \tau : 0 \le \tau < \infty \} \tag{20}$$

without any restriction on their size.

A more realistic requirement on a stopping time in search for the arbitrage-free price leads to the following optimal stopping problem:

$$V_*^T = \sup_{\tau \in \mathfrak{M}^T} \mathsf{E}\, e^{-r\tau} (K - X_\tau)^+ \tag{21}$$

where the supremum is taken over all τ belonging to the class of stopping times

$$\mathfrak{M}^T = \{ \tau : 0 \le \tau \le T \} \tag{22}$$

with the horizon T being finite.

The optimal stopping problem (21) can be reduced to a free-boundary problem that apparently cannot be solved explicitly. Its study yields that the stopping time

$$\tau_* = \inf \{ 0 \le t \le T : X_t \le b_*(t) \} \tag{23}$$

is optimal in (21), where $b_* : [0, T] \to \mathbb{R}$ is an increasing continuous function. A nonlinear Volterra integral equation can be derived which characterizes the optimal stopping boundary $t \mapsto b_*(t)$ and can be used to compute its values numerically as accurate as desired. The comments on Steps 1–4 in the diagram above made in the infinite horizon case carry over to the finite horizon case without any change.

Chapter VII studies these and other similar problems that arise from various financial interpretations of options. Chapter VIII studies *optimal prediction* problems. Fuller understanding of their scope is still incomplete at present.

3. So far we have only discussed problems A, B, C and their reformulations as optimal stopping problems. Now we want to address the methods of solution of optimal stopping problems and their reduction to free-boundary problems.

There are essentially two equivalent approaches to finding a solution of the optimal stopping problem. The first one deals with the problem

$$V_* = \sup_{\tau \in \mathfrak{M}} \mathsf{E}\, G_\tau \tag{24}$$

in the case of *infinite horizon*, or the problem

$$V_*^T = \sup_{\tau \in \mathfrak{M}^T} \mathsf{E}\, G_\tau \tag{25}$$

in the case of *finite horizon*, where $\mathfrak{M}$ and $\mathfrak{M}^T$ are the classes of stopping times defined in (20) and (22), respectively.

In this formulation it is important to realize that $G = (G_t)_{t \geq 0}$ is an arbitrary stochastic process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$, where it is assumed that G is adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$ which in turn makes each τ from $\mathfrak{M}$ or $\mathfrak{M}^T$ a stopping time. Since the method of solution to the problems (24) and (25) is based on results from the theory of martingales, the method itself is often referred to as the *martingale method*.

On the other hand, if we are to take a state space $(E, \mathcal{B})$ large enough, then one obtains the “Markov representation” $G_t = G(X_t)$ for some measurable function G, where $X = (X_t)_{t\geq 0}$ is a Markov process with values in E. Moreover, following the contemporary theory of Markov processes it is convenient to adopt the definition of a Markov process X as the *family* of Markov processes

$$((X_t)_{t\geq 0}, (\mathcal{F}_t)_{t\geq 0}, (\mathsf{P}_x)_{x\in E}) \tag{26}$$

where $\mathsf{P}_x(X_0 = x) = 1$, meaning that the process X starts at x under P_x. Such a point of view is convenient, for example, when dealing with the Kolmogorov forward or backward equations, which presuppose that the process can start at any point in the state space. Likewise, it is a profound attempt, developed in stages, to study optimal stopping problems through functions of initial points in the state space.

In this way we have arrived at the second approach which deals with the problem

$$V(x) = \sup_{\tau} \mathsf{E}_x G(X_\tau) \tag{27}$$

where the supremum is taken over $\mathfrak{M}$ or $\mathfrak{M}^T$ as above. Thus, if the Markov representation of the initial problem is valid, we will refer to the *Markovian method* of solution. Its elements will now be exposed in some detail.

Intuitively, it is clear in the Markovian setting that at time t the decision “to stop” or “to continue” with the observation should depend only on the present state X_t of the process and not on its past states X_s for $0 \leq s < t$. It is a great simplification and advantage of the Markovian setting to the general one. Indeed, in this case the problem of optimal stopping in essence becomes a problem of optimal stopping for a random path in the state space E, as opposed to the probability space Ω in the general case, while in many cases of interest we have $E = \mathbb{R}^n$ for some $n \geq 1$. This point of view makes it also clear that many Markovian problems of optimal stopping can be reformulated as problems for elliptic or parabolic equations in $\mathbb{R}^n$, because the transition density of a random path, for example in the case of a diffusion, satisfies the Kolmogorov equations. The connection to elliptic and parabolic equations just addressed will be further clarified by means of the *strong Markov property* in Chapter III.

4. To make the exposed facts more transparent, let us consider the optimal stopping problem (3) in more detail. Denote

$$X_t = |x+B_t| \tag{28}$$

for $x \geq 0$, and enable the maximum process to start at any point by setting

$$S_t = s \vee \Big(\max_{0 \leq r \leq t} X_r \Big) \tag{29}$$

for $s \geq x$. The process $S = (S_t)_{t \geq 0}$ is not Markov, but the pair $(X, S) = (X_t, S_t)_{t \geq 0}$ forms a Markov process with the state space $E = \{\, (x,s) \in \mathbb{R}^2 : 0 \leq x \leq s \,\}$. The value V_* from (3) above coincides with the value function

$$V_*(x,s) = \sup_{\tau} \mathsf{E}_{x,s}\big(S_\tau - c\tau\big) \tag{30}$$

when $x = s = 0$. The problem thus needs to be solved in this more general form.

The general theory of optimal stopping for Markov processes makes it clear that the optimal stopping time in (30) can be written in the form

$$\tau_* = \inf\, \{\, t \geq 0 : (X_t, S_t) \in D_* \} \tag{31}$$

where D_* is the *stopping set*, and $C_* = E \setminus D_*$ is the *continuation set*. In other words, if the observation of X was not stopped before time t since $X_s \in C_*$ for all $0 \leq s < t$, and we have that $X_t \in D_*$, then it is optimal to stop the observation at time t. On the other hand, if it happens that $X_t \in C_*$ as well, then the observation of X should be continued.

Heuristic considerations about the shape of the sets C_* and D_* makes it plausible to guess that there exist a point $s_* \geq 0$ and a continuous increasing function $s \mapsto g_*(s)$ with $g_*(s_*) = 0$ such that

$$D_* = \{\, (x,s) \in \mathbb{R}^2 : 0 \leq x \leq g_*(s)\,,\; s \geq s_* \,\} \tag{32}$$

(see *Figure* 1). Note that such a guess about the shape of the set D_* can be made using the following intuitive arguments. If the process (X, S) starts from a point (x,s) with small x and large s, then it is reasonable to stop immediately, because to increase the value s one needs a large time τ which in the formula (30) appears with a minus sign. At the same time it is easy to see that if x is close or equal to s, then it is reasonable to continue the observation, at least for small time Δ, because s will increase for the value $\sqrt{\Delta}$ while the cost for using this time will be $c\Delta$, and thus $\sqrt{\Delta} - c\Delta > 0$ if Δ is small enough.

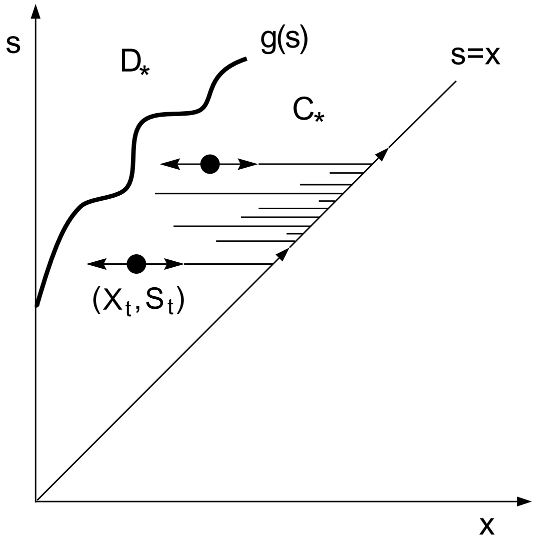


Figure 1: An illustration of the kinematics of the space-maximum process $(X_t, S_t)_{t\geq 0}$ in relation to the optimal stopping boundary g separating the continuation set C_* and the stopping set D_*.

Such an a priori analysis of the shape of the boundary between the stopping set C_* and the continuation set D_* is typical of the act of finding a solution to the optimal stopping problem. The art of guessing in this context very often plays a crucial role in solving the problem.

Having guessed that the stopping set D_* in the optimal stopping problem (30) takes the form (32), it follows that τ_* from (32) attains the supremum i.e.

$$V_*(x,s) = \mathsf{E}_{x,s}\big(S_{\tau_*} - c\tau_*\big) \tag{33}$$

for all $(x,s) \in E$. Denote by $\mathbb{L}_X = (1/2)\,\partial^2/\partial x^2$ the infinitesimal operator of the process X and consider $V_*(x,s)$ as defined by the right-hand side of (33) for (x,s) in the continuation set

$$C_* = C_*^1 \cup C_*^2 \tag{34}$$

where the two subsets are defined as follows:

$$C_*^1 = \{\, (x,s) \in \mathbb{R}^2 : 0 \leq x \leq s < s_* \,\}, \tag{35}$$

$$C_*^2 = \{\, (x,s) \in \mathbb{R}^2 : g_*(s) < x \leq s\,,\ s \geq s_* \,\}. \tag{36}$$

By the strong Markov property one finds that V_* solves the following equation:

$$\mathbb{L}_X V_*(x,s) = c \tag{37}$$

for (x,s) in C_*. Note that if the process (X,S) starts at a point (x,s) with $x<s$, then during a positive time interval the second component S of the process does not change and remains equal to s. This explains why the infinitesimal operator of the process (X,S) reduces to the infinitesimal operator of the process X in the interior of C_*. On the other hand, from the structure of the process (X,S) it follows that at the diagonal in $\mathbb{R}^2_+$ the following condition of *normal reflection* holds:

$$\frac{\partial V_*}{\partial s}(x,s)\bigg|_{x=s-} = 0. \tag{38}$$

Moreover, it is clear that for $(x,s)\in D_*$ the following condition of *instantaneous stopping* holds:

$$V_*(x,s) = s. \tag{39}$$

Finally, either by guessing or providing rigorous arguments, it is found that at the optimal boundary g_* the condition of *smooth fit* holds

$$\frac{\partial V_*}{\partial x}(x,s)\bigg|_{x=g_*(s)+} = 0. \tag{40}$$

The condition of smooth fit embodies the key principle of optimal stopping that will be discussed extensively and used frequently in the sequel.

This analysis indicates that the value function V_* and the optimal stopping boundary g_* can be obtained by searching for the *pair of functions* (V,g) solving the following *free-boundary problem*:

$$\mathbb{L}_X V(x,s) = c \quad \text{for } (x,s) \text{ in } C_g, \tag{41}$$

$$\frac{\partial V}{\partial s}(x,s)\bigg|_{x=s-} = 0 \quad \text{(normal reflection)}, \tag{42}$$

$$V(x,s) = s \quad \text{for } (x,s) \text{ in } D_g \text{ (instantaneous stopping)}, \tag{43}$$

$$\frac{\partial V}{\partial x}(x,s)\bigg|_{x=g(s)+} = 0 \quad \text{(smooth fit)}, \tag{44}$$

where the two sets are defined as follows:

$$C_g = \{\, (x,s)\in\mathbb{R}^2 : 0\le x\le s<s_0 \ \text{ or } \ g(s)<x\le s \ \text{ for } \ s\ge s_0 \,\}, \tag{45}$$

$$D_g = \{\, (x,s)\in\mathbb{R}^2 : 0\le x\le g(s)\,,\ s\ge s_0 \,\} \tag{46}$$

with $g(s_0)=0$. It turns out that this system does not have a unique solution so that an additional criterion is needed to make it unique in general (Chapter IV).

Let us briefly show how to solve the free-boundary problem (41)–(44) by picking the right solution. For more details see Chapters IV and V.

From (41) one finds that for (x,s) in C_g we have

$$V(x,s) = cx^2 + A(s)\,x + B(s) \tag{47}$$

where A and B are some functions of s. To determine A and B as well as g we can use the three conditions (42)–(44) which yield

$$g'(s) = \frac{1}{2(s - g(s))} \tag{48}$$

for $s \geq s_0$. It is easily verified that the linear function

$$g(s) = s - \frac{1}{2c} \tag{49}$$

solves (48). In this way a candidate for the optimal stopping boundary g_* is obtained.

For all $(x, s) \in E$ with $s \geq 1/2c$ one can determine $V(x, s)$ explicitly using (47) and (49). This in particular gives that $V(1/2c, 1/2c) = 3/4c$. For other points $(x, s) \in E$ when $s < 1/2c$ one can determine $V(x, s)$ using that the observation must be continued. In particular for $x = s = 0$ this yields that

$$V(0,0) = V(1/2c, 1/2c) - c\,\mathsf{E}_{0,0}(\sigma) \tag{50}$$

where σ is the first hitting time of the process (X, S) to the point $(1/2c, 1/2c)$. Because $\mathsf{E}_{0,0}(\sigma) = \mathsf{E}_{0,0}(X_\sigma^2) = (1/2c)^2$ and $V(1/2c, 1/2c) = 3/4c$, we find that

$$V(0,0) = \frac{1}{2c} \tag{51}$$

as already indicated prior to (6) above. In this way a candidate for the value function V_* is obtained.

The key role in the proof of the fact that $V = V_*$ and $g = g_*$ is played by Itô's formula (stochastic calculus) and the optional sampling theorem (martingale theory). This step forms a *verification theorem* that makes it clear that the solution of the free-boundary problem coincides with the solution of the optimal stopping problem.

5. The important point to be made in this context is that the verification theorem is usually not difficult to prove in the cases when a candidate solution to the free-boundary problem is obtained explicitly. This is quite typical for one-dimensional problems with infinite horizon, or some simpler two-dimensional problems, as the one just discussed. In the case of problems with finite horizon, however, or other multidimensional problems, the situation can be radically different. In these cases, in a manner quite opposite to the previous ones, the general results of optimal stopping can be used to prove the existence of a solution to the free-boundary problem, thus providing an alternative to analytic methods. Studies of this type will be presented in Chapters VII and VIII.

6. From the material exposed above it is clear that our basic interest concerns the case of continuous time. The theory of optimal stopping in the case

of continuous time is considerably more complicated than in the case of discrete time. However, since the former theory uses many basic ideas from the latter, we have chosen to present the case of discrete time first, both in the *martingale* and *Markovian* setting, which is then likewise followed by the case of continuous time. The two theories form Chapter I. As the methods employed throughout deal extensively with martingales and Markov processes, we have collected some of the basic facts from these theories in Chapter II. In Chapters III and IV we examine the relationship between optimal stopping problems and free-boundary problems. Finally, in Chapters V–VIII we study a number of *concrete* optimal stopping problems of *general interest* as discussed above.

Notes. To conclude the introduction we make a remark of general character about the two approaches used in optimal stopping problems (martingale and Markovian). Their similarities as well as distinction are mostly revealed by how they describe probabilistic evolution of stochastic processes which underly the optimal stopping problem.

To describe the probabilistic structure of a stochastic process in terms of general theory one commonly chooses between the following two methods which may naturally be thought of as *unconditional* and *conditional.*

In the first method one determines the probabilistic structure of a process $X = (X_t)_{t \geq 0}$ by its (unconditional) finite dimensional distributions which generate the corresponding probability distribution $P^X = \mathrm{Law}(X)$ (on the space of trajectories of X). When speaking about optimal stopping of such processes we refer to the *martingale* approach. This terminology is justified by the fact that the appropriate techniques of solution are based on concepts and methods from the theory of martingales (the most important of which is the concept of 'Snell envelope' discussed in Chapter I below).

In the second method one does not begin with the finite-dimensional distributions (which are rather complicated formations) but with a consistent family of (conditional) *transition functions* taking into account the initial state x from where the trajectories of X start. Having its origin in the 1931 paper by A. N. Kolmogorov "Analytical methods in the theory of probability" (see [111]) and leading to the (Markovian) family of probability distributions $P_x^X = \mathrm{Law}(X \mid X_0 = x)$, this approach proves to be very effective in optimal stopping problems due to powerful analytic tools provided by the theory of Markov processes (Kolmogorov forward and backward equations, theory of potential, stochastic differential equations, etc.). It is therefore natural to refer to this approach to optimal stopping problems as the *Markovian* approach.

When solving concrete problems of optimal stopping one may use either of the two approaches, and the choice certainly depends on special features of the problem. When dealing with this issue, however, it should be kept in mind that: on the one hand, any Markov process may be thought of as a special case of processes determined by unconditional probabilities; on the other hand, every process may be considered as Markov by introducing a complex state space whose elements are defined by the "past" of the underlying process.

In the present monograph we generally follow the Markovian approach since it allows us to use the well-developed analytical apparatus arising from theory and problems of differential (and integral) equations. Thus in the sequel we shall be mostly interested in free-boundary problems (Stefan problems) which arise from solving optimal stopping problems via the Markovian approach.

Chapter I.

Optimal stopping: General facts

The aim of the present chapter is to exhibit basic results of general theory of optimal stopping. Both *martingale* and *Markovian* approaches are studied first in discrete time and then in continuous time. The discrete time case, being direct and intuitively clear, provides a number of important insights into the continuous time case.

1. Discrete time

The aim of the present section is to exhibit basic results of optimal stopping in the case of discrete time. We first consider a martingale approach. This is then followed by a Markovian approach.

1.1. Martingale approach

1. Let $G = (G_n)_{n \geq 0}$ be a sequence of random variables defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \geq 0}, \mathsf{P})$. We interpret G_n as the *gain* obtained if the observation of G is stopped at time n. It is assumed that G is adapted to the filtration $(\mathcal{F}_n)_{n \geq 0}$ in the sense that each G_n is $\mathcal{F}_n$-measurable. Recall that each $\mathcal{F}_n$ is a σ-algebra of subsets of Ω such that $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}$. Typically $(\mathcal{F}_n)_{n \geq 0}$ coincides with the natural filtration $(\mathcal{F}_n^G)_{n \geq 0}$ but generally may also be larger. We interpret $\mathcal{F}_n$ as the *information* available up to time n. All our decisions in regard to optimal stopping at time n must be based on this information only (no anticipation is allowed).

The following definition formalizes the previous requirement and plays a key role in the study of optimal stopping.

Definition 1.1. A random variable $\tau : \Omega \to \{0, 1, \ldots, \infty\}$ is called a *Markov time* if $\{\, \tau \leq n \,\} \in \mathcal{F}_n$ for all $n \geq 0$. A Markov time is called a *stopping time* if $\tau < \infty$ P-a.s.

The family of all stopping times will be denoted by $\mathfrak{M}$, and the family of all Markov times will be denoted by $\bar{\mathfrak{M}}$. The following subfamilies of $\mathfrak{M}$ will be used in the present chapter:

$$\mathfrak{M}_n^N = \{\, \tau \in \mathfrak{M} : n \le \tau \le N \,\} \tag{1.1.1}$$

where $0 \le n \le N$. For simplicity we will set $\mathfrak{M}^N = \mathfrak{M}_0^N$ and $\mathfrak{M}_n = \mathfrak{M}_n^\infty$.

The *optimal stopping problem* to be studied seeks to solve

$$V_* = \sup_\tau \mathsf{E}\, G_\tau \tag{1.1.2}$$

where the supremum is taken over a family of stopping times. Note that (1.1.2) involves two tasks: (i) to compute the *value function* V_* as explicitly as possible; (ii) to exhibit an *optimal* stopping time τ_* at which the supremum is attained.

To ensure the existence of $\mathsf{E}\, G_\tau$ in (1.1.2) we need to impose additional conditions on G and τ. If the following condition is satisfied (with $G_N \equiv 0$ when $N = \infty$):

$$\mathsf{E}\Big(\sup_{n \le k \le N} |G_k| \Big) < \infty \tag{1.1.3}$$

then $\mathsf{E}\, G_\tau$ is well defined for all $\tau \in \mathfrak{M}_n^N$. Although for many results below it is possible to go beyond this condition and replace $|G_k|$ above by G_k^- or G_k^+ (or even consider only those τ for which $\mathsf{E}\, G_\tau$ is well defined) we will for simplicity assume throughout that (1.1.3) is satisfied. A more careful inspection of the proofs will easily reveal how the condition (1.1.3) can be relaxed.

With the subfamilies of stopping times $\mathfrak{M}_n^N$ introduced in (1.1.1) above we will associate the following value functions:

$$V_n^N = \sup_{\tau \in \mathfrak{M}_n^N} \mathsf{E}\, G_\tau \tag{1.1.4}$$

where $0 \le n \le N$. Again, for simplicity, we will set $V^N = V_0^N$ and $V_n = V_n^\infty$. Likewise, we will set $V = V_0^\infty$ when the supremum is taken over all τ in $\mathfrak{M}$. The main purpose of the present subsection is to study the optimal stopping problem (1.1.4) using a martingale approach.

Sometimes it is also of interest to admit that τ in (1.1.2) takes the value ∞ with positive probability, so that τ belongs to $\bar{\mathfrak{M}}$. In such a case we need to make an agreement about the value of G_τ on $\{\tau = \infty\}$. Clearly, if $\lim_{n\to\infty} G_n$ exists, then G_∞ is naturally set to take this value. Another possibility is to let G_∞ take an arbitrary but fixed value. Finally, for certain reasons of convenience, it is useful to set $G_\infty = \limsup_{n\to\infty} G_n$. In general, however, none of these choices is better than the others, and a preferred choice should always be governed by the meaning of a specific problem studied.

2. The method of backward induction. The first method for solving the problem (1.1.4) when $N < \infty$ uses backward induction first to construct a sequence of random variables $(S_n^N)_{0 \le n \le N}$ that solves the problem in a stochastic sense. Taking expectation then solves the problem in the original mean-valued sense.

Consider the optimal stopping problem (1.1.4) when $N < \infty$. Recall that (1.1.4) reads more explicitly as follows:

$$V_n^N = \sup_{n \le \tau \le N} \mathsf{E}\, G_\tau \tag{1.1.5}$$

where τ is a stopping time and $0 \le n \le N$. To solve the problem we can let the time go backward and proceed recursively as follows.

For $n = N$ we have to stop immediately and our gain S_N^N equals G_N. For $n = N-1$ we can either stop or continue. If we stop our gain S_{N-1}^N will be equal to G_{N-1}, and if we continue optimally our gain S_{N-1}^N will be equal to $\mathsf{E}\,(S_N^N \mid \mathcal{F}_{N-1})$. The latter conclusion reflects the fact that our decision about stopping or continuation at time $n = N-1$ must be based on the information contained in $\mathcal{F}_{N-1}$ only. It follows that if $G_{N-1} \ge \mathsf{E}\,(S_N^N \mid \mathcal{F}_{N-1})$ then we need to stop at time $n = N-1$, and if $G_{N-1} < \mathsf{E}\,(S_N^N \mid \mathcal{F}_{N-1})$ then we need to continue at time $n = N-1$. For $n = N-2, \ldots, 0$ the considerations are continued analogously.

The method of backward induction just explained leads to a sequence of random variables $(S_n^N)_{0 \le n \le N}$ defined recursively as follows:

$$S_n^N = G_N \quad \text{for} \quad n = N, \tag{1.1.6}$$

$$S_n^N = \max\big(G_n, \mathsf{E}\,(S_{n+1}^N \mid \mathcal{F}_n)\big) \quad \text{for} \quad n = N-1, \ldots, 0. \tag{1.1.7}$$

The method also suggests that we consider the following stopping time:

$$\tau_n^N = \inf\,\{\, n \le k \le N : S_k^N = G_k \,\} \tag{1.1.8}$$

for $0 \le n \le N$. Note that the infimum in (1.1.8) is always attained.

The first part of the following theorem shows that S_n^N and τ_n^N solve the problem in a stochastic sense. The second part of the theorem shows that this leads to a solution of the initial problem (1.1.5). The third part of the theorem provides a supermartingale characterization of the solution. The method of backward induction and the results presented in the theorem play a central role in the theory of optimal stopping.

Theorem 1.2. (Finite horizon) *Consider the optimal stopping problem* (1.1.5) *upon assuming that the condition* (1.1.3) *holds. Then for all* $0 \le n \le N$ *we have*:

$$S_n^N \ge \mathsf{E}\,(G_\tau \mid \mathcal{F}_n) \quad \textit{for each } \tau \in \mathfrak{M}_n^N, \tag{1.1.9}$$

$$S_n^N = \mathsf{E}\,(G_{\tau_n^N} \mid \mathcal{F}_n). \tag{1.1.10}$$

Moreover, if $0 \le n \le N$ *is given and fixed, then we have*:

The stopping time τ_n^N *is optimal in* (1.1.5). (1.1.11)

If τ_* *is an optimal stopping time in* (1.1.5) *then* $\tau_n^N \le \tau_*$ P*-a.s.* (1.1.12)

The sequence $(S_k^N)_{n \le k \le N}$ *is the smallest supermartingale which dominates* $(G_k)_{n \le k \le N}$. (1.1.13)

The stopped sequence $(S_{k \wedge \tau_n^N}^N)_{n \le k \le N}$ *is a martingale.* (1.1.14)

Proof. (1.1.9)–(1.1.10): The proof will be carried out by induction over $n = N$, $N-1, \ldots, 0$. Note that both relations are trivially satisfied when $n = N$ due to (1.1.6) above. Let us thus assume that (1.1.9) and (1.1.10) hold for $n = N, N-1, \ \ldots, k$ where $k \ge 1$, and let us show that (1.1.9) and (1.1.10) must then also hold for $n = k-1$.

(1.1.9): Take $\tau \in \mathfrak{M}_{k-1}^N$ and set $\bar\tau = \tau \vee k$. Then $\bar\tau \in \mathfrak{M}_k^N$ and since $\{\tau \ge k\} \in \ \mathcal{F}_{k-1}$ it follows that

$$\begin{aligned} \mathsf{E}\,(G_\tau \mid \mathcal{F}_{k-1}) &= \mathsf{E}\big(I(\tau = k-1)\, G_{k-1} \mid \mathcal{F}_{k-1}\big) + \mathsf{E}\big(I(\tau \ge k)\, G_{\bar\tau} \mid \mathcal{F}_{k-1}\big) \\ &= I(\tau = k-1)\, G_{k-1} + I(\tau \ge k)\, \mathsf{E}\big(\mathsf{E}\,(G_{\bar\tau} \mid \mathcal{F}_k) \mid \mathcal{F}_{k-1}\big). \end{aligned} \tag{1.1.15}$$

By the induction hypothesis the inequality (1.1.9) holds for $n = k$. Since $\bar\tau \in \mathfrak{M}_k^N$ this implies that $\mathsf{E}\,(G_{\bar\tau} \mid \mathcal{F}_k) \le S_k^N$. On the other hand, from (1.1.7) we see that $G_{k-1} \le S_{k-1}^N$ and $\mathsf{E}\,(S_k^N \mid \mathcal{F}_{k-1}) \le S_{k-1}^N$. Applying the preceding three inequalities to the right-hand side of (1.1.15) we get

$$\begin{aligned} \mathsf{E}\,(G_\tau \mid \mathcal{F}_{k-1}) &\le I(\tau = k-1)\, S_{k-1}^N + I(\tau \ge k)\, \mathsf{E}\,(S_k^N \mid \mathcal{F}_{k-1}) \\ &\le I(\tau = k-1)\, S_{k-1}^N + I(\tau \ge k)\, S_{k-1}^N = S_{k-1}^N. \end{aligned} \tag{1.1.16}$$

This shows that (1.1.9) holds for $n = k-1$ as claimed.

(1.1.10): To prove that (1.1.10) holds for $n = k-1$ it is enough to check that all inequalities in (1.1.15) and (1.1.16) remain equalities when $\tau = \tau_{k-1}^N$. For this, note from (1.1.8) that $\tau_{k-1}^N = \tau_k^N$ on $\{\tau_{k-1}^N \ge k\}$, so that from (1.1.15) with $\tau = \tau_{k-1}^N$ and the induction hypothesis (1.1.10) for $n = k$, we get

$$\begin{aligned} \mathsf{E}\,(G_{\tau_{k-1}^N} \mid \mathcal{F}_{k-1}) &= I(\tau_{k-1}^N = k-1)\, G_{k-1} \\ &\quad + I(\tau_{k-1}^N \ge k)\ \mathsf{E}\big(\mathsf{E}\,(G_{\tau_k^N} \mid \mathcal{F}_k) \mid \mathcal{F}_{k-1}\big) \\ &= I(\tau_{k-1}^N = k-1)\, G_{k-1} + I(\tau_{k-1}^N \ge k)\, \mathsf{E}\,(S_k^N \mid \mathcal{F}_{k-1}) \\ &= I(\tau_{k-1}^N = k-1)\, S_{k-1}^N + I(\tau_{k-1}^N \ge k)\, S_{k-1}^N = S_{k-1}^N \end{aligned} \tag{1.1.17}$$

where in the second last equality we use that $G_{k-1} = S_{k-1}^N$ on $\{\, \tau_{k-1}^N = k-1 \,\}$ by (1.1.8) as well as that $\mathsf{E}\,(S_k^N \mid \mathcal{F}_{k-1}) = S_{k-1}^N$ on $\{\, \tau_{k-1}^N \ge k \,\}$ by (1.1.8) and (1.1.7). This shows that (1.1.10) holds for $n = k-1$ as claimed.

(1.1.11): Taking E in (1.1.9) we find that $\mathsf{E}\,S_n^N \ge \mathsf{E}\,G_\tau$ for all $\tau \in \mathfrak{M}_n^N$ and hence by taking the supremum over all $\tau \in \mathfrak{M}_n^N$ we see that $\mathsf{E}\,S_n^N \ge V_n^N$. On the other hand, taking the expectation in (1.1.10) we get $\mathsf{E}\,S_n^N = \mathsf{E}\,G_{\tau_n^N}$ which shows that $\mathsf{E}\,S_n^N \le V_n^N$. The two inequalities give the equality $V_n^N = \mathsf{E}\,S_n^N$, and since $\mathsf{E}\,S_n^N = \mathsf{E}\,G_{\tau_n^N}$, we see that $V_n^N = \mathsf{E}\,G_{\tau_n^N}$ implying the claim.

(1.1.12): We claim that the optimality of τ_* implies that $S_{\tau_*}^N = G_{\tau_*}$ P-a.s. Indeed, if this would not be the case, then using that $S_k^N \ge G_k$ for all $n \le k \le N$ by (1.1.6)–(1.1.7), we see that $S_{\tau_*}^N \ge G_{\tau_*}$ with $\mathsf{P}(S_{\tau_*}^N > G_{\tau_*}) > 0$. It thus follows that $\mathsf{E}\,G_{\tau_*} < \mathsf{E}\,S_{\tau_*}^N \le \mathsf{E}\,S_n^N = V_n^N$ where the second inequality follows by the optional sampling theorem (page 60) and the supermartingale property of $(S_k^N)_{n\le k\le N}$ established in (1.1.13) below, while the final equality follows from the proof of (1.1.11) above. The strict inequality, however, contradicts the fact that τ_* is optimal. Hence $S_{\tau_*}^N = G_{\tau_*}$ P-a.s. as claimed and the fact that $\tau_n^N \le \tau_*$ P-a.s. follows from the definition (1.1.8).

(1.1.13): From (1.1.7) it follows that

$$S_k^N \ge \mathsf{E}\,(S_{k+1}^N \mid \mathcal{F}_k) \tag{1.1.18}$$

for all $n \le k \le N-1$ showing that $(S_k^N)_{n\le k\le N}$ is a supermartingale. From (1.1.6) and (1.1.7) it follows that $S_k^N \ge G_k$ P-a.s. for all $n \le k \le N$ meaning that $(S_k^N)_{n\le k\le N}$ dominates $(G_k)_{n\le k\le N}$. Moreover, if $(\widetilde{S}_k)_{n\le k\le N}$ is another supermartingale which dominates $(G_k)_{n\le k\le N}$, then the claim that $\widetilde{S}_k \ge S_k^N$ P-a.s. can be verified by induction over $k = N, N-1, \ldots, l$. Indeed, if $k = N$ then the claim follows by (1.1.6). Assuming that $\widetilde{S}_k \ge S_k^N$ P-a.s. for $k = N, N-1, \ldots, l$ with $l \ge n+1$ it follows by (1.1.7) that $S_{l-1}^N = \max(G_{l-1}, \mathsf{E}\,(S_l^N \mid \mathcal{F}_{l-1})) \le \max(G_{l-1}, \mathsf{E}\,(\widetilde{S}_l \mid \mathcal{F}_{l-1})) \le \widetilde{S}_{l-1}$ P-a.s. using the supermartingale property of $(\tilde{S}_k)_{n\le k\le N}$ and proving the claim.

(1.1.14): To verify the martingale property

$$\mathsf{E}\left(S_{(k+1)\wedge\tau_n^N}^N \mid \mathcal{F}_k\right) = S_{k\wedge\tau_n^N}^N \tag{1.1.19}$$

with $n \le k \le N-1$ given and fixed, note that

$$\begin{aligned} &\mathsf{E}\left(S_{(k+1)\wedge\tau_n^N}^N \mid \mathcal{F}_k\right) \\ &\quad= \mathsf{E}\left(I(\tau_n^N \le k)\, S_{k\wedge\tau_n^N}^N \mid \mathcal{F}_k\right) + \mathsf{E}\left(I(\tau_n^N \ge k+1)\, S_{k+1}^N \mid \mathcal{F}_k\right) \\ &\quad= I(\tau_n^N \le k)\, S_{k\wedge\tau_n^N}^N + I(\tau_n^N \ge k+1)\, \mathsf{E}\,(S_{k+1}^N \mid \mathcal{F}_k) \\ &\quad= I(\tau_n^N \le k)\, S_{k\wedge\tau_n^N}^N + I(\tau_n^N \ge k+1)\, S_k^N = S_{k\wedge\tau_n^N}^N \end{aligned} \tag{1.1.20}$$

where the second last equality follows from the fact that $S_k^N = \mathsf{E}\,(S_{k+1}^N \mid \mathcal{F}_k)$ on $\{\tau_n^N \ge k+1\}$, while $\{\tau_n^N \ge k+1\} \in \mathcal{F}_k$ since τ_n^N is a stopping time. This establishes (1.1.19) and the proof of the theorem is complete. □

Note that (1.1.9) can also be derived from the supermartingale property (1.1.13), and that (1.1.10) can also be derived from the martingale property (1.1.14), both by means of the optional sampling theorem (page 60).

It follows from Theorem 1.2 that the optimal stopping problem V_0^N is solved inductively by solving the problems V_n^N for $n = N, N-1, \ldots, 0$. Moreover, the optimal stopping rule τ_n^N for V_n^N satisfies $\tau_n^N = \tau_k^N$ on $\{\tau_n^N \geq k\}$ for $0 \leq n \leq k \leq N$ where τ_k^N is the optimal stopping rule for V_k^N. This, in other words, means that if it was not optimal to stop within the time set $\{n, n+1, \ldots, k-1\}$ then the same optimality rule applies in the time set $\{k, k+1, \ldots, N\}$. In particular, when specialized to the problem V_0^N, the following general principle is obtained: If the stopping rule τ_0^N is optimal for V_0^N and it was not optimal to stop within the time set $\{0, 1, \ldots, n-1\}$, then starting the observation at time n and being based on the information $\mathcal{F}_n$, the same stopping rule is still optimal for the problem V_n^N. This principle of solution for optimal stopping problems has led to the general principle of *dynamic programming* in the theory of optimal stochastic control (often referred to as *Bellman's principle*).

3. The method of essential supremum. The method of backward induction by its nature requires that the horizon N be finite so that the case of infinite horizon N remains uncovered. It turns out, however, that the random variables S_n^N defined by the recurrent relations (1.1.6)–(1.1.7) above admit a different characterization which can be directly extended to the case of infinite horizon N. This characterization forms the basis for the second method that will now be presented.

With this aim note that (1.1.9) and (1.1.10) in Theorem 1.2 above suggest that the following identity should hold:

$$S_n^N = \sup_{\tau \in \mathfrak{M}_n^N} \mathsf{E}\big(G_\tau \mid \mathcal{F}_n\big). \tag{1.1.21}$$

A difficulty arises, however, from the fact that both (1.1.9) and (1.1.10) hold only P-a.s. so that the exceptional P-null set may depend on the given $\tau \in \mathfrak{M}_n^N$. Thus, if the supremum in (1.1.21) is taken over uncountably many τ, then the right-hand side need not define a measurable function, and the identity (1.1.21) may fail as well. To overcome this difficulty it turns out that the concept of *essential supremum* proves useful.

Lemma 1.3. (Essential supremum) *Let $\{\, Z_\alpha : \alpha \in I \,\}$ be a family of random variables defined on $(\Omega, \mathcal{G}, \mathsf{P})$ where the index set I can be arbitrary. Then there exists a countable subset J of I such that the random variable $Z^* : \Omega \to \bar{\mathbb{R}}$ defined by*

$$Z^* = \sup_{\alpha \in J} Z_\alpha \tag{1.1.22}$$

satisfies the following two properties:

$$\mathsf{P}(Z_\alpha \le Z^*) = 1 \ \ \textit{for each } \alpha \in I. \tag{1.1.23}$$

If $\tilde{Z} : \Omega \to \bar{\mathbb{R}}$ *is another random variable satisfying* (1.1.23) *in place of* Z^**, then* $\mathsf{P}(Z^* \le \tilde{Z}) = 1$. (1.1.24)

The random variable Z^* is called the *essential supremum* of $\{ Z_\alpha : \alpha \in I \}$ relative to P and is denoted by $Z^* = \operatorname{ess\,sup}_{\alpha \in I} Z_\alpha$. It is determined by the properties (1.1.23) and (1.1.24) uniquely up to a P-null set.

Moreover, if the family $\{Z_\alpha : \alpha \in I\}$ *is upwards directed in the sense that*

For any α *and* β *in* I *there exists* γ *in* I *such that* $Z_\alpha \vee Z_\beta \le Z_\gamma$ P-*a.s.* (1.1.25)

then the countable set $J = \{\alpha_n : n \ge 1\}$ *can be chosen so that*

$$Z^* = \lim_{n\to\infty} Z_{\alpha_n} \quad \mathsf{P}\text{-}a.s. \tag{1.1.26}$$

where $Z_{\alpha_1} \le Z_{\alpha_2} \le \cdots$ P-*a.s.*

Proof. Since $x \mapsto (2/\pi)\arctan(x)$ is a strictly increasing function from $\bar{\mathbb{R}}$ to $[-1, 1]$, it is no restriction to assume that $|Z_\alpha| \le 1$ for all $\alpha \in I$. Otherwise, replace Z_α by $(2/\pi)\arctan(Z_\alpha)$ for $\alpha \in I$ and proceed as in the rest of the proof.

Let $\mathcal{C}$ denote the family of all countable subsets C of I. Choose an increasing sequence $\{ C_n : n \ge 1 \}$ in $\mathcal{C}$ such that

$$a = \sup_{C\in\mathcal{C}} \mathsf{E}\Big(\sup_{\alpha\in C} Z_\alpha\Big) = \sup_{n\ge 1} \mathsf{E}\Big(\sup_{\alpha\in C_n} Z_\alpha\Big). \tag{1.1.27}$$

Then $J := \bigcup_{n=1}^\infty C_n$ is a countable subset of I and we claim that Z^* defined by (1.1.22) satisfies (1.1.23) and (1.1.24).

To verify these claims take $\alpha \in I$ arbitrarily and note the following. If $\alpha \in J$ then $Z_\alpha \le Z^*$ so that (1.1.23) holds. On the other hand, if $\alpha \notin J$ and we assume that $\mathsf{P}(Z_\alpha > Z^*) > 0$, then $a < \mathsf{E}(Z^* \vee Z_\alpha) \le a$ since $a = \mathsf{E}\,Z^* \in [-1, 1]$ (by the monotone convergence theorem) and $J \cup \{\alpha\}$ belongs to $\mathcal{C}$. As the strict inequality is clearly impossible, we see that (1.1.23) holds for all $\alpha \in I$ as claimed. Moreover, it is obvious that (1.1.24) follows from (1.1.22) and (1.1.23) since J is countable.

Finally, if (1.1.25) is satisfied then the initial countable set $J = \{\alpha_1^0, \alpha_2^0, \dots\}$ can be replaced by a new countable set $J = \{ \alpha_1, \alpha_2, \dots \}$ if we initially set $\alpha_1 = \alpha_1^0$, and then inductively choose $\alpha_{n+1} \ge \alpha_n \vee \alpha_{n+1}^0$ for $n \ge 1$, where $\gamma \ge \alpha \vee \beta$ corresponds to Z_α, Z_β and Z_γ such that $Z_\gamma \ge Z_\alpha \vee Z_\beta$ P-a.s.

The concluding claim in (1.1.26) is then obvious, and the proof of the lemma is complete. □

With the concept of essential supremum we may now rewrite (1.1.9) and (1.1.10) in Theorem 1.2 above as follows:

$$S_n^N = \operatorname*{ess\,sup}_{n \le \tau \le N} \mathsf{E}\,(G_\tau \mid \mathcal{F}_n) \tag{1.1.28}$$

for all $0 \le n \le N$. This identity provides an additional characterization of the sequence of random variables $(S_n^N)_{0 \le n \le N}$ introduced initially by means of the recurrent relations (1.1.6)–(1.1.7). Its advantage in comparison with the recurrent relations lies in the fact that the identity (1.1.28) can naturally be extended to the case of infinite horizon N. This programme will now be described.

Consider the optimal stopping problem (1.1.4) when $N = \infty$. Recall that (1.1.4) reads more explicitly as follows:

$$V_n = \sup_{\tau \ge n} \mathsf{E}\, G_\tau \tag{1.1.29}$$

where τ is a stopping time and $n \ge 0$. To solve the problem we will consider the sequence of random variables $(S_n)_{n \ge 0}$ defined as follows:

$$S_n = \operatorname*{ess\,sup}_{\tau \ge n} \mathsf{E}\,(G_\tau \mid \mathcal{F}_n) \tag{1.1.30}$$

as well as the following stopping time:

$$\tau_n = \inf\,\{\, k \ge n : S_k = G_k \} \tag{1.1.31}$$

for $n \ge 0$ where $\inf \varnothing = \infty$ by definition. The sequence $(S_n)_{n \ge 0}$ is often referred to as the *Snell envelope* of G.

The first part of the following theorem shows that $(S_n)_{n \ge 0}$ satisfies the same recurrent relations as $(S_n^N)_{0 \le n \le N}$. The second part of the theorem shows that S_n and τ_n solve the problem in a stochastic sense. The third part of the theorem shows that this leads to a solution of the initial problem (1.1.29). The fourth part of the theorem provides a supermartingale characterization of the solution.

Theorem 1.4. (Infinite horizon) *Consider the optimal stopping problem* (1.1.29) *upon assuming that the condition* (1.1.3) *holds. Then the following recurrent relations hold:*

$$S_n = \max\big(G_n, \mathsf{E}\,(S_{n+1} \mid \mathcal{F}_n)\big) \tag{1.1.32}$$

for all $n \ge 0$. *Assume moreover when required below that*

$$\mathsf{P}(\tau_n < \infty) = 1 \tag{1.1.33}$$

where $n \geq 0$. *Then for all* $n \geq 0$ *we have:*

$$S_n \geq \mathsf{E}\,(G_\tau \mid \mathcal{F}_n) \;\; \textit{for each } \tau \in \mathfrak{M}_n, \tag{1.1.34}$$

$$S_n = \mathsf{E}\,(G_{\tau_n} \mid \mathcal{F}_n). \tag{1.1.35}$$

Moreover, if $n \geq 0$ *is given and fixed, then we have*:

The stopping time τ_n *is optimal in* (1.1.29). (1.1.36)

If τ_* *is an optimal stopping time in* (1.1.29) *then* $\tau_n \leq \tau_*$ P*-a.s.* (1.1.37)

The sequence $(S_k)_{k\geq n}$ *is the smallest supermartingale which dominates* $(G_k)_{k\geq n}$ *.* (1.1.38)

The stopped sequence $(S_{k\wedge\tau_n})_{k\geq n}$ *is a martingale.* (1.1.39)

Finally, if the condition (1.1.33) *fails so that* $\mathsf{P}(\tau_n = \infty) > 0$ *, then there is no optimal stopping time* (*with probability* 1) *in* (1.1.29).

Proof. (1.1.32): Let us first show that the left-hand side is smaller than the right-hand side when $n \geq 0$ is given and fixed.

For this, take $\tau \in \mathfrak{M}_n$ and set $\bar\tau = \tau \vee (n+1)$. Then $\bar\tau \in \mathfrak{M}_{n+1}$ and since $\{\tau \geq n+1\} \in \mathcal{F}_n$ we have

$$\begin{aligned} \mathsf{E}\,(G_\tau \mid \mathcal{F}_n) &= \mathsf{E}\,\big(I(\tau = n)\, G_n \mid \mathcal{F}_n\big) + \mathsf{E}\,\big(I(\tau \geq n+1)\, G_{\bar\tau} \mid \mathcal{F}_n\big) \\ &= I(\tau = n)\, G_n + I(\tau \geq n+1)\, \mathsf{E}\,(G_{\bar\tau} \mid \mathcal{F}_n) \\ &= I(\tau = n)\, G_n + I(\tau \geq n+1)\, \mathsf{E}\,\big(\mathsf{E}\,(G_{\bar\tau} \mid \mathcal{F}_{n+1}) \mid \mathcal{F}_n\big) \\ &\leq I(\tau = n)\, G_n + I(\tau \geq n+1)\, \mathsf{E}\,(S_{n+1} \mid \mathcal{F}_n) \\ &\leq \max\big(G_n, \mathsf{E}\,(S_{n+1} \mid \mathcal{F}_n)\big). \end{aligned} \tag{1.1.40}$$

From this inequality it follows that

$$\operatorname*{ess\,sup}_{\tau\geq n} \mathsf{E}\,(G_\tau \mid \mathcal{F}_n) \leq \max\big(G_n, \mathsf{E}\,(S_{n+1} | \mathcal{F}_n)\big) \tag{1.1.41}$$

which is the desired inequality.

To prove the reverse inequality, let us first note that $S_n \geq G_n$ P-a.s. by the definition of S_n so that it is enough to show that

$$S_n \geq \mathsf{E}\,(S_{n+1} \mid \mathcal{F}_n) \tag{1.1.42}$$

which is the supermartingale property of $(S_n)_{n\geq 0}$. To verify this inequality, let us first show that the family $\{\mathsf{E}\,(G_\tau \mid \mathcal{F}_{n+1}) : \tau \in \mathfrak{M}_{n+1}\}$ is upwards directed in the sense that (1.1.25) is satisfied. For this, note that if σ_1 and σ_2 are from $\mathfrak{M}_{n+1}$ and we set $\sigma_3 = \sigma_1 I_A + \sigma_2 I_{A^c}$ where $A = \{\mathsf{E}\,(G_{\sigma_1} \mid \mathcal{F}_{n+1}) \geq \mathsf{E}\,(G_{\sigma_2} \mid \mathcal{F}_{n+1})\}$, then σ_3 belongs to $\mathfrak{M}_{n+1}$ and we have

$$\begin{aligned} \mathsf{E}\,(G_{\sigma_3} \mid \mathcal{F}_{n+1}) &= \mathsf{E}\,(G_{\sigma_1} I_A + G_{\sigma_2} I_{A^c} \mid \mathcal{F}_{n+1}) \\ &= I_A\, \mathsf{E}\,(G_{\sigma_1} \mid \mathcal{F}_{n+1}) + I_{A^c}\, \mathsf{E}\,(G_{\sigma_2} \mid \mathcal{F}_{n+1}) \\ &= \mathsf{E}\,(G_{\sigma_1} \mid \mathcal{F}_{n+1}) \vee \mathsf{E}\,(G_{\sigma_2} \mid \mathcal{F}_{n+1}) \end{aligned} \tag{1.1.43}$$

implying (1.1.25) as claimed. Hence by (1.1.26) there exists a sequence $\{\sigma_k : k \geq 1\}$ in $\mathfrak{M}_{n+1}$ such that

$$\operatorname*{ess\,sup}_{\tau \geq n+1} \mathsf{E}\,(G_\tau \mid \mathcal{F}_{n+1}) = \lim_{k\to\infty} \mathsf{E}\,(G_{\sigma_k} \mid \mathcal{F}_{n+1}) \tag{1.1.44}$$

where $\mathsf{E}\,(G_{\sigma_1} \mid \mathcal{F}_{n+1}) \leq \mathsf{E}\,(G_{\sigma_2} \mid \mathcal{F}_{n+1}) \leq \cdots$ P-a.s. Since the left-hand side in (1.1.44) equals S_{n+1}, by the conditional monotone convergence theorem we get

$$\begin{aligned} \mathsf{E}\,(S_{n+1} \mid \mathcal{F}_n) &= \mathsf{E}\Big(\lim_{k\to\infty} \mathsf{E}\,(G_{\sigma_k} \mid \mathcal{F}_{n+1}) \mid \mathcal{F}_n\Big) \\ &= \lim_{k\to\infty} \mathsf{E}\big(\mathsf{E}\,(G_{\sigma_k} \mid \mathcal{F}_{n+1}) \mid \mathcal{F}_n\big) \\ &= \lim_{k\to\infty} \mathsf{E}\,(G_{\sigma_k} \mid \mathcal{F}_n) \leq S_n \end{aligned} \tag{1.1.45}$$

where the final inequality follows from the definition of S_n. This establishes (1.1.42) and the proof of (1.1.32) is complete.

(1.1.34): This inequality follows directly from the definition (1.1.30).

(1.1.35): The proof of (1.1.39) below shows that the stopped sequence $(S_{k\wedge\tau_n})_{k\geq n}$ is a martingale. Moreover, setting $G_n^* = \sup_{k\geq n} |G_k|$ we have

$$|S_k| \leq \operatorname*{ess\,sup}_{\tau\geq k} \mathsf{E}\,\big(|G_\tau| \mid \mathcal{F}_k\big) \leq \mathsf{E}\,(G_n^* \mid \mathcal{F}_k) \tag{1.1.46}$$

for all $k \geq n$. Since G_n^* is integrable due to (1.1.3), it follows from (1.1.46) that $(S_k)_{k\geq n}$ is uniformly integrable. Thus the optional sampling theorem (page 60) can be applied to the martingale $(M_k)_{k\geq n} = (S_{k\wedge\tau_n})_{k\geq n}$ and the stopping time τ_n yielding

$$M_n = \mathsf{E}\,(M_{\tau_n} \mid \mathcal{F}_n). \tag{1.1.47}$$

Since $M_n = S_n$ and $M_{\tau_n} = S_{\tau_n}$ we see that (1.1.47) is the same as (1.1.35).

(1.1.36): This is proved using (1.1.34) and (1.1.35) in exactly the same way as (1.1.11) above using (1.1.9) and (1.1.10).

(1.1.37): This is proved in exactly the same way as (1.1.12) above.

(1.1.38): It was shown in (1.1.42) that $(S_k)_{k\geq n}$ is a supermartingale. Moreover, it follows from (1.1.30) that $S_k \geq G_k$ P-a.s. for all $k \geq n$ meaning that $(S_k)_{k\geq n}$ dominates $(G_k)_{k\geq n}$. Finally, if $(\tilde{S}_k)_{k\geq n}$ is another supermartingale which dominates $(G_k)_{k\geq n}$, then by (1.1.35) we find

$$S_k = \mathsf{E}\,(G_{\tau_k} \mid \mathcal{F}_k) \leq \mathsf{E}\,(\tilde{S}_{\tau_k} \mid \mathcal{F}_k) \leq \tilde{S}_k \tag{1.1.48}$$

for all $k \geq n$ where the final inequality follows by the optional sampling theorem (page 60) being applicable since $\tilde{S}_k^- \leq G_k^- \leq G_n^*$ for all $k \geq n$ with G_n^* integrable.

(1.1.39): This is proved in exactly the same way as (1.1.14) above.

Finally, note that the final claim follows directly from (1.1.37). This completes the proof of the theorem. □

4. In the last part of this subsection we will briefly explore a connection between the two methods above when the horizon N tends to infinity in the former.

For this, note from (1.1.28) that $N \mapsto S_n^N$ and $N \mapsto \tau_n^N$ are increasing, so that

$$S_n^\infty = \lim_{N\to\infty} S_n^N \quad \text{and} \quad \tau_n^\infty = \lim_{N\to\infty} \tau_n^N \tag{1.1.49}$$

exist P-a.s. for each $n \ge 0$. Note also from (1.1.5) that $N \mapsto V_n^N$ is increasing, so that

$$V_n^\infty = \lim_{N\to\infty} V_n^N \tag{1.1.50}$$

exists for each $n \ge 0$. From (1.1.28) and (1.1.30) we see that

$$S_n^\infty \le S_n \quad \text{and} \quad \tau_n^\infty \le \tau_n \tag{1.1.51}$$

P-a.s. for each $n \ge 0$. Similarly, from (1.1.10) and (1.1.35) we find that

$$V_n^\infty \le V_n \tag{1.1.52}$$

for each $n \ge 0$. The following simple example shows that in the absence of the condition (1.1.3) above the inequalities in (1.1.51) and (1.1.52) can be strict.

Example 1.5. Let $G_n = \sum_{k=0}^n \varepsilon_k$ for $n \ge 0$ where $(\varepsilon_k)_{k\ge0}$ is a sequence of independent and identically distributed random variables with $\mathsf{P}(\varepsilon_k = -1) = \mathsf{P}(\varepsilon_k = 1) = 1/2$ for $k \ge 0$. Setting $\mathcal{F}_n = \sigma(\varepsilon_1, \ldots, \varepsilon_n)$ for $n \ge 0$ it follows that $(G_n)_{n\ge0}$ is a martingale with respect to $(\mathcal{F}_n)_{n\ge0}$. From (1.1.28) using the optional sampling theorem (page 60) one sees that $S_n^N = G_n$ and hence $\tau_n^N = n$ as well as $V_n^N = 0$ for all $0 \le n \le N$. On the other hand, if we make use of the stopping times $\sigma_m = \inf\{\, k \ge n : G_k = m \,\}$ upon recalling that $\mathsf{P}(\sigma_m < \infty) = 1$ whenever $m \ge 1$, it follows by (1.1.30) that $S_n \ge m$ P-a.s. for all $m \ge 1$. From this one sees that $S_n = \infty$ P-a.s. and hence $\tau_n = \infty$ P-a.s. as well as $V_n = \infty$ for all $n \ge 0$. Thus, in this case, all inequalities in (1.1.51) and (1.1.52) are strict.

Theorem 1.6. (From finite to infinite horizon) *Consider the optimal stopping problems* (1.1.5) *and* (1.1.29) *upon assuming that the condition* (1.1.3) *holds. Then equalities in* (1.1.51) *and* (1.1.52) *hold for all* $n \ge 0$.

Proof. Letting $N \to \infty$ in (1.1.7) and using the conditional monotone convergence theorem one finds that the following recurrent relations hold:

$$S_n^\infty = \max\big(G_n, \mathsf{E}\,(S_{n+1}^\infty \mid \mathcal{F}_n)\big) \tag{1.1.53}$$

for all $n \geq 0$. In particular, it follows that $(S_n^\infty)_{n\geq 0}$ is a supermartingale. Since $S_n^\infty \geq G_n$ P-a.s. we see that $(S_n^\infty)^- \leq G_n^- \leq \sup_{n\geq 0} G_n^-$ P-a.s. for all $n \geq 0$ from where by means of (1.1.3) we see that $((S_n^\infty)^-)_{n\geq 0}$ is uniformly integrable. Thus by the optional sampling theorem (page 60) we get

$$S_n^\infty \geq \mathsf{E}\,(S_\tau^\infty \mid \mathcal{F}_n) \tag{1.1.54}$$

for all $\tau \in \mathfrak{M}_n$. Moreover, since $S_k^\infty \geq G_k$ P-a.s. for all $k \geq n$, it follows that $S_\tau^\infty \geq G_\tau$ P-a.s. for all $\tau \in \mathfrak{M}_n$, and hence

$$\mathsf{E}\,(S_\tau^\infty \mid \mathcal{F}_n) \geq \mathsf{E}\,(G_\tau \mid \mathcal{F}_n) \tag{1.1.55}$$

for all $\tau \in \mathfrak{M}_n$. Combining (1.1.54) and (1.1.55) we see by (1.1.30) that $S_n^\infty \geq S_n$ P-a.s. for all $n \geq 0$. Since the reverse inequality holds in general as shown in (1.1.51) above, this establishes that $S_n^\infty = S_n$ P-a.s. for all $n \geq 0$. From this it also follows that $\tau_n^\infty = \tau_n$ P-a.s. for all $n \geq 0$. Finally, the third identity $V_n^\infty = V_n$ follows by the monotone convergence theorem. The proof of the theorem is complete. □

1.2. Markovian approach

In this subsection we will present basic results of optimal stopping when the time is discrete and the process is Markovian. (Basic definitions and properties of such processes are given in Subsections 4.1 and 4.2.)

1. Throughout we consider a time-homogeneous Markov chain $X = (X_n)_{n\geq 0}$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, \mathsf{P}_x)$ and taking values in a measurable space $(E, \mathcal{B})$ where for simplicity we assume that $E = \mathbb{R}^d$ for some $d \geq 1$ and $\mathcal{B} = \mathcal{B}(\mathbb{R}^d)$ is the Borel σ-algebra on $\mathbb{R}^d$. It is assumed that the chain X starts at x under P_x for $x \in E$. It is also assumed that the mapping $x \mapsto \mathsf{P}_x(F)$ is measurable for each $F \in \mathcal{F}$. It follows that the mapping $x \mapsto \mathsf{E}_x(Z)$ is measurable for each random variable Z. Finally, without loss of generality we assume that $(\Omega, \mathcal{F})$ equals the canonical space $(E^{\mathbb{N}_0}, \mathcal{B}^{\mathbb{N}_0})$ so that the shift operator $\theta_n : \Omega \to \Omega$ is well defined by $\theta_n(\omega)(k) = \omega(n{+}k)$ for $\omega = (\omega(k))_{k\geq 0} \in \Omega$ and $n, k \geq 0$. (Recall that $\mathbb{N}_0$ stands for $\mathbb{N} \cup \{0\}$.)

2. Given a measurable function $G : E \to \mathbb{R}$ satisfying the following condition (with $G(X_N) = 0$ if $N = \infty$):

$$\mathsf{E}_x\Big(\sup_{0\leq n\leq N} |G(X_n)|\Big) < \infty \tag{1.2.1}$$

for all $x \in E$, we consider the optimal stopping problem

$$V^N(x) = \sup_{0\leq \tau\leq N} \mathsf{E}_x G(X_\tau) \tag{1.2.2}$$

where $x \in E$ and the supremum is taken over all stopping times τ of X. The latter means that τ is a stopping time with respect to the natural filtration of X given by $\mathcal{F}_n^X = \sigma(X_k : 0 \le k \le n)$ for $n \ge 0$. Since the same results remain valid if we take the supremum in (1.2.2) over stopping times τ with respect to $(\mathcal{F}_n)_{n \ge 0}$, and this assumption makes final conclusions more powerful (at least formally), we will assume in the sequel that the supremum in (1.2.2) is taken over this larger class of stopping times. Note also that in (1.2.2) we admit that N can be $+\infty$ as well. In this case, however, we still assume that the supremum is taken over stopping times τ, i.e. over Markov times τ satisfying $\tau < \infty$ P-a.s. In this way any specification of $G(X_\infty)$ becomes irrelevant for the problem (1.2.2).

3. To solve the problem (1.2.2) in the case when $N < \infty$ we may note that by setting

$$G_n = G(X_n) \tag{1.2.3}$$

for $n \ge 0$ the problem (1.2.2) reduces to the problem (1.1.5) where instead of P and E we have P_x and E_x for $x \in E$. Introducing the expectation in (1.2.2) with respect to P_x under which $X_0 = x$ and studying the resulting problem by means of the mapping $x \mapsto V^N(x)$ for $x \in E$ constitutes a profound step which most directly aims to exploit the Markovian structure of the problem. (The same remark applies in the theory of optimal stochastic control in contrast to classical methods developed in calculus of variations.)

Having identified the problem (1.2.2) as the problem (1.1.5) we can apply the method of backward induction (1.1.6)–(1.1.7) which leads to a sequence of random variables $(S_n^N)_{0 \le n \le N}$ and a stopping time τ_n^N defined in (1.1.8). The key identity is

$$S_n^N = V^{N-n}(X_n) \tag{1.2.4}$$

for $0 \le n \le N$. This will be established in the proof of the next theorem. Once (1.2.4) is known to hold, the results of Theorem 1.2 translate immediately into the present setting and get a more transparent form as follows.

In the sequel we set

$$C_n = \{\, x \in E : V^{N-n}(x) > G(x) \,\}, \tag{1.2.5}$$

$$D_n = \{\, x \in E : V^{N-n}(x) = G(x) \,\} \tag{1.2.6}$$

for $0 \le n \le N$. We define

$$\tau_D = \inf \{\, 0 \le n \le N : X_n \in D_n \,\}. \tag{1.2.7}$$

Finally, the transition operator T of X is defined by

$$TF(x) = \mathsf{E}_x F(X_1) \tag{1.2.8}$$

for $x \in E$ whenever $F : E \to \mathbb{R}$ is a measurable function so that $F(X_1)$ is integrable with respect to P_x for all $x \in E$.

Theorem 1.7. (Finite horizon: The time-homogeneous case) *Consider the optimal stopping problem* (1.2.2) *upon assuming that the condition* (1.2.1) *holds. Then the value function* V^n *satisfies the* Wald–Bellman equations

$$V^n(x) = \max(G(x), TV^{n-1}(x)) \qquad (x \in E) \tag{1.2.9}$$

for $n = 1, \ldots, N$ *where* $V^0 = G$ *. Moreover, we have*:

The stopping time τ_D *is optimal in* (1.2.2). (1.2.10)

If τ_* *is an optimal stopping time in* (1.2.2) *then* $\tau_D \le \tau_*$ P_x*-a.s. for every* $x \in E$. (1.2.11)

The sequence $(V^{N-n}(X_n))_{0 \le n \le N}$ *is the smallest supermartingale which dominates* $(G(X_n))_{0 \le n \le N}$ *under* P_x *for* $x \in E$ *given and fixed.* (1.2.12)

The stopped sequence $(V^{N-n\wedge\tau_D}(X_{n\wedge\tau_D}))_{0 \le n \le N}$ *is a martingale under* P_x *for every* $x \in E$. (1.2.13)

Proof. To verify (1.2.4) recall from (1.1.10) that

$$S_n^N = \mathsf{E}_x\big(G(X_{\tau_n^N}) \mid \mathcal{F}_n\big) \tag{1.2.14}$$

for $0 \le n \le N$. Since $S_k^{N-n} \circ \theta_n = S_{n+k}^N$ we get that τ_n^N satisfies

$$\tau_n^N = \inf\{\, n \le k \le N : S_k^N = G(X_k)\,\} = n + \tau_0^{N-n} \circ \theta_n \tag{1.2.15}$$

for $0 \le n \le N$. Inserting (1.2.15) into (1.2.14) and using the Markov property we obtain

$$\begin{aligned} S_n^N &= \mathsf{E}_x\big(G(X_{n+\tau_0^{N-n}\circ\theta_n}) \mid \mathcal{F}_n\big) = \mathsf{E}_x\big(G(X_{\tau_0^{N-n}}) \circ \theta_n \mid \mathcal{F}_n\big) \\ &= \mathsf{E}_{X_n} G(X_{\tau_0^{N-n}}) = V^{N-n}(X_n) \end{aligned} \tag{1.2.16}$$

where the final equality follows by (1.1.9)–(1.1.10) which imply

$$\mathsf{E}_x S_0^{N-n} = \mathsf{E}_x G(X_{\tau_0^{N-n}}) = \sup_{0 \le \tau \le N-n} \mathsf{E}_x G(X_\tau) = V^{N-n}(x) \tag{1.2.17}$$

for $0 \le n \le N$ and $x \in E$. Thus (1.2.4) holds as claimed.

To verify (1.2.9) note that (1.1.7) using (1.2.4) and the Markov property reads as follows:

$$\begin{aligned} V^{N-n}(X_n) &= \max\big(G(X_n), \mathsf{E}_x\big(V^{N-n-1}(X_{n+1}) \mid \mathcal{F}_n\big)\big) \\ &= \max\big(G(X_n), \mathsf{E}_x\big(V^{N-n-1}(X_1) \circ \theta_n \mid \mathcal{F}_n\big)\big) \\ &= \max\big(G(X_n), \mathsf{E}_{X_n}\big(V^{N-n-1}(X_1)\big)\big) \\ &= \max\big(G(X_n), TV^{N-n-1}(X_n)\big) \end{aligned} \tag{1.2.18}$$

for all $0 \le n \le N$. Letting $n = 0$ and using that $X_0 = x$ under P_x we see that (1.2.18) yields (1.2.9).

The remaining statements of the theorem follow directly from Theorem 1.2 above. The proof is complete. □

4. The Wald–Bellman equations (1.2.9) can be written in a more compact form as follows. Introduce the operator Q by setting

$$QF(x) = \max(G(x), TF(x)) \tag{1.2.19}$$

for $x \in E$ where $F : E \to \mathbb{R}$ is a measurable function for which $F(X_1) \in L^1(\mathsf{P}_x)$ for $x \in E$. Then (1.2.9) reads as follows:

$$V^n(x) = Q^n G(x) \tag{1.2.20}$$

for $1 \le n \le N$ where Q^n denotes the n-th power of Q. The recursive relations (1.2.20) form a constructive method for finding V^N when $\text{Law}(X_1 \mid \mathsf{P}_x)$ is known for $x \in E$.

5. Let us now discuss the case when X is a time-inhomogeneous Markov chain. Setting $Z_n = (n, X_n)$ for $n \ge 0$ one knows that $Z = (Z_n)_{n\ge 0}$ is a time-homogeneous Markov chain. Given a measurable function $G : \{0, 1, \ldots, N\} \times E \to \mathbb{R}$ satisfying the following condition:

$$\mathsf{E}_{n,x}\Big(\sup_{0\le k\le N-n} |G(n{+}k, X_{n+k})|\Big) < \infty \tag{1.2.1$'$}$$

for all $0 \le n \le N$ and $x \in E$, the optimal stopping problem (1.2.2) therefore naturally extends as follows:

$$V^N(n, x) = \sup_{0\le\tau\le N-n} \mathsf{E}_{n,x} G(n{+}\tau, X_{n+\tau}) \tag{1.2.2$'$}$$

where the supremum is taken over stopping times τ of X and $X_n = x$ under $\mathsf{P}_{n,x}$ with $0 \le n \le N$ and $x \in E$ given and fixed.

As above one verifies that

$$S^N_{n+k} = V^N(n{+}k, X_{n+k}) \tag{1.2.21}$$

under $\mathsf{P}_{n,x}$ for $0 \le n \le N - n$. Moreover, inserting this into (1.1.7) and using the Markov property one finds

$$\begin{aligned} &V^N(n{+}k, X_{n+k}) \qquad\qquad (1.2.22)\\ &\quad = \max\Big(G(n{+}k, X_{n+k}), \mathsf{E}_{n,x}\big(V^N(n{+}k{+}1, X_{n+k+1}) \mid \mathcal{F}_{n+k}\big)\Big)\\ &\quad = \max\Big(G(Z_{n+k}), \mathsf{E}_z\big(V^N(Z_{n+k+1}) \mid \mathcal{F}_{n+k}\big)\Big)\\ &\quad = \max\Big(G(Z_{n+k}), \mathsf{E}_z\big(V^N(Z_1) \circ \theta_{n+k} \mid \mathcal{F}_{n+k}\big)\Big)\\ &\quad = \max\Big(G(Z_{n+k}), \mathsf{E}_{Z_{n+k}}\big(V^N(Z_1)\big)\Big) \end{aligned}$$

for $0 \le k \le N-n-1$ where $z = (n,x)$ with $0 \le n \le N$ and $x \in E$. Letting $k = 0$ and using that $Z_n = z = (n,x)$ under P_z, one gets

$$V^N(n,x) = \max\big(G(n,x), TV^N(n,x)\big) \tag{1.2.23}$$

for $n = N-1, \dots, 1, 0$ where $TV^N(N-1,x) = \mathsf{E}_{N-1,x} G(N, X_N)$ and T is the transition operator of Z given by

$$TF(n,x) = \mathsf{E}_{n,x} F(n+1, X_{n+1}) \tag{1.2.24}$$

for $0 \le n \le N$ and $x \in E$ whenever the right-hand side in (1.2.24) is well defined (finite).

The recursive relations (1.2.23) are the *Wald–Bellman equations* corresponding to the time-inhomogeneous problem $(1.2.2')$. Note that when X is time-homogeneous (and $G = G(x)$ only) we have $V^N(n,x) = V^{N-n}(x)$ and (1.2.23) reduces to (1.2.9). In order to present a reformulation of the property (1.2.12) in Theorem 1.7 above we will proceed as follows.

6. The following definition plays a fundamental role in finding a solution to the optimal stopping problem $(1.2.2')$.

Definition 1.8. A measurable function $F : \{0,1,\dots,N\} \times E \to \mathbb{R}$ is said to be *superharmonic* if

$$TF(n,x) \le F(n,x) \tag{1.2.25}$$

for all $(n,x) \in \{0,1,\dots,N\} \times E$.

It is assumed in (1.2.25) that $TF(n,x)$ is well defined i.e. that $F(n+1, X_{n+1}) \in L^1(\mathsf{P}_{n,x})$ for all (n,x) as above. Moreover, if $F(n{+}k, X_{n+k}) \in L^1(\mathsf{P}_{n,x})$ for all $0 \le k \le N-n$ and all (n,x) as above, then one verifies directly by the Markov property that the following stochastic characterization of superharmonic functions holds:

F is superharmonic if and only if $(F(n{+}k, X_{n+k}))_{0 \le k \le N-n}$ is a supermartingale under $\mathsf{P}_{n,x}$ for all $(n,x) \in \{0,1,\dots,N-1\} \times E$. (1.2.26)

The proof of this fact is simple and will be given in a more general case following (1.2.40) below.

Introduce the continuation set

$$C = \big\{(n,x) \in \{0,1,\dots,N\} \times E : V(n,x) > G(n,x)\big\} \tag{1.2.27}$$

and the stopping set

$$D = \big\{(n,x) \in \{0,1,\dots,N\} \times E : V(n,x) = G(n,x)\big\}. \tag{1.2.28}$$

Introduce the first entry time τ_D into D by setting

$$\tau_D = \inf\{\, n \le k \le N : (n{+}k, X_{n+k}) \in D \,\} \tag{1.2.29}$$

under $\mathsf{P}_{n,x}$ where $(n,x) \in \{0,1,\dots,N\} \times E$.

The preceding considerations may now be summarized in the following extension of Theorem 1.7.

Theorem 1.9. (Finite horizon: The time-inhomogeneous case) *Consider the optimal stopping problem* (1.2.2′) *upon assuming that the condition* (1.2.1′) *holds. Then the value function* V^N *satisfies the* Wald–Bellman equations

$$V^N(n,x) = \max\big(G(n,x), TV^N(n,x)\big) \tag{1.2.30}$$

for $n = N-1, \ldots, 1, 0$ *where* $TV^N(N-1,x) = \mathsf{E}_{N-1,x}G(N, X_N)$ *and* $x \in E$. *Moreover, we have*:

The stopping time τ_D *is optimal in* (1.2.2′). (1.2.31)

If τ_* *is an optimal stopping time in* (1.2.2′) *then* $\tau_D \le \tau_*$ $\mathsf{P}_{n,x}$*-a.s. for every* $(n,x) \in \{0,1,\ldots,N\} \times E$. (1.2.32)

The value function V^N *is the smallest superharmonic function which dominates the gain function* G *on* $\{0,1,\ldots,N\} \times E$. (1.2.33)

The stopped sequence $\big(V^N((n+k)\wedge\tau_D, X_{(n+k)\wedge\tau_D})\big)_{0\le k\le N-n}$ *is a martingale under* $\mathsf{P}_{n,x}$ *for every* $(n,x) \in \{0,1,\ldots,N\} \times E$. (1.2.34)

Proof. The proof is carried out in exactly the same way as the proof of Theorem 1.7 above. The key identity linking the problem (1.2.2′) to the problem (1.1.5) is (1.2.21). This yields (1.2.23) i.e. (1.2.30) as shown above. Note that (1.2.33) refines (1.2.12) and follows by (1.2.26). The proof is complete. □

7. Consider the optimal stopping problem (1.2.2) when $N = \infty$. Recall that (1.2.2) reads as follows:

$$V(x) = \sup_\tau \mathsf{E}_x G(X_\tau) \tag{1.2.35}$$

where τ is a stopping time of X and $\mathsf{P}_x(X_0 = x) = 1$ for $x \in E$.

Introduce the continuation set

$$C = \{\, x \in E : V(x) > G(x)\,\} \tag{1.2.36}$$

and the stopping set

$$D = \{\, x \in E : V(x) = G(x)\,\}. \tag{1.2.37}$$

Introduce the first entry time τ_D into D by setting

$$\tau_D = \inf\{\, t \ge 0 : X_t \in D\,\}. \tag{1.2.38}$$

8. The following definition plays a fundamental role in finding a solution to the optimal stopping problem (1.2.35). Note that Definition 1.8 above may be viewed as a particular case of this definition.

Definition 1.10. A measurable function $F : E \to \mathbb{R}$ is said to be *superharmonic* if

$$TF(x) \le F(x) \tag{1.2.39}$$

for all $x \in E$.

It is assumed in (1.2.39) that $TF(x)$ is well defined by (1.2.8) above i.e. that $F(X_1) \in L^1(\mathsf{P}_x)$ for all $x \in E$. Moreover, if $F(X_n) \in L^1(\mathsf{P}_x)$ for all $n \geq 0$ and all $x \in E$, then the following stochastic characterization of superharmonic functions holds (recall (1.2.26) above):

F is superharmonic if and only if $(F(X_n))_{n \geq 0}$ is a supermartingale under P_x for every $x \in E$. (1.2.40)

The proof of this equivalence relation is simple. Suppose first that F is superharmonic. Then (1.2.39) holds for all $x \in E$ and therefore by the Markov property we get

$$\begin{aligned} TF(X_n) &= \mathsf{E}_{X_n}(F(X_1)) = \mathsf{E}_x(F(X_1) \circ \theta_n \mid \mathcal{F}_n) \\ &= \mathsf{E}_x(F(X_{n+1}) \mid \mathcal{F}_n) \leq F(X_n) \end{aligned} \tag{1.2.41}$$

for all $n \geq 0$ proving the supermartingale property of $(F(X_n))_{n \geq 0}$ under P_x for every $x \in E$. Conversely, if $(F(X_n))_{n \geq 0}$ is a supermartingale under P_x for every $x \in E$, then the final inequality in (1.2.41) holds for all $n \geq 0$. Letting $n = 0$ and taking E_x on both sides gives (1.2.39). Thus F is superharmonic as claimed.

9. In the case of infinite horizon (i.e. when $N = \infty$ in (1.2.2) above) it is not necessary to treat the time-inhomogeneous case separately from the time-homogeneous case as we did it for clarity in the case of finite horizon (i.e. when $N < \infty$ in (1.2.2) above). This is due to the fact that the state space E may be general anyway (two-dimensional) and the passage from the time-inhomogeneous process $(X_n)_{n \geq 0}$ to the time-homogeneous process $(n, X_n)_{n \geq 0}$ does not affect the time set in which the stopping times of X take values (by altering the remaining time).

Theorem 1.11. (Infinite horizon) *Consider the optimal stopping problem* (1.2.35) *upon assuming that the condition* (1.2.1) *holds. Then the value function* V *satisfies the* Wald–Bellman equation

$$V(x) = \max(G(x), TV(x)) \tag{1.2.42}$$

for $x \in E$. *Assume moreover when required below that*

$$\mathsf{P}_x(\tau_D < \infty) = 1 \tag{1.2.43}$$

for all $x \in E$. *Then we have*:

The stopping time τ_D *is optimal in* (1.2.35). (1.2.44)

If τ_* *is an optimal stopping time in* (1.2.35) *then* $\tau_D \leq \tau_*$ P_x*-a.s. for every* $x \in E$. (1.2.45)

The value function V *is the smallest superharmonic function which dominates the gain function* G *on* E. (1.2.46)

The stopped sequence $(V(X_{n \wedge \tau_D}))_{n \geq 0}$ *is a martingale under* P_x *for every* $x \in E$. (1.2.47)

Finally, if the condition (1.2.43) *fails so that* $\mathsf{P}_x(\tau_D = \infty) > 0$ *for some* $x \in E$ *, then there is no optimal stopping time* (*with probability* 1) *in* (1.2.35).

Proof. The key identity in reducing the problem (1.2.35) to the problem (1.1.29) is

$$S_n = V(X_n) \tag{1.2.48}$$

for $n \geq 0$. This can be proved by passing to the limit for $N \to \infty$ in (1.2.4) and using the result of Theorem 1.6 above. In exactly the same way one derives (1.2.42) from (1.2.9). The remaining statements follow from Theorem 1.4 above. Note also that (1.2.46) refines (1.1.38) and follows by (1.2.40). The proof is complete. □

Corollary 1.12. (Iterative method) *Under the initial hypothesis of Theorem* 1.11 *we have*

$$V(x) = \lim_{n\to\infty} Q^n G(x) \tag{1.2.49}$$

for all $x \in E$ *.*

Proof. It follows from (1.2.9) and Theorem 1.6 above. □

The relation (1.2.49) offers a constructive method for finding the value function V . (Note that $n \mapsto Q^n G(x)$ is increasing on $\{0, 1, 2, \dots\}$ for every $x \in E$.)

10. We have seen in Theorem 1.7 and Theorem 1.9 that the Wald–Bellman equations (1.2.9) and (1.2.30) characterize the value function V^N when the horizon N is finite (i.e. these equations cannot have other solutions). This is due to the fact that V^N equals G in the "end of time" N . When the horizon N is infinite, however, this characterization is no longer true for the Wald–Bellman equation (1.2.42). For example, if G is identically equal to a constant c then any other constant C larger than c will define a function solving (1.2.42). On the other hand, it is evident from (1.2.42) that every solution of this equation is superharmonic and dominates G . By (1.2.46) we thus see that a minimal solution of (1.2.42) will coincide with the value function. This "minimality condition" (over all points) can be replaced by a single condition as the following theorem shows. From the standpoint of finite horizon such a "boundary condition at infinity" is natural.

Theorem 1.13. (Uniqueness in the Wald–Bellman equation)
Under the hypothesis of Theorem 1.11 *suppose that* $F : E \to \mathbb{R}$ *is a function solving the* Wald–Bellman equation

$$F(x) = \max(G(x), TF(x)) \tag{1.2.50}$$

for $x \in E$ *.* (*It is assumed that* F *is measurable and* $F(X_1) \in L^1(\mathsf{P}_x)$ *for all* $x \in E$ *.*) *Suppose moreover that* F *satisfies*

$$\mathsf{E}\Big(\sup_{n\geq 0} F(X_n)\Big) < \infty. \tag{1.2.51}$$

Then F equals the value function V if and only if the following "boundary condition at infinity" holds:

$$\limsup_{n\to\infty} F(X_n) = \limsup_{n\to\infty} G(X_n) \qquad \mathsf{P}_x\text{-}a.s. \tag{1.2.52}$$

for every $x \in E$. (In this case the $\limsup$ *on the left-hand side of* (1.2.52) *equals the* $\liminf$, *i.e. the sequence $(F(X_n))_{n\ge 0}$ is convergent P_x-a.s. for every $x \in E$.)*

Proof. If $F = V$ then by (1.2.46) we know that F is the smallest superharmonic function which dominates G on E. Let us show (the fact of independent interest) that any such function F must satisfy (1.2.52). Note that the condition (1.2.51) is not needed for this implication.

Since $F \ge G$ we see that the left-hand side in (1.2.52) is evidently larger than the right-hand side. To prove the reverse inequality, consider the function $H : E \to \mathbb{R}$ defined by

$$H(x) = \mathsf{E}_x\Big(\sup_{n\ge 0} G(X_n)\Big) \tag{1.2.53}$$

for $x \in E$. Then the key property of H stating that

$$H \text{ is superharmonic} \tag{1.2.54}$$

can be verified as follows. By the Markov property we have

$$\begin{aligned} TH(x) &= \mathsf{E}_x H(X_1) = \mathsf{E}_x\Big(\mathsf{E}_{X_1}\Big(\sup_{n\ge 0} G(X_n)\Big)\Big) \\ &= \mathsf{E}_x\Big(\mathsf{E}_x\Big(\sup_{n\ge 0} G(X_n)\circ\theta_1 \,\Big|\, \mathcal{F}_1\Big)\Big) = \mathsf{E}_x\Big(\sup_{n\ge 0} G(X_{n+1})\Big) \\ &\le H(x) \end{aligned} \tag{1.2.55}$$

for all $x \in E$ proving (1.2.54). Moreover, since $X_0 = x$ under P_x we see that $H(x) \ge G(x)$ for all $x \in E$. Hence $F(x) \le H(x)$ for all $x \in E$ by assumption. By the Markov property it thus follows that

$$\begin{aligned} F(X_n) \le H(X_n) &= \mathsf{E}_{X_n}\Big(\sup_{k\ge 0} G(X_k)\Big) = \mathsf{E}_x\Big(\sup_{k\ge 0} G(X_k)\circ\theta_n \,\Big|\, \mathcal{F}_n\Big) \\ &= \mathsf{E}_x\Big(\sup_{k\ge 0} G(X_{k+n}) \,\Big|\, \mathcal{F}_n\Big) \le \mathsf{E}_x\Big(\sup_{l\ge m} G(X_l) \,\Big|\, \mathcal{F}_n\Big) \end{aligned} \tag{1.2.56}$$

for any $m \le n$ given and fixed where $x \in E$. The final expression in (1.2.56) defines a (generalized) martingale for $n \ge 1$ under P_x which is known to converge P_x-a.s. as $n \to \infty$ for every $x \in E$ with the limit satisfying the following inequality:

$$\lim_{n\to\infty} \mathsf{E}_x\Big(\sup_{l\ge m} G(X_l) \,\Big|\, \mathcal{F}_n\Big) \le \mathsf{E}_x\Big(\sup_{l\ge m} G(X_l) \,\Big|\, \mathcal{F}_\infty\Big) = \sup_{l\ge m} G(X_l) \tag{1.2.57}$$

where the final identity follows from the fact that $\sup_{l \ge m} G(X_l)$ is $\mathcal{F}_\infty$-measurable. Letting $n \to \infty$ in (1.2.56) and using (1.2.57) we find

$$\limsup_{n \to \infty} F(X_n) \le \sup_{l \ge m} G(X_l) \qquad \mathsf{P}_x\text{-a.s.} \tag{1.2.58}$$

for all $m \ge 0$ and $x \in E$. Letting finally $m \to \infty$ in (1.2.58) we end up with (1.2.52). This ends the first part of the proof.

Conversely, suppose that F satisfies (1.2.50)–(1.2.52) and let us show that F must then be equal to V. For this, first note that (1.2.50) implies that F is superharmonic and that $F \ge G$. Hence by (1.2.46) we see that $V \le F$. To show that $V \ge F$ consider the stopping time

$$\tau_{D_\varepsilon} = \inf\{\, n \ge 0 : F(X_n) \le G(X_n) + \varepsilon \,\} \tag{1.2.59}$$

where $\varepsilon > 0$ is given and fixed. Then by (1.2.52) we see that $\tau_{D_\varepsilon} < \infty$ P_x-a.s. for $x \in E$. Moreover, we claim that $\big(F(X_{\tau_{D_\varepsilon} \wedge n})\big)_{n \ge 0}$ is a martingale under P_x for $x \in E$. For this, note that the Markov property and (1.2.50) imply

$$\begin{aligned}
&\mathsf{E}_x\big(F(X_{\tau_{D_\varepsilon} \wedge n}) \,|\, \mathcal{F}_{n-1}\big) \qquad (1.2.60)\\
&= \mathsf{E}_x\big(F(X_n) I(\tau_{D_\varepsilon} \ge n) \,|\, \mathcal{F}_{n-1}\big) + \mathsf{E}_x\big(F(X_{\tau_{D_\varepsilon}}) I(\tau_{D_\varepsilon} < n) \,|\, \mathcal{F}_{n-1}\big)\\
&= \mathsf{E}_x\big(F(X_n) \,|\, \mathcal{F}_{n-1}\big) I(\tau_{D_\varepsilon} \ge n) + \mathsf{E}_x\big(\textstyle\sum_{k=0}^{n-1} F(X_k) I(\tau_{D_\varepsilon} = k) \,|\, \mathcal{F}_{n-1}\big)\\
&= \mathsf{E}_{X_{n-1}}\big(F(X_1)\big)\, I(\tau_{D_\varepsilon} \ge n) + \textstyle\sum_{k=0}^{n-1} F(X_k)\, I(\tau_{D_\varepsilon} = k)\\
&= TF(X_{n-1})\, I(\tau_{D_\varepsilon} \ge n) + F(X_{\tau_{D_\varepsilon}})\, I(\tau_{D_\varepsilon} < n)\\
&= F(X_{n-1})\, I(\tau_{D_\varepsilon} \ge n) + F(X_{\tau_{D_\varepsilon}})\, I(\tau_{D_\varepsilon} < n)\\
&= F(X_{\tau_{D_\varepsilon} \wedge (n-1)})\, I(\tau_{D_\varepsilon} \ge n) + F(X_{\tau_{D_\varepsilon} \wedge (n-1)})\, I(\tau_{D_\varepsilon} < n)\\
&= F(X_{\tau_{D_\varepsilon} \wedge (n-1)})
\end{aligned}$$

for all $n \ge 1$ and $x \in E$ proving the claim. Hence

$$\mathsf{E}_x\big(F(X_{\tau_{D_\varepsilon} \wedge n})\big) = F(x) \tag{1.2.61}$$

for all $n \ge 0$ and $x \in E$. Next note that

$$\mathsf{E}_x\big(F(X_{\tau_{D_\varepsilon} \wedge n})\big) = \mathsf{E}_x\big(F(X_{\tau_{D_\varepsilon}})\, I(\tau_{D_\varepsilon} \le n)\big) + \mathsf{E}_x\big(F(X_n)\, I(\tau_{D_\varepsilon} > n)\big) \tag{1.2.62}$$

for all $n \ge 0$. Letting $n \to \infty$, using (1.2.51) and (1.2.1) with $F \ge G$, we get

$$\mathsf{E}_x\big(F(X_{\tau_{D_\varepsilon}})\big) = F(x) \tag{1.2.63}$$

for all $x \in E$. This fact is of independent interest.

Finally, since V is superharmonic, we find using (1.2.63) that

$$V(x) \geq \mathsf{E}_x V(X_{\tau_{D_\varepsilon}}) \geq \mathsf{E}_x G(X_{\tau_{D_\varepsilon}}) \geq \mathsf{E}_x F(X_{\tau_{D_\varepsilon}}) - \varepsilon = F(x) - \varepsilon \tag{1.2.64}$$

for all $\varepsilon > 0$ and $x \in E$. Letting $\varepsilon \downarrow 0$ we get $V \geq F$ as needed and the proof is complete. □

11. Given $\alpha \in (0,1]$ and (bounded) measurable functions $g : E \to \mathbb{R}$ and $c : E \to \mathbb{R}_+$, consider the optimal stopping problem

$$V(x) = \sup_\tau \mathsf{E}_x \Big(\alpha^\tau g(X_\tau) - \sum_{k=1}^{\tau} \alpha^{k-1} c(X_{k-1}) \Big) \tag{1.2.65}$$

where τ is a stopping time of X and $\mathsf{P}_x(X_0 = x) = 1$.

The value $c(x)$ is interpreted as the cost of making the next observation of X when X equals x. The sum in (1.2.65) by definition equals 0 when τ equals 0.

The problem formulation (1.2.65) goes back to a problem formulation due to Bolza in classic calculus of variation (a more detailed discussion will be given in Chapter III below). Let us briefly indicate how the problem (1.2.65) can be reduced to the setting of Theorem 1.11 above.

For this, let $\widetilde{X} = (\widetilde{X}_n)_{n \geq 0}$ denote the Markov chain X killed at rate α. It means that the transition operator of $\widetilde{X}$ is given by

$$\widetilde{T} F(x) = \alpha\, T F(x) \tag{1.2.66}$$

for $x \in E$ whenever $F(X_1) \in L^1(\mathsf{P}_x)$. The problem (1.2.65) then reads

$$V(x) = \sup_\tau \mathsf{E}_x \Big(g(\widetilde{X}_\tau) - \sum_{k=1}^{\tau} c(\widetilde{X}_{k-1}) \Big) \tag{1.2.65$'$}$$

where τ is a stopping time of $\widetilde{X}$ and $\mathsf{P}_x(\widetilde{X}_0 = x) = 1$.

Introduce the sequence

$$\widetilde{I}_n = a + \sum_{k=1}^{n} c(\widetilde{X}_{k-1}) \tag{1.2.67}$$

for $n \geq 1$ with $\widetilde{I}_0 = a$ in $\mathbb{R}$. Then $\widetilde{Z}_n = (\widetilde{X}_n, \widetilde{I}_n)$ defines a Markov chain for $n \geq 0$ with $\widetilde{Z}_0 = (\widetilde{X}_0, \widetilde{I}_0) = (x, a)$ under P_x so that we may write $\mathsf{P}_{x,a}$ instead of P_x. (The latter can be justified rigorously by passage to the canonical probability space.) The transition operator of $\widetilde{Z} = (\widetilde{X}, \widetilde{I}\,)$ equals

$$T_{\widetilde{Z}}\, F(x,a) = \mathsf{E}_{x,a} F(\widetilde{X}_1, \widetilde{I}_1) \tag{1.2.68}$$

for $(x,a) \in E \times \mathbb{R}$ whenever $F(\widetilde{X}_1, \widetilde{I}_1) \in L^1(\mathsf{P}_{x,a})$.

The problem $(1.2.65')$ may now be rewritten as follows:

$$W(x,a) = \sup_{\tau} \mathsf{E}_{x,a} G(Z_\tau) \tag{1.2.65''}$$

where we set

$$G(z) = g(x) - a \tag{1.2.69}$$

for $z = (x,a) \in E \times \mathbb{R}$. Obviously by subtracting a on both sides of $(1.2.65')$ we set that

$$W(x,a) = V(x) - a \tag{1.2.70}$$

for all $(x,a) \in E \times \mathbb{R}$.

The problem $(1.2.65'')$ is of the same type as the problem (1.2.35) above and thus Theorem 1.11 is applicable. To write down (1.2.42) more explicitly note that

$$\begin{aligned} T_{\widetilde{Z}}\, W(x,a) &= \mathsf{E}_{x,a} W(\widetilde{X}_1, \widetilde{I}_1) = \mathsf{E}_{x,a}\big(V(\widetilde{X}_1) - \widetilde{I}_1\big) \\ &= \mathsf{E}_x V(\widetilde{X}_1) - a - c(x) = \alpha T V(x) - a - c(x) \end{aligned} \tag{1.2.71}$$

so that (1.2.42) reads

$$V(x) - a = \max\big(g(x) - a\,,\ \alpha T V(x) - a - c(x)\big) \tag{1.2.72}$$

where we used (1.2.70), (1.2.69) and (1.2.71). Clearly a can be removed from (1.2.72) showing finally that the Wald–Bellman equation (1.2.42) takes the following form:

$$V(x) = \max\big(g(x)\,,\ \alpha T V(x) - c(x)\big) \tag{1.2.73}$$

for $x \in E$. Note also that (1.2.39) takes the following form:

$$\alpha T F(x) - c(x) \le F(x) \tag{1.2.74}$$

for $x \in E$. Thus F satisfies (1.2.74) if and only if $(x,a) \mapsto F(x) - a$ is superharmonic relative to the Markov chain $\widetilde{Z} = (\widetilde{X}, \widetilde{I})$. Having (1.2.73) and (1.2.74) set out explicitly the remaining statements of Theorem 1.11 and Corollary 1.12 are directly applicable and we shall omit further details. It may be noted above that $L = T - I$ is the generator of the Markov chain X. More general problems of this type (involving also the maximum functional) will be discussed in Chapter III below. We will conclude this section by giving an illustrative example.

12. The following example illustrates general results of optimal stopping theory for Markov chains when applied to a nontrivial problem in order to determine the value function and an optimal Markov time (in the class $\bar{\mathfrak{M}}$).

Example 1.14. Let $\xi, \xi_1, \xi_2, \dots$ be independent and identically distributed random variables, defined on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$, with $\mathsf{E}\xi < 0$. Put $S_0 = 0$,

$S_n = \xi_1 + \cdots + \xi_n$ for $n \ge 1$; $X_0 = x$, $X_n = x + S_n$ for $n \ge 1$, and $M = \sup_{n \ge 0} S_n$. Let P_x be the probability distribution of the sequence $(X_n)_{n\ge 0}$ with $X_0 = x$ from $\mathbb{R}$. It is clear that the sequence $(X_n)_{n\ge 0}$ is a Markov chain started at x.

For any $n \ge 1$ define the gain function $G_n(x) = (x^+)^n$ where $x^+ = \max(x, 0)$ for $x \in \mathbb{R}$, and let

$$V_n(x) = \sup_{\tau \in \mathfrak{M}} \mathsf{E}_x G_n(X_\tau) \tag{1.2.75}$$

where the supremum is taken over the class $\mathfrak{M}$ of all Markov (stopping) times τ satisfying $\mathsf{P}_x(\tau < \infty) = 1$ for all $x \in \mathbb{R}$. Let us also denote

$$\bar{V}_n(x) = \sup_{\tau \in \bar{\mathfrak{M}}} \mathsf{E}_x G_n(X_\tau) I(\tau < \infty) \tag{1.2.76}$$

where the supremum is taken over the class $\bar{\mathfrak{M}}$ of all Markov times.

The problem of finding the value functions $V_n(x)$ and $\bar{V}_n(x)$ is of interest for the theory of American options because these functions represent arbitrage-free (fair, rational) prices of "Power options" under the assumption that any exercise time τ belongs to the class $\mathfrak{M}$ or $\bar{\mathfrak{M}}$ respectively. In the present case we have $V_n(x) = \bar{V}_n(x)$ for $n \ge 1$ and $x \in \mathbb{R}$, and it will be clear from what follows below that an optimal Markov time exists in the class $\bar{\mathfrak{M}}$ (but does not belong to the class $\mathfrak{M}$ of stopping times).

We follow [144] where the authors solved the formulated problems (see also [119]). First of all let us introduce the notion of the Appell polynomial which will be used in the formulation of the basic results.

Let $\eta = \eta(\omega)$ be a random variable with $\mathsf{E} e^{\lambda|\eta|} < \infty$ for some $\lambda > 0$. Consider the Esscher transform

$$x \rightsquigarrow \frac{e^{ux}}{\mathsf{E} e^{u\eta}} \qquad |u| \le \lambda, \quad x \in \mathbb{R}, \tag{1.2.77}$$

and the decomposition

$$\frac{e^{ux}}{\mathsf{E} e^{u\eta}} = \sum_{k=0}^{\infty} \frac{u^k}{k!} Q_k^{(\eta)}(x). \tag{1.2.78}$$

Polynomials $Q_k^{(\eta)}(x)$ are called the *Appell polynomials* for the random variable η. (If $\mathsf{E}|\eta|^n < \infty$ for some $n \ge 1$ then the polynomials $Q_k^{(\eta)}(x)$ are uniquely defined for all $k \le n$.)

The polynomials $Q_k^{(\eta)}(x)$ can be expressed through the semi-invariants $\varkappa_1$, $\varkappa_2, \ldots$ of the random variable η. For example,

$$\begin{aligned} &Q_0^{(\eta)}(x) = 1, && Q_2^{(\eta)}(x) = (x-\varkappa_1)^2 - \varkappa_2, \\ &Q_1^{(\eta)}(x) = x - \varkappa_1, && Q_3^{(\eta)}(x) = (x-\varkappa_1)^3 - 3\varkappa_2(x-\varkappa_1) - \varkappa_3, \end{aligned} \quad \ldots \tag{1.2.79}$$

where (as is well known) the semi-invariants $\varkappa_1, \varkappa_2, \ldots$ are expressed via the moments $\mu_1, \mu_2, \ldots$ of η:

$$\varkappa_1 = \mu_1, \quad \varkappa_2 = \mu_2 - \mu_1^2, \quad \varkappa_3 = 2\mu_1^3 - 3\mu_1\mu_2 + \mu_3, \quad \ldots. \tag{1.2.80}$$

Let us also mention the following property of the Appell polynomials: if $\mathsf{E}\,|\eta|^n < \infty$ then for $k \le n$ we have

$$\frac{d}{dx} Q_k^{(\eta)}(x) = k Q_{k-1}^{(\eta)}(x), \tag{1.2.81}$$

$$\mathsf{E}\, Q_k^{(\eta)}(x+\eta) = x^k. \tag{1.2.82}$$

For simplicity of notation we will use $\mathbb{Q}_k(s)$ to denote the polynomials $Q_k^{(M)}(x)$ for the random variable $M = \sup_{n \ge 0} S_n$. Every polynomial $\mathbb{Q}_k(x)$ has a unique positive root a_k^*. Moreover, $\mathbb{Q}_k(x) \le 0$ for $0 \le x < a_k^*$ and $\mathbb{Q}_k(x)$ increases for $x \ge a_k^*$.

In accordance with the characteristic property (1.2.46) recall that *the value function* $V_n(x)$ *is the smallest superharmonic* (*excessive*) *function which dominates the gain function* $G_n(x)$ *on* $\mathbb{R}$. Thus, one method to find $V_n(x)$ is to search for the smallest excessive majorant of the function $G_n(x)$. In [144] this method is realized as follows.

For every $a \ge 0$ introduce the Markov time

$$\tau_a = \inf\{n \ge 0 : X_n \ge a\} \tag{1.2.83}$$

and for each $n \ge 1$ consider the new optimal stopping problem:

$$\widehat{V}(x) = \sup_{a \ge 0} \mathsf{E}_x G_n(X_{\tau_a}) I(\tau_a < \infty). \tag{1.2.84}$$

It is clear that $G_n(X_{\tau_a}) = (X_{\tau_a}^+)^n = X_{\tau_a}^n$ (on the set $\{\tau_a < \infty\}$). Hence

$$\widehat{V}(x) = \sup_{a \ge 0} \mathsf{E}_x X_{\tau_a}^n I(\tau_a < \infty). \tag{1.2.85}$$

The identity (1.2.82) prompts that the following property should be valid: if $\mathsf{E}\,|M|^n < \infty$ then

$$\mathsf{E}\,\mathbb{Q}_n(x+M) I(x+M \ge a) = \mathsf{E}_x X_{\tau_a}^n I(\tau_a < \infty). \tag{1.2.86}$$

This formula and properties of the Appell polynomials imply that

$$\widehat{V}(x) = \sup_{a \ge 0} \mathsf{E}\,\mathbb{Q}_n(x+M) I(x+M \ge a) = \mathsf{E}\,\mathbb{Q}_n(x+M) I(x+M \ge a_n^*). \tag{1.2.87}$$

From this we see that $\tau_{a_n^*}$ is an optimal Markov time for the problem (1.2.84).

It is clear that $\bar{V}_n(x) \geq \widehat{V}_n(x)$. From (1.2.87) and properties of the Appell polynomials we obtain that $\widehat{V}_n(x)$ is an excessive majorant of the gain function ($\widehat{V}_n(x) \geq \mathsf{E}_x \widehat{V}_n(X_1)$ and $\widehat{V}_n(x) \geq G_n(x)$ for $x \in \mathbb{R}$). But $\bar{V}_n(x)$ is the *smallest* excessive majorant of $G_n(x)$. Thus $\bar{V}_n(x) \leq \widehat{V}_n(x)$.

On the whole we obtain the following result (for further details see [144]):

Suppose that $\mathsf{E}\,(\xi^+)^{n+1} < \infty$ and a_n^* is the largest root of the equation $\mathbb{Q}_n(x) = 0$ for $n \geq 1$ fixed. Denote $\tau_n^* = \inf\{\, k \geq 0 : X_k \geq a_n^* \,\}$. Then the Markov time τ_n^* is optimal:

$$V_n(x) = \sup_{\tau \in \bar{\mathfrak{M}}} \mathsf{E}_x (X_\tau^+)^n I(\tau < \infty) = \mathsf{E}_x \big(X_{\tau_n^*}^+\big)^n I(\tau < \infty). \tag{1.2.88}$$

Moreover,

$$V_n(x) = \mathsf{E}\,\mathbb{Q}_n(x{+}M) I(x{+}M \geq a_n^*). \tag{1.2.89}$$

Remark 1.15. In the cases $n = 1$ and $n = 2$ we have

$$a_1^* = \mathsf{E}\,M \quad \text{and} \quad a_2^* = \mathsf{E}\,M + \sqrt{\mathsf{D}M}. \tag{1.2.90}$$

Remark 1.16. If $\mathsf{P}(\xi = 1) = p$, $\mathsf{P}(\xi = -1) = q$ and $p < q$, then $M := \sup_{n \geq 0} S_n$ (with $S_0 = 0$ and $S_n = \xi_1 + \cdots + \xi_n$) has geometric distribution:

$$\mathsf{P}(M \geq k) = \Big(\frac{p}{q}\Big)^k \tag{1.2.91}$$

for $k \geq 0$. Hence

$$\mathsf{E}\,M = \frac{q}{q-p}. \tag{1.2.92}$$

2. Continuous time

The aim of the present section is to exhibit basic results of optimal stopping in the case of continuous time. We first consider a martingale approach (cf. Subsection 1.1 above). This is then followed by a Markovian approach (cf. Subsection 1.2 above).

2.1. Martingale approach

1. Let $G = (G_t)_{t \geq 0}$ be a stochastic process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathsf{P})$. We interpret G_t as the *gain* if the observation of G is stopped at time t. It is assumed that G is adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$ in the sense that each G_t is $\mathcal{F}_t$-measurable. Recall that each $\mathcal{F}_t$ is a σ-algebra of subsets of Ω such that $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for $s \leq t$. Typically $(\mathcal{F}_t)_{t \geq 0}$ coincides with the natural filtration $(\mathcal{F}_t^G)_{t \geq 0}$ but generally may also be larger. We interpret $\mathcal{F}_t$ as the *information* available up to time t. All our decisions in regard to optimal stopping at time t must be based on this information only (no anticipation is allowed).

The following definition formalizes the previous requirement and plays a key role in the study of optimal stopping (cf. Definition 1.1).

Definition 2.1. A random variable $\tau : \Omega \to [0, \infty]$ is called a *Markov time* if $\{\tau \le t\} \in \mathcal{F}_t$ for all $t \ge 0$. A Markov time is called a *stopping time* if $\tau < \infty$ P-a.s.

In the sequel we will only consider stopping times. We refer to Subsection 1.1 above for other similar comments which translate to the present setting of continuous time without major changes.

2. We will assume that the process G is right-continuous and left-continuous over stopping times (if τ_n and τ are stopping times such that $\tau_n \uparrow \tau$ as $n \to \infty$ then $G_{\tau_n} \to G_\tau$ P-a.s. as $n \to \infty$). We will also assume that the following condition is satisfied (with $G_T = 0$ when $T = \infty$):

$$\mathsf{E}\Big(\sup_{0 \le t \le T} |G_t| \Big) < \infty. \tag{2.1.1}$$

Just as in the case of discrete time (Subsection 1.1) here too it is possible to go beyond this condition in both theory and applications of optimal stopping, however, none of the conclusions will essentially be different and we thus work with (2.1.1) throughout.

In order to invoke a theorem on the existence of a right-continuous modification of a given supermartingale, we will assume in the sequel that the filtration $(\mathcal{F}_t)_{t \ge 0}$ is right-continuous and that each $\mathcal{F}_t$ contains all P-null sets from $\mathcal{F}$. This is a technical requirement and its enforcement has no significant impact on interpretations of the optimal stopping problem under consideration and its solution to be presented.

3. We consider the optimal stopping problem

$$V_t^T = \sup_{t \le \tau \le T} \mathsf{E}\, G_\tau \tag{2.1.2}$$

where τ is a stopping time and $0 \le t \le T$. In (2.1.2) we admit that T can be $+\infty$ as well. In this case, however, we assume that the supremum is still taken over stopping times τ, i.e. over Markov times τ satisfying $t \le \tau < \infty$. In this case we will set $V_t = V_t^\infty$ for $t \ge 0$. Moreover, for certain reasons of convenience we will also drop T from V_t^T in (2.1.1) even if the horizon T is finite.

4. By analogy with the results of Subsection 1.1 above (discrete time case) there are two possible ways to tackle the problem (2.1.2). The first method consists of replacing the time interval $[0, T]$ by sets $\mathcal{D}_n = \{t_0^n, t_1^n, \ldots, t_n^n\}$ where $\mathcal{D}_n \uparrow \mathcal{D}$

as $n \to \infty$ and $\mathcal{D}$ is a (countable) dense subset of $[0, T]$, applying the results of Subsection 1.1 (the method of backward induction) to each $G^n = (G_{t_i^n})_{0 \le i \le n}$, and then passing to the limit as $n \to \infty$. In this context it is useful to know that each stopping time τ can be obtained as a decreasing limit of the discrete stopping times $\tau_n = \sum_{i=1}^{n} t_i^n \, I(t_{i-1}^n \le \tau < t_i^n)$ as $n \to \infty$. The methodology described becomes useful for getting numerical approximations for the solution but we will omit further details in this direction. The second method aims directly to extend the method of essential supremum in Subsection 1.1 above from the discrete time case to the continuous time case. This programme will now be addressed.

5. Since there is no essential difference in the treatment of either finite or infinite horizon T, we will treat both cases at the same time by setting

$$V_t = V_t^T \tag{2.1.3}$$

for simplicity of notation.

To solve the problem (2.1.2) we will (by analogy with the results of Subsection 1.1) consider the process $S = (S_t)_{t \ge 0}$ defined as follows:

$$S_t = \operatorname*{ess\,sup}_{\tau \ge t} \mathsf{E}\,(G_\tau \,|\, \mathcal{F}_t) \tag{2.1.4}$$

where τ is a stopping time. In the case of a finite horizon T we also require in (2.1.4) that τ is smaller than or equal to T. We will see in the proof of Theorem 2.2 below that there is no restriction to assume that the process S is right-continuous. The process S is often referred to as the *Snell envelope* of G.

For the same reasons we will consider the following stopping time:

$$\tau_t = \inf\{\, s \ge t : S_s = G_s \,\} \tag{2.1.5}$$

for $t \ge 0$ where $\inf \varnothing = \infty$ by definition. In the case of a finite horizon T we also require in (2.1.5) that s is smaller than or equal to T.

Regarding the initial part of Theorems 1.2 and 1.4 (the Wald–Bellman equation) one should observe that Theorem 2.2 below implies that

$$S_t \ge \max\big(G_t, \mathsf{E}\,(S_s \mid \mathcal{F}_t)\big) \tag{2.1.6}$$

for $s \ge t$. The reverse inequality, however, is not true in general. The reason roughly speaking lies in the fact that, unlike in discrete time, in continuous time there is no smallest unit of time, so that no matter how close s to t is (when strictly larger) the values S_u can still wander far away from S_t when $u \in (t, s)$. Note however that Theorem 2.2 below implies that the following refinement of the Wald–Bellman equation still holds:

$$S_t = \max\big(G_t, \mathsf{E}\,(S_{\sigma \wedge \tau_t} \mid \mathcal{F}_t)\big) \tag{2.1.7}$$

for every stopping time σ larger than or equal to t (note that σ can also be identically equal to any $s \geq t$) where τ_t is given in (2.1.5) above.

The other three parts of Theorems 1.2 and 1.4 (pages 3 and 8) extend to the present case with no significant change. Thus the first part of the following theorem shows that $(S_s)_{s \geq t}$ and τ_t solve the problem in a stochastic sense. The second part of the theorem shows that this leads to a solution of the initial problem (2.1.2). The third part of the theorem provides a supermartingale characterization of the solution.

Theorem 2.2. *Consider the optimal stopping problem* (2.1.2) *upon assuming that the condition* (2.1.1) *holds. Assume moreover when required below that*

$$\mathsf{P}(\tau_t < \infty) = 1 \tag{2.1.8}$$

where $t \geq 0$ *.* (*Note that this condition is automatically satisfied when the horizon* T *is finite.*) *Then for all* $t \geq 0$ *we have*:

$$S_t \geq \mathsf{E}\,(G_\tau \mid \mathcal{F}_t) \quad \textit{for each } \tau \in \mathfrak{M}_t, \tag{2.1.9}$$

$$S_t = \mathsf{E}\,(G_{\tau_t} \mid \mathcal{F}_t) \tag{2.1.10}$$

where $\mathfrak{M}_t$ *denotes the family of all stopping times* τ *satisfying* $\tau \geq t$ (*being also smaller than or equal to* T *when the latter is finite*)*. Moreover, if* $t \geq 0$ *is given and fixed, then we have*:

The stopping time τ_t *is optimal in* (2.1.2). (2.1.11)

If τ_* *is an optimal stopping time in* (2.1.2) *then* $\tau_t \leq \tau_*$ P*-a.s.* (2.1.12)

The process $(S_s)_{s \geq t}$ *is the smallest right-continuous supermartingale which dominates* $(G_s)_{s \geq t}$*.* (2.1.13)

The stopped process $(S_{s \wedge \tau_t})_{s \geq t}$ *is a right-continuous martingale.* (2.1.14)

Finally, if the condition (2.1.8) *fails so that* $\mathsf{P}(\tau_t = \infty) > 0$ *, then there is no optimal stopping time* (*with probability* 1) *in* (2.1.2).

Proof. 1°. Let us first show that $S = (S_t)_{t \geq 0}$ defined by (2.1.4) above is a supermartingale. For this, fix $t \geq 0$ and let us show that the family $\big\{\mathsf{E}\,(G_\tau \mid \mathcal{F}_t) : \tau \in \mathfrak{M}_t\big\}$ is upwards directed in the sense that (1.1.25) is satisfied. Indeed, note that if σ_1 and σ_2 are from $\mathfrak{M}_t$ and we set $\sigma_3 = \sigma_1 I_A + \sigma_2 I_{A^c}$ where $A = \big\{\mathsf{E}\,(G_{\sigma_1} \mid \mathcal{F}_t) \geq \mathsf{E}\,(G_{\sigma_2} \mid \mathcal{F}_t)\big\}$, then σ_3 belongs to $\mathfrak{M}_t$ and we have

$$\begin{aligned} \mathsf{E}\,(G_{\sigma_3} \mid \mathcal{F}_t) &= \mathsf{E}\,(G_{\sigma_1} I_A + G_{\sigma_2} I_{A^c} \mid \mathcal{F}_t) \\ &= I_A \mathsf{E}\,(G_{\sigma_1} \mid \mathcal{F}_t) + I_{A^c} \mathsf{E}\,(G_{\sigma_2} \mid \mathcal{F}_t) \\ &= \mathsf{E}\,(G_{\sigma_1} \mid \mathcal{F}_t) \vee \mathsf{E}\,(G_{\sigma_2} \mid \mathcal{F}_t) \end{aligned} \tag{2.1.15}$$

implying (1.1.25) as claimed. Hence by (1.1.26) there exists a sequence $\{\sigma_k : k \geq 1\}$ in $\mathfrak{M}_t$ such that

$$\operatorname*{ess\,sup}_{\tau \in \mathfrak{M}_t} \mathsf{E}\,(G_\tau \mid \mathcal{F}_t) = \lim_{k \to \infty} \mathsf{E}\,(G_{\sigma_k} \mid \mathcal{F}_t) \tag{2.1.16}$$

where $\mathsf{E}(G_{\sigma_1} \mid \mathcal{F}_t) \le \mathsf{E}(G_{\sigma_2} \mid \mathcal{F}_t) \le \cdots$ P-a.s. Since the left-hand side in (2.1.16) equals S_t, by the conditional monotone convergence theorem using (2.1.1) above, we find for any $s \in [0,t]$ that

$$\begin{aligned} \mathsf{E}(S_t \mid \mathcal{F}_s) &= \mathsf{E}\Big(\lim_{k\to\infty} \mathsf{E}(G_{\sigma_k} \mid \mathcal{F}_t) \mid \mathcal{F}_s\Big) \\ &= \lim_{k\to\infty} \mathsf{E}\big(\mathsf{E}(G_{\sigma_k} \mid \mathcal{F}_t) \mid \mathcal{F}_s\big) \\ &= \lim_{k\to\infty} \mathsf{E}(G_{\sigma_k} \mid \mathcal{F}_s) \le S_s \end{aligned} \tag{2.1.17}$$

where the final inequality follows by the definition of S_s given in (2.1.4) above. This shows that $(S_t)_{t\ge 0}$ is a supermartingale as claimed. Note also that (2.1.4) and (2.1.16) using the monotone convergence theorem and (2.1.1) imply that

$$\mathsf{E}\,S_t = \sup_{\tau \ge t} \mathsf{E}\,G_\tau \tag{2.1.18}$$

where τ is a stopping time and $t \ge 0$.

2°. Let us next show that the supermartingale S admits a right-continuous modification $\widetilde{S} = (\widetilde{S}_t)_{t\ge 0}$. A well-known result in martingale theory (see e.g. [134]) states that the latter is possible to achieve if and only if

$$t \mapsto \mathsf{E}\,S_t \quad \text{is right-continuous on } \mathbb{R}_+ . \tag{2.1.19}$$

To verify (2.1.19) note that by the supermartingale property of S we have $\mathsf{E}\,S_t \ge \cdots \ge \mathsf{E}\,S_{t_2} \ge \mathsf{E}\,S_{t_1}$ so that $L := \lim_{n\to\infty} \mathsf{E}\,S_{t_n}$ exists and $\mathsf{E}\,S_t \ge L$ whenever $t_n \downarrow t$ as $n \to \infty$ is given and fixed. To prove the reverse inequality, fix $\varepsilon > 0$ and by means of (2.1.18) choose $\sigma \in \mathfrak{M}_t$ such that

$$\mathsf{E}\,G_\sigma \ge \mathsf{E}\,S_t - \varepsilon. \tag{2.1.20}$$

Fix $\delta > 0$ and note that there is no restriction to assume that $t_n \in [t, t+\delta]$ for all $n \ge 1$. Define a stopping time σ_n by setting

$$\sigma_n = \begin{cases} \sigma & \text{if} \quad \sigma > t_n, \\ t+\delta & \text{if} \quad \sigma \le t_n \end{cases} \tag{2.1.21}$$

for $n \ge 1$. Then for all $n \ge 1$ we have

$$\mathsf{E}\,G_{\sigma_n} = \mathsf{E}\,G_\sigma I(\sigma > t_n) + \mathsf{E}\,G_{t+\delta} I(\sigma \le t_n) \le \mathsf{E}\,S_{t_n} \tag{2.1.22}$$

since $\sigma_n \in \mathfrak{M}_{t_n}$ and (2.1.18) holds. Letting $n \to \infty$ in (2.1.22) and using (2.1.1) we get

$$\mathsf{E}\,G_\sigma I(\sigma > t) + \mathsf{E}\,G_{t+\delta} I(\sigma = t) \le L \tag{2.1.23}$$

for all $\delta > 0$. Letting now $\delta \downarrow 0$ and using that G is right-continuous we finally obtain

$$\mathsf{E}\,G_\sigma I(\sigma > t) + \mathsf{E}\,G_t I(\sigma = t) = \mathsf{E}\,G_\sigma \le L. \tag{2.1.24}$$

From (2.1.20) and (2.1.24) we see that $L \geq \mathsf{E}\, S_t - \varepsilon$ for all $\varepsilon > 0$. Hence $L \geq \mathsf{E}\, S_t$ and thus $L = \mathsf{E}\, S_t$ showing that (2.1.19) holds. It follows that S admits a right-continuous modification $\widetilde{S} = (\widetilde{S}_t)_{t\geq 0}$ which we also denote by S throughout.

3°. Let us show that (2.1.13) holds. For this, let $\widehat{S} = (\widehat{S}_s)_{s\geq t}$ be another right-continuous supermartingale which dominates $G = (G_s)_{s\geq t}$. Then by the optional sampling theorem (page 60) using (2.1.1) above we have

$$\widehat{S}_s \geq \mathsf{E}\,(\widehat{S}_\tau \mid \mathcal{F}_s) \geq \mathsf{E}\,(G_\tau \mid \mathcal{F}_s) \tag{2.1.25}$$

for all $\tau \in \mathfrak{M}_s$ when $s \geq t$. Hence by the definition of S_s given in (2.1.4) above we find that $S_s \leq \widehat{S}_s$ P-a.s. for all $s \geq t$. By the right-continuity of S and $\widehat{S}$ this further implies that $\mathsf{P}(S_s \leq \widehat{S}_s \text{ for all } s \geq t) = 1$ as claimed.

4°. Noticing that (2.1.9) follows at once from (2.1.4) above, let us now show that (2.1.10) holds. For this, let us first consider the case when $G_t \geq 0$ for all $t \geq 0$.

For each $\lambda \in (0,1)$ introduce the stopping time

$$\tau_t^\lambda = \inf\{\, s \geq t : \lambda S_s \leq G_s \,\} \tag{2.1.26}$$

where $t \geq 0$ is given and fixed. For further reference note that by the right-continuity of S and G we have:

$$\lambda S_{\tau_t^\lambda} \leq G_{\tau_t^\lambda}, \tag{2.1.27}$$

$$\tau_{t+}^\lambda = \tau_t^\lambda \tag{2.1.28}$$

for all $\lambda \in (0,1)$. In exactly the same way we find:

$$S_{\tau_t} = G_{\tau_t}, \tag{2.1.29}$$

$$\tau_{t+} = \tau_t \tag{2.1.30}$$

for τ_t defined in (2.1.5) above.

Next note that the optional sampling theorem (page 60) using (2.1.1) above implies

$$S_t \geq \mathsf{E}\,(S_{\tau_t^\lambda} \mid \mathcal{F}_t) \tag{2.1.31}$$

since τ_t^λ is a stopping time greater than or equal to t. To prove the reverse inequality

$$S_t \leq \mathsf{E}\,(S_{\tau_t^\lambda} \mid \mathcal{F}_t) \tag{2.1.32}$$

consider the process

$$R_t = \mathsf{E}\,(S_{\tau_t^\lambda} \mid \mathcal{F}_t) \tag{2.1.33}$$

for $t \geq 0$. We claim that $R = (R_t)_{t\geq 0}$ is a supermartingale. Indeed, for $s < t$ we have

$$\mathsf{E}\,(R_t \mid \mathcal{F}_s) = \mathsf{E}\big(\mathsf{E}\,(S_{\tau_t^\lambda} \mid \mathcal{F}_t) \mid \mathcal{F}_s\big) = \mathsf{E}\,(S_{\tau_t^\lambda} \mid \mathcal{F}_s) \leq \mathsf{E}\,(S_{\tau_s^\lambda} \mid \mathcal{F}_s) = R_s \tag{2.1.34}$$

where the inequality follows by the optional sampling theorem (page 60) using (2.1.1) above since $\tau_t^\lambda \geq \tau_s^\lambda$ when $s < t$. This shows that R is a supermartingale as claimed. Hence $\mathsf{E}\,R_{t+h}$ increases when h decreases and $\lim_{h\downarrow 0} \mathsf{E}\,R_{t+h} \leq \mathsf{E}\,R_t$. On the other hand, note by Fatou's lemma using (2.1.1) above that

$$\lim_{h\downarrow 0} \mathsf{E}\,R_{t+h} = \lim_{h\downarrow 0} \mathsf{E}\,S_{\tau_{t+h}^\lambda} \geq \mathsf{E}\,S_{\tau_t^\lambda} = \mathsf{E}\,R_t \tag{2.1.35}$$

where we also use (2.1.28) above together with the facts that τ_{t+h}^λ decreases when h decreases and S is right-continuous. This shows that $t \mapsto \mathsf{E}\,R_t$ is right-continuous on $\mathbb{R}_+$ and hence R admits a right-continuous modification which we also denote by R in the sequel. It follows that there is no restriction to assume that the supermartingale R is right-continuous.

To prove (2.1.32) i.e. that $S_t \leq R_t$ P-a.s. consider the right-continuous supermartingale defined as follows:

$$L_t = \lambda S_t + (1-\lambda) R_t \tag{2.1.36}$$

for $t \geq 0$. We then claim that

$$L_t \geq G_t \quad \mathsf{P}\text{-a.s.} \tag{2.1.37}$$

for all $t \geq 0$. Indeed, we have

$$\begin{aligned} L_t &= \lambda S_t + (1-\lambda) R_t = \lambda S_t + (1-\lambda) R_t I(\tau_t^\lambda = t) \\ &\quad + (1-\lambda) R_t I(\tau_t^\lambda > t) \\ &= \lambda S_t + (1-\lambda) \mathsf{E}\big(S_t I(\tau_t^\lambda = t) \mid \mathcal{F}_t\big) + (1-\lambda) R_t I(\tau_t^\lambda > t) \\ &= \lambda S_t I(\tau_t^\lambda = t) + (1-\lambda) S_t I(\tau_t^\lambda = t) + \lambda S_t I(\tau_t^\lambda > t) \\ &\quad + (1-\lambda) R_t I(\tau_t^\lambda > t) \\ &\geq S_t I(\tau_t^\lambda = t) + \lambda S_t I(\tau_t^\lambda > t) \geq G_t I(\tau_t^\lambda = t) + G_t I(\tau_t^\lambda > t) = G_t \end{aligned} \tag{2.1.38}$$

where in the second last inequality we used that $R_t \geq 0$ and in the last inequality we used the definition of τ_t^λ given in (2.1.26) above. Thus (2.1.37) holds as claimed. Finally, since S is the smallest right-continuous supermartingale which dominates G, we see that (2.1.37) implies that

$$S_t \leq L_t \quad \mathsf{P}\text{-a.s.} \tag{2.1.39}$$

from where by (2.1.36) we conclude that $S_t \leq R_t$ P-a.s. Thus (2.1.32) holds as claimed. Combining (2.1.31) and (2.1.32) we get

$$S_t = \mathsf{E}\,(S_{\tau_t^\lambda} \mid \mathcal{F}_t) \tag{2.1.40}$$

for all $\lambda \in (0,1)$. From (2.1.40) and (2.1.27) we find

$$S_t \leq \frac{1}{\lambda} \mathsf{E}\,(G_{\tau_t^\lambda} \mid \mathcal{F}_t) \tag{2.1.41}$$

for all $\lambda \in (0,1)$. Letting $\lambda \uparrow 1$, using the conditional Fatou's lemma and (2.1.1) above together with the fact that G is left-continuous over stopping times, we obtain

$$S_t \le \mathsf{E}\,(G_{\tau_t^1} \mid \mathcal{F}_t) \tag{2.1.42}$$

where τ_t^1 is a stopping time given by

$$\tau_t^1 = \lim_{\lambda \uparrow 1} \tau_t^\lambda. \tag{2.1.43}$$

(Note that τ_t^λ increases when λ increases.) Since by (2.1.4) we know that the reverse inequality in (2.1.42) is always fulfilled, we may conclude that

$$S_t = \mathsf{E}\,(G_{\tau_t^1} \mid \mathcal{F}_t) \tag{2.1.44}$$

for all $t \ge 0$. Thus to complete the proof of (2.1.10) it is enough to verify that

$$\tau_t^1 = \tau_t \tag{2.1.45}$$

where τ_t is defined in (2.1.5) above. For this, note first that $\tau_t^\lambda \le \tau_t$ for all $\lambda \in (0,1)$ so that $\tau_t^1 \le \tau_t$. On the other hand, if $\tau_t(\omega) > t$ (the case $\tau_t(\omega) = t$ being obvious) then there exists $\varepsilon > 0$ such that $\tau_t(\omega) - \varepsilon > t$ and $S_{\tau_t(\omega)-\varepsilon} > G_{\tau_t(\omega)-\varepsilon} \ge 0$. Hence one can find $\lambda \in (0,1)$ (close enough to 1) such that $\lambda S_{\tau_t(\omega)-\varepsilon} > G_{\tau_t(\omega)-\varepsilon}$ showing that $\tau_t^\lambda(\omega) \ge \tau_t(\omega) - \varepsilon$. Letting first $\lambda \uparrow 1$ and then $\varepsilon \downarrow 0$ we conclude that $\tau_t^1 \ge \tau_t$. Hence (2.1.45) holds as claimed and the proof of (2.1.10) is complete in the case when $G_t \ge 0$ for all $t \ge 0$.

5°. In the case of general G satisfying (2.1.1) we can set

$$H = \inf_{t \ge 0} G_t \tag{2.1.46}$$

and introduce the right-continuous martingale

$$M_t = \mathsf{E}\,(H \mid \mathcal{F}_t) \tag{2.1.47}$$

for $t \ge 0$ so as to replace the initial gain process G by a new gain process $\widetilde{G} = (\widetilde{G}_t)_{t \ge 0}$ defined by

$$\widetilde{G}_t = G_t - M_t \tag{2.1.48}$$

for $t \ge 0$. Note that $\widetilde{G}$ need not satisfy (2.1.1) due to the existence of M, but M itself is uniformly integrable since $H \in L^1(\mathsf{P})$. Similarly, $\widetilde{G}$ is right-continuous and not necessarily left-continuous over stopping times due to the existence of M, but M itself is a (uniformly integrable) martingale so that the optional sampling theorem (page 60) is applicable. Finally, it is clear that $\widetilde{G}_t \ge 0$ and the optional sampling theorem implies that

$$\widetilde{S}_t = \operatorname*{ess\,sup}_{\tau \in \mathfrak{M}_t} \mathsf{E}\left(\widetilde{G}_\tau \mid \mathcal{F}_t\right) = \operatorname*{ess\,sup}_{\tau \in \mathfrak{M}_t} \mathsf{E}\,(G_\tau - M_\tau \mid \mathcal{F}_t) = S_t - M_t \tag{2.1.49}$$

for all $t \geq 0$. A closer inspection based on the new properties of $\widetilde{G}$ displayed above instead of the old ones imposed on G when $G_t \geq 0$ for all $t \geq 0$ shows that the proof above can be applied to $\widetilde{G}$ and $\widetilde{S}$ to yield the same conclusions implying (2.1.10) in the general case.

6°. Noticing that (2.1.11) follows by taking expectation in (2.1.10) and using (2.1.18), let us now show that (2.1.12) holds. We claim that the optimality of τ_* implies that $S_{\tau_*} = G_{\tau_*}$ P-a.s. Indeed, if this would not be the case then we would have $S_{\tau_*} \geq G_{\tau_*}$ P-a.s. with $\mathsf{P}(S_{\tau_*} > G_{\tau_*}) > 0$. It would then follow that $\mathsf{E}\, G_{\tau_*} < \mathsf{E}\, S_{\tau_*} \leq \mathsf{E}\, S_t = V_t$ where the second inequality follows by the optional sampling theorem (page 60) and the supermartingale property of $(S_s)_{s \geq t}$ using (2.1.1) above, while the final equality is stated in (2.1.18) above. The strict inequality, however, contradicts the fact that τ_* is optimal. Hence $S_{\tau_*} = G_{\tau_*}$ P-a.s. as claimed and the fact that $\tau_t \leq \tau_*$ P-a.s. follows from the definition (2.1.5) above.

7°. To verify the martingale property (2.1.14) it is enough to prove that

$$\mathsf{E}\, S_{\sigma \wedge \tau_t} = \mathsf{E}\, S_t \tag{2.1.50}$$

for all (bounded) stopping times σ greater than or equal to t. For this, note first that the optional sampling theorem (page 60) using (2.1.1) above implies

$$\mathsf{E}\, S_{\sigma \wedge \tau_t} \leq \mathsf{E}\, S_t. \tag{2.1.51}$$

On the other hand, from (2.1.10) and (2.1.29) we likewise see that

$$\mathsf{E}\, S_t = \mathsf{E}\, G_{\tau_t} = \mathsf{E}\, S_{\tau_t} \leq \mathsf{E}\, S_{\sigma \wedge \tau_t}. \tag{2.1.52}$$

Combining (2.1.51) and (2.1.52) we see that (2.1.50) holds and thus $(S_{s \wedge \tau_t})_{s \geq t}$ is a martingale (right-continuous by (2.1.13) above). This completes the proof of (2.1.14).

Finally, note that the final claim follows directly from (2.1.12). This completes the proof of the theorem. □

2.2. Markovian approach

In this subsection we will present basic results of optimal stopping when the time is continuous and the process is Markovian. (Basic definitions and properties of such processes are given in Subsection 4.3.)

1. Throughout we will consider a strong Markov process $X = (X_t)_{t \geq 0}$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathsf{P}_x)$ and taking values in a measurable space $(E, \mathcal{B})$ where for simplicity we will assume that $E = \mathbb{R}^d$ for some $d \geq 1$ and $\mathcal{B} = \mathcal{B}(\mathbb{R}^d)$ is the Borel σ-algebra on $\mathbb{R}^d$. It is assumed that the process X starts at x under P_x for $x \in E$ and that the sample paths of X are right-continuous and left-continuous over stopping times (if $\tau_n \uparrow \tau$ are stopping times, then $X_{\tau_n} \to X_\tau$ P_x-a.s. as $n \to \infty$). It is also assumed that the filtration $(\mathcal{F}_t)_{t \ge 0}$ is right-continuous (implying that the first entry times to open and closed sets are stopping times). In addition, it is assumed that the mapping $x \mapsto \mathsf{P}_x(F)$ is measurable for each $F \in \mathcal{F}$. It follows that the mapping $x \mapsto \mathsf{E}_x(Z)$ is measurable for each (bounded or non-negative) random variable Z. Finally, without loss of generality we will assume that $(\Omega, \mathcal{F})$ equals the canonical space $(E^{[0,\infty)}, \mathcal{B}^{[0,\infty)})$ so that the shift operator $\theta_t : \Omega \to \Omega$ is well defined by $\theta_t(\omega)(s) = \omega(t+s)$ for $\omega = (\omega(s))_{s \ge 0} \in \Omega$ and $t, s \ge 0$.

2. Given a measurable function $G : E \to \mathbb{R}$ satisfying the following condition (with $G(X_T) = 0$ if $T = \infty$):

$$\mathsf{E}_x\Big(\sup_{0 \le t \le T} |G(X_t)|\Big) < \infty \tag{2.2.1}$$

for all $x \in E$, we consider the optimal stopping problem

$$V(x) = \sup_{0 \le \tau \le T} \mathsf{E}_x G(X_\tau) \tag{2.2.2}$$

where $x \in E$ and the supremum is taken over stopping times τ of X. The latter means that τ is a stopping time with respect to the natural filtration of X given by $\mathcal{F}_t^X = \sigma(X_s : 0 \le s \le t)$ for $t \ge 0$. Since the same results remain valid if we take the supremum in (2.2.2) over stopping times τ with respect to $(\mathcal{F}_t)_{t \ge 0}$, and this assumption makes certain conclusions more elegant (the optimal stopping time will be attained), we will assume in the sequel that the supremum in (2.2.2) is taken over this larger class of stopping times. Note also that in (2.2.2) we admit that T can be ∞ as well (infinite horizon). In this case, however, we still assume that the supremum is taken over stopping times τ, i.e. over Markov times satisfying $0 \le \tau < \infty$. In this way any specification of $G(X_\infty)$ becomes irrelevant for the problem (2.2.2).

3. Recall that V is called the *value function* and G is called the *gain function.* To solve the optimal stopping problem (2.2.2) means two things. Firstly, we need to exhibit an *optimal* stopping time, i.e. a stopping time τ_* at which the supremum is attained. Secondly, we need to compute the value $V(x)$ for $x \in E$ as explicitly as possible.

Let us briefly comment on what one expects to be a solution to the problem (2.2.2) (recall also Subsection 1.2 above). For this note that being Markovian means that the process X always starts afresh. Thus following the sample path $t \mapsto X_t(\omega)$ for $\omega \in \Omega$ given and fixed and evaluating $G(X_t(\omega))$ it is naturally expected that at each time t we shall be able optimally to decide either to continue with the observation or to stop it. In this way the state space E naturally splits into the *continuation set* C and the *stopping set* $D = E \setminus C$. It follows that as soon as the observed value $X_t(\omega)$ enters D, the observation should be stopped and an optimal stopping time is obtained. The central question thus arises as how to determine the sets C and D. (Note that the same arguments also hold in the discrete-time case of Subsection 1.2 above.)

In comparison with the general optimal stopping problem of Subsection 2.1 above, it may be noted that the description of the optimal stopping time just given does not involve any probabilistic construction (of a new stochastic process $S = (S_t)_{t \ge 0}$) but is purely deterministic (obtained by splitting E into two disjoint subsets defined by the deterministic functions G and V).

4. In the sequel we will treat the *finite horizon* formulation ($T < \infty$) and the *infinite horizon* formulation ($T = \infty$) of the optimal stopping problem (2.2.2) at the same time. It should be noted that in the former case ($T < \infty$) we need to replace the process X_t by the process $Z_t = (t, X_t)$ for $t \ge 0$ so that the problem reads

$$V(t,x) = \sup_{0 \le \tau \le T-t} \mathsf{E}_{t,x} G(t+\tau, X_{t+\tau}) \tag{2.2.2$'$}$$

where the "rest of time" $T - t$ changes when the initial state $(t, x) \in [0, T] \times E$ changes in its first argument. It turns out, however, that no argument below is more seriously affected by this change, and the results obtained for the problem (2.2.2) with $T = \infty$ will automatically hold for the problem (2.2.2$'$) if we simply think of X to be Z (with a new "two-dimensional" state space E equal to $\mathbb{R}_+ \times E$). Moreover, it may be noted in (2.2.2$'$) that at time T we have the "terminal" condition $V(T, x) = G(T, x)$ for all $x \in E$ so that the first entry time of Z to the stopping set D, denoted below by τ_D, will always be smaller than or equal to T and thus finite. This works to a technical advantage of the finite horizon formulation (2.2.2$'$) over the infinite horizon formulation (2.2.2) (where instead of the condition $V(T, x) = G(T, x)$ for all $x \in E$ another "boundary condition at infinity" such as (2.2.52) may hold).

5. Consider the optimal stopping problem (2.2.2) when $T = \infty$. Recall that (2.2.2) reads as follows:

$$V(x) = \sup_{\tau} \mathsf{E}_x G(X_\tau) \tag{2.2.3}$$

where τ is a stopping time (with respect to $(\mathcal{F}_t)_{t \ge 0}$) and $\mathsf{P}_x(X_0 = x) = 1$ for $x \in E$. Introduce the continuation set

$$C = \{x \in E : V(x) > G(x)\} \tag{2.2.4}$$

and the stopping set

$$D = \{x \in E : V(x) = G(x)\} \tag{2.2.5}$$

Note that if V is lsc (lower semicontinuous) and G usc (upper semicontinuous) then C is open and D is closed. Introduce the first entry time τ_D of X into D by setting

$$\tau_D = \inf\{\, t \ge 0 : X_t \in D \,\}. \tag{2.2.6}$$

Note that τ_D is a stopping (Markov) time with respect to $(\mathcal{F}_t)_{t\geq 0}$ when D is closed since both X and $(\mathcal{F}_t)_{t\geq 0}$ are right-continuous.

6. The following definition plays a fundamental role in solving the optimal stopping problem (2.2.3).

Definition 2.3. A measurable function $F : E \to \mathbb{R}$ is said to be *superharmonic* if

$$\mathsf{E}_x F(X_\sigma) \leq F(x) \tag{2.2.7}$$

for all stopping times σ and all $x \in E$.

It is assumed in (2.2.7) that the left-hand side is well defined (and finite) i.e. that $F(X_\sigma) \in L^1(\mathsf{P}_x)$ for all $x \in E$ whenever σ is a stopping time. Moreover, it will be verified in the proof of Theorem 2.4 below that the following stochastic characterization of superharmonic functions holds (recall also (1.2.40)):

F is superharmonic if and only if $(F(X_t))_{t\geq 0}$ is a right-continuous supermartingale under P_x for every $x \in E$ (2.2.8)

whenever F is lsc and $(F(X_t))_{t\geq 0}$ is uniformly integrable.

7. The following theorem presents *necessary conditions* for the existence of an optimal stopping time.

Theorem 2.4. *Let us assume that there exists an optimal stopping time* τ_* *in* (2.2.3), *i.e. let*

$$V(x) = \mathsf{E}_x G(X_{\tau_*}) \tag{2.2.9}$$

for all $x \in E$. *Then we have*:

The value function V *is the smallest superharmonic function which dominates the gain function* G *on* E. (2.2.10)

Let us in addition to (2.2.9) *assume that* V *is lsc and* G *is usc. Then we have*:

The stopping time τ_D *satisfies* $\tau_D \leq \tau_*$ P_x*-a.s. for all* $x \in E$ *and is optimal in* (2.2.3). (2.2.11)

The stopped process $(V(X_{t\wedge\tau_D}))_{t\geq 0}$ *is a right-continuous martingale under* P_x *for every* $x \in E$. (2.2.12)

Proof. (2.2.10): To show that V is superharmonic note that by the strong Markov property we have:

$$\begin{aligned} \mathsf{E}_x V(X_\sigma) &= \mathsf{E}_x \mathsf{E}_{X_\sigma} G(X_{\tau_*}) = \mathsf{E}_x \mathsf{E}_x \big(G(X_{\tau_*}) \circ \theta_\sigma \mid \mathcal{F}_\sigma\big) \\ &= \mathsf{E}_x G(X_{\sigma+\tau_*\circ\theta_\sigma}) \leq \sup_\tau \mathsf{E}_x G(X_\tau) = V(x) \end{aligned} \tag{2.2.13}$$

for each stopping time σ and all $x \in E$. This establishes (2.2.7) and proves the initial claim.

Let F be a superharmonic function which dominates G on E. Then we have

$$\mathsf{E}_x G(X_\tau) \leq \mathsf{E}_x F(X_\tau) \leq F(x) \tag{2.2.14}$$

for each stopping time τ and all $x \in E$. Taking the supremum over all τ in (2.2.14) we find that $V(x) \leq F(x)$ for all $x \in E$. Since V is superharmonic itself, this proves the final claim.

(2.2.11): We claim that $V(X_{\tau_*}) = G(X_{\tau_*})$ P_x-a.s. for all $x \in E$. Indeed, if $\mathsf{P}_x\big(V(X_{\tau_*}) > G(X_{\tau_*})\big) > 0$ for some $x \in E$, then $\mathsf{E}_x G(X_{\tau_*}) < \mathsf{E}_x V(X_{\tau_*}) \leq V(x)$ since V is superharmonic, leading to a contradiction with the fact that τ_* is optimal. From the identity just verified it follows that $\tau_D \leq \tau_*$ P_x-a.s. for all $x \in E$ as claimed.

To make use of the previous inequality we may note that setting $\sigma \equiv s$ in (2.2.7) and using the Markov property we get

$$V(X_t) \geq \mathsf{E}_{X_t} V(X_s) = \mathsf{E}_x\big(V(X_{t+s}) \,|\, \mathcal{F}_t\big) \tag{2.2.15}$$

for all $t, s \geq 0$ and all $x \in E$. This shows:

The process $(V(X_t))_{t\geq 0}$ is a supermartingale under P_x for each $x \in E$. (2.2.16)

Moreover, to indicate the argument as clearly as possible, let us for the moment assume that V is continuous. Then obviously it follows that $(V(X_t))_{t\geq 0}$ is right-continuous. Thus, by the optional sampling theorem (page 60) using (2.2.1) above, we see that (2.2.7) extends as follows:

$$\mathsf{E}_x V(X_\tau) \leq \mathsf{E}_x V(X_\sigma) \tag{2.2.17}$$

for stopping times σ and τ such that $\sigma \leq \tau$ P_x-a.s. with $x \in E$. In particular, since $\tau_D \leq \tau_*$ P_x-a.s. by (2.2.17) we get

$$V(x) = \mathsf{E}_x G(X_{\tau_*}) = \mathsf{E}_x V(X_{\tau_*}) \leq \mathsf{E}_x V(X_{\tau_D}) = \mathsf{E}_x G(X_{\tau_D}) \leq V(x) \tag{2.2.18}$$

for $x \in E$ upon using that $V(X_{\tau_D}) = G(X_{\tau_D})$ since V is lsc and G is usc. This shows that τ_D is optimal if V is continuous. Finally, if V is only known to be lsc, then by Proposition 2.5 below we know that $(V(X_t))_{t\geq 0}$ is right-continuous P_x-a.s. for each $x \in E$, and the proof can be completed as above. This shows that τ_D is optimal if V is lsc as claimed.

(2.2.12): By the strong Markov property we have

$$\begin{aligned}
\mathsf{E}_x\big(V(X_{t\wedge\tau_D})\,|\,\mathcal{F}_{s\wedge\tau_D}\big) &= \mathsf{E}_x\Big(\mathsf{E}_{X_{t\wedge\tau_D}}G(X_{\tau_D})\,|\,\mathcal{F}_{s\wedge\tau_D}\Big) \qquad (2.2.19)\\
&= \mathsf{E}_x\Big(\mathsf{E}_x\big(G(X_{\tau_D})\circ\theta_{t\wedge\tau_D}\,|\,\mathcal{F}_{t\wedge\tau_D}\big)\,|\,\mathcal{F}_{s\wedge\tau_D}\Big)\\
&= \mathsf{E}_x\Big(\mathsf{E}_x\big(G(X_{\tau_D})\,|\,\mathcal{F}_{t\wedge\tau_D}\big)\,|\,\mathcal{F}_{s\wedge\tau_D}\Big) = \mathsf{E}_x\big(G(X_{\tau_D})\,|\,\mathcal{F}_{s\wedge\tau_D}\big)\\
&= \mathsf{E}_{X_{s\wedge\tau_D}}\big(G(X_{\tau_D})\big) = V(X_{s\wedge\tau_D})
\end{aligned}$$

for all $0 \le s \le t$ and all $x \in E$ proving the martingale property. The right-continuity of $\big(V(X_{t\wedge\tau_D})\big)_{t\ge 0}$ follows from the right-continuity of $(V(X_t))_{t\ge0}$ and the proof is complete. □

The following fact was needed in the proof above to extend the result from continuous to lsc V.

Proposition 2.5. *If a superharmonic function $F : E \to \mathbb{R}$ is lsc (lower semicontinuous), then the supermartingale $(F(X_t))_{t\ge0}$ is right-continuous P_x-a.s. for every $x \in E$.*

Proof. Firstly, we will show that

$$F(X_\tau) = \lim_{h\downarrow 0} F(X_{\tau+h}) \qquad \mathsf{P}_x\text{-a.s.} \qquad (2.2.20)$$

for any given stopping time τ and $x \in E$. For this, note that the right-continuity of X and the ls-continuity of F, we get

$$F(X_\tau) \le \liminf_{h\downarrow 0} F(X_{\tau+h}) \qquad \mathsf{P}_x\text{-a.s.} \qquad (2.2.21)$$

To prove the reverse inequality we will first derive it for $\tau \equiv 0$, i.e. we have

$$\limsup_{h\downarrow 0} F(X_h) \le F(x) \qquad \mathsf{P}_x\text{-a.s.} \qquad (2.2.22)$$

For this, note by Blumenthal's 0-1 law (cf. page 97) that $\limsup_{h\downarrow0} F(X_h)$ is equal P_x-a.s. to a constant $c \in \mathbb{R}$. Let us assume that $c > F(x)$. Then there is $\varepsilon > 0$ such that $c > F(x)+\varepsilon$. Set $A_\varepsilon = \{\, y \in E : F(y) > F(x)+\varepsilon \,\}$ and consider the stopping time $\tau_\varepsilon = \inf\{\, h \ge 0 : X_h \in A_\varepsilon \,\}$. By definition of c and A_ε we see that $\tau_\varepsilon = 0$ P_x-a.s. Note however that A_ε is open (since F is lsc) and that we cannot claim a priori that X_{τ_ε}, i.e. x, belongs to A_ε as one would like to reach a contradiction. For this reason choose an increasing sequence of closed sets K_n for $n \ge 1$ such that $\bigcup_{n=1}^\infty K_n = A_\varepsilon$. Consider the stopping time $\tau_n = \inf\{\, h \ge 0 : X_h \in K_n \,\}$ for $n \ge 1$. Then $\tau_n \downarrow \tau_\varepsilon$ as $n \to \infty$ and since K_n is closed we see that $X_{\tau_n} \in K_n$ for all $n \ge 1$. Hence $X_{\tau_n} \in A_\varepsilon$ i.e. $F(X_{\tau_n}) > F(x) + \varepsilon$ for all $n \ge 1$. Using that F is superharmonic this implies

$$\begin{aligned}
F(x) &\ge \mathsf{E}_x F(X_{\tau_n\wedge 1}) = \mathsf{E}_x F(X_{\tau_n}) I(\tau_n \le 1) + \mathsf{E}_x F(X_1) I(\tau_n > 1) \qquad (2.2.23)\\
&\ge (F(x)+\varepsilon)\mathsf{P}(\tau_n \le 1) + \mathsf{E}_x F(X_1) I(\tau_n > 1) \to F(x)+\varepsilon
\end{aligned}$$

as $n \to \infty$ since $\tau_n \downarrow 0$ P_x-a.s. as $n \to \infty$ and $F(X_1) \in L^1(\mathsf{P}_x)$. As clearly (2.2.23) is impossible, we may conclude that (2.2.22) holds as claimed.

To treat the case of a general stopping time τ, take E_x on both sides of (2.2.22) and insert $x = X_\tau$. This by the strong Markov property gives

$$\begin{aligned} F(X_\tau) &\geq \mathsf{E}_{X_\tau}\Big(\limsup_{h \downarrow 0} F(X_h)\Big) = \mathsf{E}_x\Big(\limsup_{h \downarrow 0} F(X_h) \circ \theta_\tau \mid \mathcal{F}_\tau\Big) \\ &= \mathsf{E}_x\Big(\limsup_{h \downarrow 0} F(X_{\tau+h}) \mid \mathcal{F}_\tau\Big) = \limsup_{h \downarrow 0} F(X_{\tau+h}) \quad \mathsf{P}_x\text{-a.s.} \end{aligned} \tag{2.2.24}$$

since $\limsup_{h \downarrow 0} F(X_{\tau+h})$ is $\mathcal{F}_{\tau+}$-measurable and $\mathcal{F}_\tau = \mathcal{F}_{\tau+}$ by the right-continuity of $(\mathcal{F}_t)_{t \geq 0}$. Combining (2.2.21) and (2.2.24) we get (2.2.20). In particular, taking $\tau \equiv t$ we see that

$$\lim_{h \downarrow 0} F(X_{t+h}) = F(X_t) \qquad \mathsf{P}_x\text{-a.s.} \tag{2.2.25}$$

for all $t \geq 0$. Note that the P_x-null set in (2.2.25) does depend on the given t.

Secondly, by means of (2.2.20) we will now show that a single P_x-null set can be selected so that the convergence relation in (2.2.25) holds on its complement simultaneously for all $t \geq 0$. For this, set $\tau_0 = 0$ and define the stopping time

$$\tau_n = \inf\{\, t \geq \tau_{n-1} : |F(X_t) - F(X_{\tau_{n-1}})| > \varepsilon/2 \,\} \tag{2.2.26}$$

for $n = 1, 2, \ldots$ where $\varepsilon > 0$ is given and fixed. By (2.2.20) we see that for each $n \geq 1$ there is a P_x-null set N_n such that $\tau_n > \tau_{n-1}$ on $\Omega \setminus N_n$. Continuing the procedure (2.2.26) by transfinite induction over countable ordinals (there can be at most countably many disjoint intervals in $\mathbb{R}_+$) and calling the union of the countably many P_x-null set by N_ε, it follows that for each $\omega \in \Omega \setminus N_\varepsilon$ and each $t \geq 0$ there is a countable ordinal α such that $\tau_\alpha(\omega) \leq t < \tau_{\alpha+1}(\omega)$. Hence for every $s \in [\tau_\alpha(\omega), \tau_{\alpha+1}(\omega))$ we have $|F(X_t(\omega)) - F(X_s(\omega))| \leq |F(X_t(\omega)) - F(X_{\tau_\alpha}(\omega))| + |F(X_s(\omega)) - F(X_\alpha(\omega))| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$. This shows that $\limsup_{s \downarrow t} |F(X_t) - F(X_s)| \leq \varepsilon$ on $\Omega \setminus N_\varepsilon$. Setting $N = \bigcup_{n=1}^\infty N_{1/n}$ we see that $\mathsf{P}_x(N) = 0$ and $\lim_{s \downarrow t} F(X_s) = F(X_t)$ on $\Omega \setminus N$ completing the proof. □

Remark 2.6. The result and proof of Theorem 2.4 above extend in exactly the same form (by slightly changing the notation only) to the finite horizon problem (2.2.2′). We will omit further details in this direction.

8. The following theorem provides *sufficient condition* for the existence of an optimal stopping time.

Theorem 2.7. *Consider the optimal stopping problem* (2.2.3) *upon assuming that the condition* (2.2.1) *is satisfied. Let us assume that there exists the smallest superharmonic function* $\widehat{V}$ *which dominates the gain function* G *on* E. *Let us in* *addition assume that* $\widehat{V}$ *is lsc and* G *is usc. Set* $D = \{x \in E : \widehat{V}(x) = G(x)\}$ *and let* τ_D *be defined by* (2.2.6) *above. We then have*:

If $\mathsf{P}_x(\tau_D < \infty) = 1$ *for all* $x \in E$*, then* $\widehat{V} = V$ *and* τ_D *is optimal in* (2.2.3). (2.2.27)

If $\mathsf{P}_x(\tau_D < \infty) < 1$ *for some* $x \in E$*, then there is no optimal stopping time* (*with probability* 1) *in* (2.2.3). (2.2.28)

Proof. Since $\widehat{V}$ is superharmonic, we have

$$\mathsf{E}_x G(X_\tau) \le \mathsf{E}_x \widehat{V}(X_\tau) \le \widehat{V}(x) \tag{2.2.29}$$

for all stopping times τ and all $x \in E$. Taking the supremum in (2.2.17) over all τ we find that

$$G(x) \le V(x) \le \widehat{V}(x) \tag{2.2.30}$$

for all $x \in E$. Assuming that $\mathsf{P}_x(\tau_D < \infty) = 1$ for all $x \in E$, we will now present two *different* proofs of the fact that $\widehat{V} = V$ implying also that τ_D is optimal in (2.2.3).

First proof. Let us assume that G is *bounded.* With $\varepsilon > 0$ given and fixed, consider the sets:

$$C_\varepsilon = \{\, x \in E : \widehat{V}(x) > G(x) + \varepsilon \,\}, \tag{2.2.31}$$

$$D_\varepsilon = \{\, x \in E : \widehat{V}(x) \le G(x) + \varepsilon \,\}. \tag{2.2.32}$$

Since $\widehat{V}$ is lsc and G is usc we see that C_ε is open and D_ε is closed. Moreover, it is clear that $C_\varepsilon \uparrow C$ and $D_\varepsilon \downarrow D$ as $\varepsilon \downarrow 0$ where C and D are defined by (2.2.4) and (2.2.5) above respectively.

Define the stopping time

$$\tau_{D_\varepsilon} = \inf\{\, t \ge 0 : X_t \in D_\varepsilon \}. \tag{2.2.33}$$

Since $D \subseteq D_\varepsilon$ and $\mathsf{P}_x(\tau_D < \infty) = 1$ for all $x \in E$, we see that $\mathsf{P}_x(\tau_{D_\varepsilon} < \infty) = 1$ for all $x \in E$. The latter fact can also be derived directly (without assuming the former fact) by showing that $\limsup_{t \to \infty} \widehat{V}(X_t) = \limsup_{t \to \infty} G(X_t)$ P_x-a.s. for all $x \in E$. This can be done in exactly the same way as in the first part of the proof of Theorem 1.13.

In order to show that

$$\mathsf{E}_x \widehat{V}\big(X_{\tau_{D_\varepsilon}}\big) = \widehat{V}(x) \tag{2.2.34}$$

for all $x \in E$, we will first show that

$$G(x) \le \mathsf{E}_x \widehat{V}\big(X_{\tau_{D_\varepsilon}}\big) \tag{2.2.35}$$

for all $x \in E$. For this, set

$$c = \sup_{x \in E} \big(G(x) - \mathsf{E}_x \widehat{V}(X_{\tau_{D_\varepsilon}}) \big) \tag{2.2.36}$$

and note that

$$G(x) \le c + \mathsf{E}_x \widehat{V}\big(X_{\tau_{D_\varepsilon}}\big) \tag{2.2.37}$$

for all $x \in E$. (Observe that c is finite since G is bounded implying also that $\widehat{V}$ is bounded.)

Next by the strong Markov property we find

$$\begin{aligned} \mathsf{E}_x \mathsf{E}_{X_\sigma} \widehat{V}\big(X_{\tau_{D_\varepsilon}}\big) &= \mathsf{E}_x \mathsf{E}_x \Big(\widehat{V}\big(X_{\tau_{D_\varepsilon}}\big) \circ \theta_\sigma \mid \mathcal{F}_\sigma \Big) \\ &= \mathsf{E}_x \mathsf{E}_x \Big(\widehat{V}\big(X_{\sigma + \tau_{D_\varepsilon} \circ \theta_\sigma}\big) \mid \mathcal{F}_\sigma \Big) \\ &= \mathsf{E}_x \widehat{V}\big(X_{\sigma + \tau_{D_\varepsilon} \circ \theta_\sigma}\big) \le \mathsf{E}_x \widehat{V}\big(X_{\tau_{D_\varepsilon}}\big) \end{aligned} \tag{2.2.38}$$

using that $\widehat{V}$ is superharmonic and lsc (recall Proposition 2.5 above) and $\sigma + \tau_{D_\varepsilon} \circ \theta_\sigma \ge \tau_{D_\varepsilon}$ since τ_{D_ε} is the first entry time to a set. This shows that the function

$$x \mapsto \mathsf{E}_x \widehat{V}\big(X_{\tau_{D_\varepsilon}}\big) \quad \text{is superharmonic} \tag{2.2.39}$$

from E to $\mathbb{R}$. Hence the function of the right-hand side of (2.2.37) is also superharmonic so that by the definition of $\widehat{V}$ we can conclude that

$$\widehat{V}(x) \le c + \mathsf{E}_x \widehat{V}\big(X_{\tau_{D_\varepsilon}}\big) \tag{2.2.40}$$

for all $x \in E$.

Given $0 < \delta \le \varepsilon$ choose $x_\delta \in E$ such that

$$G(x_\delta) - \mathsf{E}_{x_\delta} \widehat{V}\big(X_{\tau_{D_\varepsilon}}\big) \ge c - \delta. \tag{2.2.41}$$

Then by (2.2.40) and (2.2.41) we get

$$\widehat{V}(x_\delta) \le c + \mathsf{E}_{x_\delta} \widehat{V}\big(X_{\tau_{D_\varepsilon}}\big) \le G(x_\delta) + \delta \le G(x_\delta) + \varepsilon. \tag{2.2.42}$$

This shows that $x_\delta \in D_\varepsilon$ and thus $\tau_{D_\varepsilon} \equiv 0$ under P_{x_δ}. Inserting the latter conclusion into (2.2.41) we get

$$c - \delta \le G(x_\delta) - \widehat{V}(x_\delta) \le 0. \tag{2.2.43}$$

Letting $\delta \downarrow 0$ we see that $c \le 0$ thus proving (2.2.35) as claimed. Using the definition of $\widehat{V}$ and (2.2.39) we see that (2.2.34) follows directly from (2.2.35).

Finally, from (2.2.34) we get

$$\widehat{V}(x) = \mathsf{E}_x \widehat{V}\big(X_{\tau_{D_\varepsilon}}\big) \le \mathsf{E}_x G\big(X_{\tau_{D_\varepsilon}}\big) + \varepsilon \le V(x) + \varepsilon \tag{2.2.44}$$

for all $x \in E$ upon using that $\widehat{V}\big(X_{\tau_{D_\varepsilon}}\big) \le G\big(X_{\tau_{D_\varepsilon}}\big) + \varepsilon$ since $\widehat{V}$ is lsc and G is usc. Letting $\varepsilon \downarrow 0$ in (2.2.44) we see that $\widehat{V} \le V$ and thus by (2.2.30) we can conclude that $\widehat{V} = V$. From (2.2.44) we thus also see that

$$V(x) \le \mathsf{E}_x G\big(X_{\tau_{D_\varepsilon}}\big) + \varepsilon \tag{2.2.45}$$

for all $x \in E$.

Letting $\varepsilon \downarrow 0$ and using that $D_\varepsilon \downarrow D$ we see that $\tau_{D_\varepsilon} \uparrow \tau_0$ where τ_0 is a stopping time satisfying $\tau_0 \le \tau_D$. Since V is lsc and G is usc it is easily seen from the definition of τ_{D_ε} that $V\big(X_{\tau_{D_\varepsilon}}\big) \le G\big(X_{\tau_{D_\varepsilon}}\big) + \varepsilon$ for all $\varepsilon > 0$. Letting $\varepsilon \downarrow 0$ and using that X is left-continuous over stopping times it follows that $V(X_{\tau_0}) \le G(X_{\tau_0})$ since V is lsc and G is usc. This shows that $V(X_{\tau_0}) = G(X_{\tau_0})$ and therefore $\tau_D \le \tau_0$ showing that $\tau_0 = \tau_D$. Thus $\tau_{D_\varepsilon} \uparrow \tau_D$ as $\varepsilon \downarrow 0$.

Making use of the latter fact in (2.2.34) upon letting $\varepsilon \downarrow 0$ and applying Fatou's lemma, we get

$$\begin{aligned} V(x) &\le \limsup_{\varepsilon \downarrow 0} \mathsf{E}_x G\big(X_{\tau_{D_\varepsilon}}\big) \le \mathsf{E}_x \limsup_{\varepsilon \downarrow 0} G\big(X_{\tau_{D_\varepsilon}}\big) \\ &\le \mathsf{E}_x G\Big(\limsup_{\varepsilon \downarrow 0} X_{\tau_{D_\varepsilon}}\Big) = \mathsf{E}_x G(X_{\tau_D}) \end{aligned} \tag{2.2.46}$$

using that G is usc. This shows that τ_D is optimal in the case when G is bounded.

Second proof. We will divide the second proof in two parts depending on if G is bounded (from below) or not.

1°. Let us assume that G is *bounded from below*. It means that $c := \inf_{x\in E} G(x) > -\infty$. Replacing G by $G - c$ and $\widehat{V}$ by $\widehat{V} - c$ when $c < 0$ we see that there is no restriction to assume that $G(x) \ge 0$ for all $x \in E$.

By analogy with (2.2.31) and (2.2.32), with $0 < \lambda < 1$ given and fixed, consider the sets

$$C_\lambda = \{\, x \in E : \lambda \widehat{V}(x) > G(x)\,\}, \tag{2.2.47}$$

$$D_\lambda = \{\, x \in E : \lambda \widehat{V}(x) \le G(x)\,\}. \tag{2.2.48}$$

Since $\widehat{V}$ is lsc and G is usc we see that C_λ is open and D is closed. Moreover, it is clear that $C_\lambda \uparrow C$ and $D_\lambda \downarrow D$ as $\lambda \uparrow 1$ where C and D are defined by (2.2.4) and (2.2.5) above respectively.

Define the stopping time

$$\tau_{D_\lambda} = \inf\{\, t \ge 0 : X_t \in D_\lambda \,\}. \tag{2.2.49}$$

Since $D \subseteq D_\lambda$ and $\mathsf{P}_x(\tau_D < \infty) = 1$ for all $x \in E$, we see that $\mathsf{P}_x(\tau_{D_\lambda} < \infty) = 1$ for all $x \in E$. (The latter fact can also be derived directly as in the remark following (2.2.33) above.)

In order to show that

$$\mathsf{E}_x \widehat{V}\big(X_{\tau_{D_\lambda}}\big) = \widehat{V}(x) \tag{2.2.50}$$

for all $x \in E$, we will first note that

$$G(x) \le \lambda \widehat{V}(x) + (1-\lambda)\, \mathsf{E}_x \widehat{V}\big(X_{\tau_{D_\lambda}}\big) \tag{2.2.51}$$

for all $x \in E$. Indeed, if $x \in C_\lambda$ then $G(x) < \lambda \widehat{V}(x) \le \lambda \widehat{V}(x) + (1-\lambda) \mathsf{E}_x \widehat{V}\big(X_{\tau_{D_\lambda}}\big)$ since $\widehat{V} \ge G \ge 0$ on E. On the other hand, if $x \in D_\lambda$ then $\tau_{D_\lambda} \equiv 0$ and (2.2.51) follows since $G \le \widehat{V}$ on E.

Next in exactly the same way as in (2.2.38) above one verifies that the function

$$x \mapsto \mathsf{E}_x \widehat{V}\big(X_{\tau_{D_\lambda}}\big) \quad \text{is superharmonic} \tag{2.2.52}$$

from E to $\mathbb{R}$. Hence the function on the right-hand side of(2.2.51) is superharmonic so that by the definition of V we can conclude that

$$\widehat{V}(x) \le \lambda \widehat{V}(x) + (1-\lambda)\, \mathsf{E}_x \widehat{V}\big(X_{\tau_{D_\lambda}}\big) \tag{2.2.53}$$

for all $x \in E$. This proves (2.2.50) as claimed.

From (2.2.50) we get

$$\widehat{V}(x) = \mathsf{E}_x \widehat{V}\big(X_{\tau_{D_\lambda}}\big) \le \frac{1}{\lambda}\, \mathsf{E}_x G\big(X_{\tau_{D_\lambda}}\big) \le \frac{1}{\lambda}\, V(x) \tag{2.2.54}$$

for all $x \in E$ upon using that $\widehat{V}\big(X_{\tau_{D_\lambda}}\big) \le (1/\lambda)\, G\big(X_{\tau_{D_\lambda}}\big)$ since $\widehat{V}$ is lsc and G is usc. Letting $\lambda \uparrow 1$ in (2.2.54) we see that $\widehat{V} \le V$ and thus by (2.2.30) we can conclude that $\widehat{V} = V$. From (2.2.54) we thus see that

$$V(x) \le \frac{1}{\lambda}\, \mathsf{E}_x G\big(X_{\tau_{D_\lambda}}\big) \tag{2.2.55}$$

for all $x \in E$ and all $0 \le \lambda < 1$.

Letting $\lambda \uparrow 1$ and using that $D_\lambda \downarrow D$ we see that $\tau_{D_\lambda} \uparrow \tau_1$ where τ_1 is a stopping time satisfying $\tau_1 \le \tau_D$. Since V is lsc and G is usc it is easily seen from the definition of τ_{D_λ} that $V(\tau_{D_\lambda}) \le (1/\lambda)\, G(\tau_{D_\lambda})$ for all $0 < \lambda < 1$. Letting $\lambda \uparrow 1$ and using that X is left-continuous over stopping times it follows that $V(X_{\tau_1}) \le G(X_{\tau_1})$ since V is lsc and G is usc. This shows that $V(X_{\tau_1}) = G(X_{\tau_1})$ and therefore $\tau_D \le \tau_1$ showing that $\tau_1 = \tau_D$. Thus $\tau_{D_\lambda} \uparrow \tau_1$ as $\lambda \uparrow 1$.

Making use of the latter fact in (2.2.55) upon letting $\lambda \uparrow 1$ and applying Fatou's lemma, we get

$$\begin{aligned} V(x) &\le \limsup_{\lambda \uparrow 1} \mathsf{E}_x G\big(X_{\tau_{D_\lambda}}\big) \le \mathsf{E}_x \limsup_{\lambda \uparrow 1} G\big(X_{\tau_{D_\lambda}}\big) \\ &\le \mathsf{E}_x G\Big(\limsup_{\lambda \uparrow 1} X_{\tau_{D_\lambda}}\Big) = \mathsf{E}_x G(X_{\tau_D}) \end{aligned} \tag{2.2.56}$$

using that G is usc. This shows that τ_D is optimal in the case when G is bounded from below.

2°. Let us assume that G is a (general) measurable function satisfying (2.2.1) (i.e. not necessarily bounded or bounded from below). Then Part 1° of the proof can be extended by means of the function $h : E \to \mathbb{R}$ defined by

$$h(x) = \mathsf{E}_x\Big(\inf_{t\geq 0} G(X_t)\Big) \tag{2.2.57}$$

for $x \in E$. The key observation is that $-h$ is superharmonic which is seen as follows (recall (2.2.57)):

$$\begin{aligned}\mathsf{E}_x(-h(X_\sigma)) &= \mathsf{E}_x\mathsf{E}_{X_\sigma}\sup_{t\geq 0}(-G(X_t)) = \mathsf{E}_x\mathsf{E}_x\Big(\sup_{t\geq 0}(-G(X_t))\circ\theta_\sigma \mid \mathcal{F}_\sigma\Big) \\ &= \mathsf{E}_x\mathsf{E}_x\Big(\sup_{t\geq 0}(-G(X_{\sigma+t}))\Big) \leq -h(x)\end{aligned} \tag{2.2.58}$$

for all $x \in E$ proving the claim. Moreover, it is obvious that $\widehat{V} - h \geq G - h \geq 0$ on E. Knowing this we can define sets C_λ and D_λ by extending (2.2.47) and (2.2.48) as follows:

$$C_\lambda = \big\{\, x \in E : \lambda\big(\widehat{V}(x) - h(x)\big) > G(x) - h(x) \,\big\} \tag{2.2.59}$$

$$D_\lambda = \big\{\, x \in E : \lambda\big(\widehat{V}(x) - h(x)\big) \leq G(x) - h(x) \,\big\} \tag{2.2.60}$$

for $0 < \lambda < 1$.

We then claim that

$$G(x) - h(x) \leq \lambda\big(\widehat{V}(x) - h(x)\big) + (1-\lambda)\,\mathsf{E}_x\big(\widehat{V}(X_{\tau_{D_\lambda}}) - h(X_{\tau_{D_\lambda}})\big) \tag{2.2.61}$$

for all $x \in E$. Indeed, if $x \in C_\lambda$ then (2.2.61) follows by the fact that $\widehat{V} \geq h$ on E. On the other hand, if $x \in D_\lambda$ then $\tau_{D_\lambda} = 0$ and the inequality (2.2.61) reduces to the trivial inequality that $G \leq \widehat{V}$. Thus (2.2.61) holds as claimed.

Since $-h$ is superharmonic we have

$$-h(x) \geq -\lambda h(x) + (1-\lambda)\,\mathsf{E}_x\big(-h(X_{\tau_{D_\lambda}})\big) \tag{2.2.62}$$

for all $x \in E$. From (2.2.61) and (2.2.62) we see that

$$G(x) \leq \lambda\widehat{V}(x) + (1-\lambda)\,\mathsf{E}_x\widehat{V}\big(X_{\tau_{D_\lambda}}\big) \tag{2.2.63}$$

for all $x \in E$. Upon noting that $D_\lambda \downarrow D$ as $\lambda \uparrow 1$ the rest of the proof can be carried out in exactly the same way as in Part 1° above. (If h does not happen to be lsc, then C_λ and D_λ are still measurable sets and thus τ_{D_λ} is a stopping time (with respect to the completion of $(\mathcal{F}_t^X)_{t\geq 0}$ by the family of all P_x-null sets from $\mathcal{F}_\infty^X$ for $x \in E$). Moreover, it is easily verified using the strong Markov property of X and the conditional Fatou lemma that

$$h(X_{\tau_D}) \le \limsup_{\lambda \downarrow 0} h\big(X_{\tau_{D_\lambda}}\big) \quad \mathsf{P}_x\text{-a.s.} \tag{2.2.64}$$

for all $x \in E$, which is sufficient for the proof.)

The final claim of the theorem follows from (2.2.11) in Theorem 2.4 above. The proof is complete. □

Remark 2.8. The result and proof of Theorem 2.7 above extend in exactly the same form (by slightly changing the notation only) to the finite horizon problem (2.2.2′). Note moreover in this case that $\tau_D \le T < \infty$ (since $V(T,x) = G(T,x)$ and thus $(T,x) \in D$ for all $x \in E$) so that the condition $\mathsf{P}_x(\tau_D < \infty) = 1$ is automatically satisfied for all $x \in E$ and need not be assumed.

9. The following corollary is an elegant tool for tackling the optimal stopping problem in the case when one can prove *directly* from definition of V that V is lsc. Note that the result is particularly useful in the case of finite horizon since it provides the existence of an optimal stopping time τ_* by simply identifying it with τ_D from (2.2.6) above.

Corollary 2.9. (The existence of an optimal stopping time)

Infinite horizon. *Consider the optimal stopping problem* (2.2.3) *upon assuming that the condition* (2.2.1) *is satisfied. Suppose that* V *is lsc and* G *is usc. If* $\mathsf{P}_x(\tau_D < \infty) = 1$ *for all* $x \in E$, *then* τ_D *is optimal in* (2.2.3). *If* $\mathsf{P}_x(\tau_D < \infty) < 1$ *for some* $x \in E$, *then there is no optimal stopping time* (*with probability* 1) *in* (2.2.3).

Finite horizon. *Consider the optimal stopping problem* (2.2.2′) *upon assuming that the corresponding condition* (2.2.1) *is satisfied. Suppose that* V *is lsc and* G *is usc. Then* τ_D *is optimal in* (2.2.2′).

Proof. The case of finite horizon can be proved in exactly the same way as the case of infinite horizon if the process (X_t) is replaced by the process (t, X_t) for $t \ge 0$. A proof in the case of infinite horizon can be given as follows.

The key is to show that V is superharmonic. For this, note that V is measurable (since it is lsc) and thus so is the mapping

$$V(X_\sigma) = \sup_\tau \mathsf{E}_{X_\sigma} G(X_\tau) \tag{2.2.65}$$

for any stopping time σ which is given and fixed. On the other hand, by the strong Markov property we have

$$\mathsf{E}_{X_\sigma} G(X_\tau) = \mathsf{E}_x\big(G(X_{\sigma+\tau\circ\theta_\sigma}) \mid \mathcal{F}_\sigma\big) \tag{2.2.66}$$

for every stopping time τ and $x \in E$. From (2.2.65) and (2.2.66) we see that

$$V(X_\sigma) = \operatorname*{ess\,sup}_\tau \mathsf{E}_x\big(G(X_{\sigma+\tau\circ\theta_\sigma}) \,|\, \mathcal{F}_\sigma\big) \tag{2.2.67}$$

under P_x where $x \in E$ is given and fixed.

Next we will show that the family

$$\big\{\, \mathsf{E}_x(X_{\sigma+\tau\circ\theta_\sigma} \,|\, \mathcal{F}_\sigma\big) : \tau \text{ is a stopping time} \,\big\} \tag{2.2.68}$$

is upwards directed in the sense of (1.1.25). Indeed, if τ_1 and τ_2 are stopping times given and fixed, set $\rho_1 = \sigma + \tau_1 \circ \theta_\sigma$ and $\rho_2 = \sigma + \tau_2 \circ \theta_\sigma$, and define

$$B = \big\{\, \mathsf{E}_x(X_{\rho_1} \,|\, \mathcal{F}_\sigma) \ge \mathsf{E}_x(X_{\rho_2} \,|\, \mathcal{F}_\sigma) \,\big\}. \tag{2.2.69}$$

Then $B \in \mathcal{F}_\sigma$ and the mapping

$$\rho = \rho_1 I_B + \rho_2 I_{B^c} \tag{2.2.70}$$

is a stopping time. To verify this let us note that $\{\rho \le t\} = (\{\rho_1 \le t\} \cap B) \cup (\{\rho_2 \le t\} \cap B^c) = (\{\rho_1 \le t\} \cap B \cap \{\sigma \le t\}) \cup (\{\rho_2 \le t\} \cap B^c \cap \{\sigma \le t\}) \in \mathcal{F}_t$ since B and B^c belong to $\mathcal{F}_\sigma$ proving the claim. Moreover, the stopping time ρ can be written as

$$\rho = \sigma + \tau \circ \theta_\sigma \tag{2.2.71}$$

for some stopping time τ. Indeed, setting

$$A = \big\{\, \mathsf{E}_{X_0} G(X_{\tau_1}) \ge \mathsf{E}_{X_0} G(X_{\tau_2}) \,\big\} \tag{2.2.72}$$

we see that $A \in \mathcal{F}_0$ and $B = \theta_\sigma^{-1}(A)$ upon recalling (2.2.66). Hence from (2.2.70) we get

$$\begin{aligned} \rho &= (\sigma + \tau_1 \circ \theta_\sigma) I_B + (\sigma + \tau_2 \circ \theta_\sigma) I_{B^c} \\ &= \sigma + \big((\tau_1 \circ \theta_\sigma)(I_A \circ \theta_\sigma) + (\tau_2 \circ \theta_\sigma)(I_{A^c} \circ \theta_\sigma)\big) \\ &= \sigma + (\tau_1 I_A + \tau_2 I_{A^c}) \circ \theta_\sigma \end{aligned} \tag{2.2.73}$$

which implies that (2.2.71) holds with the stopping time $\tau = \tau_1 I_A + \tau_2 I_{A^c}$. (The latter is a stopping time since $\{\tau \le t\} = (\{\tau_1 \le t\} \cap A) \cup (\{\tau_2 \le t\} \cap A^c) \in \mathcal{F}_t$ for all $t \ge 0$ due to the fact that $A \in \mathcal{F}_0 \subseteq \mathcal{F}_t$ for all $t \ge 0$.) Finally, we have

$$\begin{aligned} \mathsf{E}(X_\rho \,|\, \mathcal{F}_\sigma) &= \mathsf{E}(X_{\rho_1} \,|\, \mathcal{F}_\sigma) I_B + \mathsf{E}(X_{\rho_2} \,|\, \mathcal{F}_\sigma) I_{B^c} \\ &= \mathsf{E}(X_{\rho_1} \,|\, \mathcal{F}_\sigma) \vee \mathsf{E}(X_{\rho_2} \,|\, \mathcal{F}_\sigma) \end{aligned} \tag{2.2.74}$$

proving that the family (2.2.68) is upwards directed as claimed.

From the latter using (1.1.25) and (1.1.26) we can conclude that there exists a sequence of stopping times $\{\tau_n : n \ge 1\}$ such that

$$V(X_\sigma) = \lim_{n\to\infty} \mathsf{E}_x\big(G(X_{\sigma+\tau_n\circ\theta_\sigma}) \,|\, \mathcal{F}_\sigma\big) \tag{2.2.75}$$

where the sequence $\{\, \mathsf{E}_x\big(G(X_{\sigma+\tau_n\circ\theta_\sigma})\,|\,\mathcal{F}_\sigma\big) : n \geq 1 \,\}$ is increasing P_x-a.s. By the monotone convergence theorem using (2.2.1) above we can therefore conclude

$$\mathsf{E}_x V(X_\sigma) = \lim_{n\to\infty} \mathsf{E}_x G(X_{\sigma+\tau_n\circ\theta_\sigma}) \leq V(x) \tag{2.2.76}$$

for all stopping times σ and all $x \in E$. This proves that V is superharmonic. (Note that the only a priori assumption on V used so far is that V is measurable.) As evidently V is the smallest superharmonic function which dominates G on E (recall (2.2.14) above) we see that the remaining claims of the corollary follow directly from Theorem 2.7 above. This completes the proof. □

Remark 2.10. Note that the assumption of lsc on V and usc on G is natural, since the supremum of lsc functions defines an lsc function, and since every usc function attains its supremum on a compact set. To illustrate the former claim note that if the function

$$x \mapsto \mathsf{E}_x G(X_\tau) \tag{2.2.77}$$

is continuous (or lsc) for every stopping time τ, then $x \mapsto V(x)$ is lsc and the results of Corollary 2.9 are applicable. This yields a powerful existence result by simple means (both in finite and infinite horizon). We will exploit the latter in our study of finite horizon problems in Chapters VI–VIII below. On the other hand, if X is a one-dimensional diffusion, then V is continuous whenever G is measurable (see Subsection 9.3 below). Note finally that if X_t converges to X_∞ as $t \to \infty$ then there is no essential difference between infinite and finite horizon, and the second half of Corollary 2.9 above (Finite horizon) applies in this case as well, no matter if τ_D is finite or not. In the latter case one sees that τ_D is an optimal Markov time (recall Example 1.14 above).

Remark 2.11. Theorems 2.4 and 2.7 above have shown that the optimal stopping problem (2.2.2) is equivalent to the problem of finding the smallest superharmonic function $\widehat{V}$ which dominates G on E. Once $\widehat{V}$ is found it follows that $V = \widehat{V}$ and τ_D from (2.2.6) is optimal (if no obvious contradiction arises).

There are two traditional ways for finding $\widehat{V}$:

(i) *Iterative procedure* (constructive but non-explicit),

(ii) *Free-boundary problem* (explicit or non-explicit).

Note that Corollary 2.9 and Remark 2.10 present yet another way for finding $\widehat{V}$ simply by identifying it with V when the latter is known to be sufficiently regular (lsc).

The book [196, Ch. 3] provides numerous examples of (i) under various conditions on G and X. For example, it is known that if G is lsc and $\mathsf{E}_x \inf_{t\geq 0} G(X_t) > -\infty$ for all $x \in E$, then $\widehat{V}$ can be computed as follows:

$$Q_n G(x) := G(x) \vee \mathsf{E}_x G(X_{1/2^n}), \tag{2.2.78}$$

$$\widehat{V}(x) = \lim_{n\to\infty} \lim_{N\to\infty} Q_n^N G(x) \tag{2.2.79}$$

for $x \in E$ where Q_n^N is the N-th power of Q_n. The method of proof relies upon discretization of the time set $\mathbb{R}_+$ and making use of discrete-time results of optimal stopping reviewed in Subsection 1.2 above. It follows that

$$\text{If } G \text{ is continuous and } X \text{ is a Feller process, then } V \text{ is lsc.} \tag{2.2.80}$$

The present book studies various examples of (ii). The basic idea (following from the results of Theorems 2.4 and 2.7) is that $\widehat{V}$ and C (or D) should solve the free-boundary problem:

$$\mathbb{L}_X \widehat{V} \le 0 \quad (\widehat{V} \text{ minimal}), \tag{2.2.81}$$

$$\widehat{V} \ge G \quad (\widehat{V} > G \text{ on } C \;\;\&\;\; \widehat{V} = G \text{ on } D) \tag{2.2.82}$$

where $\mathbb{L}_X$ is the characteristic (infinitesimal) operator of X (cf. Chapter II below).

Assuming that G is smooth in a neighborhood of ∂C the following "rule of thumb" is valid. If X after starting at ∂C enters immediately into $\operatorname{int}(D)$ (e.g. when X is a diffusion process and ∂C is sufficiently nice) then the condition (2.2.81) (under (2.2.82) above) splits into the two conditions:

$$\mathbb{L}_X \widehat{V} = 0 \quad \text{in } C, \tag{2.2.83}$$

$$\frac{\partial \widehat{V}}{\partial x}\Big|_{\partial C} = \frac{\partial G}{\partial x}\Big|_{\partial C} \quad (\textit{smooth fit}). \tag{2.2.84}$$

On the other hand, if X after starting at ∂C does not enter immediately into $\operatorname{int}(D)$ (e.g. when X has jumps and no diffusion component while ∂C may still be sufficiently nice) then the condition (2.2.81) (under (2.2.82) above) splits into the two conditions:

$$\mathbb{L}_X \widehat{V} = 0 \quad \text{in } C, \tag{2.2.85}$$

$$\widehat{V}\big|_{\partial C} = G\big|_{\partial C} \quad (\textit{continuous fit}). \tag{2.2.86}$$

A more precise meaning of these conditions will be discussed in Chapter IV below (and through numerous examples throughout).

Remark 2.12. (Linear programming) A *linear programming problem* may be defined as the problem of maximizing or minimizing a linear function subject to linear constraints.

Optimal stopping problems may be viewed as linear programming problems (cf. [55, p. 107]). Indeed, we have seen in Theorems 2.4 and 2.7 that the optimal stopping problem (2.2.2) is equivalent to finding the smallest superharmonic function $\hat{V}$ which dominates G on E. Letting L denote the linear space of all superharmonic functions, letting the *constrained set* be defined by $L_G = \{ V \in L : V \ge G \}$, and letting the *objective function* be defined by

$F(V) = V$ for $V \in L$, the optimal stopping problem (2.2.2) is equivalent to the linear programming problem

$$\hat{V} = \inf_{V \in L_G} F(V) . \tag{2.2.87}$$

Clearly, this formulation/interpretation extends to the martingale setting of Section 2.1 (where instead of superharmonic functions we need to deal with supermartingales) as well as to discrete time of both martingale and Markovian settings (Sections 1.1 and 1.2). Likewise, the free-boundary problem (2.2.81)–(2.2.82) may be viewed as a linear programming problem.

A *dual problem* to the *primal problem* (2.2.87) can be obtained using the fact that the first hitting time τ_* of $\hat{S}_t = \hat{V}(X_t)$ to $G_t = G(X_t)$ is optimal, so that

$$\sup_t (G_t - \hat{S}_t) = 0 \tag{2.2.88}$$

since $\hat{S}_t \geq G_t$ for all t. It follows that

$$\inf_S \mathsf{E} \sup_t (G_t - S_t) = 0 \tag{2.2.89}$$

where the infimum is taken over all supermartingales S satisfying $S_t \geq G_t$ for all t. (Note that (2.2.89) holds without the expectation sign as well.) Moreover, the infimum in (2.2.89) can equivalently be taken over all supermartingales S such that $\mathsf{E}S_0 = \mathsf{E}\hat{S}_0$ (where we recall that $\mathsf{E}\hat{S}_0 = \sup_\tau \mathsf{E}G_\tau$). Indeed, this follows since by the supermartingale property we have $\mathsf{E}S_{\tau_*} \leq \mathsf{E}S_0$ so that

$$\mathsf{E} \sup_t (G_t - S_t) \geq \mathsf{E}(G_{\tau_*} - S_{\tau_*}) \geq \mathsf{E}G_{\tau_*} - \mathsf{E}S_0 = \mathsf{E}G_{\tau_*} - \mathsf{E}\hat{S}_0 = 0 . \tag{2.2.90}$$

Finally, since $(\hat{S}_{t \wedge \tau_*})_{t \geq 0}$ is a martingale, we see that (2.2.89) can also be written as

$$\inf_M \mathsf{E} \sup_t (G_t - M_t) = 0 \tag{2.2.91}$$

where the infimum is taken over all martingales M satisfying $\mathsf{E}M_0 = \mathsf{E}\hat{S}_0$. In particular, the latter claim can be rewritten as

$$\sup_\tau \mathsf{E}G_\tau = \inf_M \mathsf{E} \sup_t (G_t - M_t) \tag{2.2.92}$$

where the infimum is taken over all martingales M satisfying $\mathsf{E}M_0 = 0$.

Notes. Optimal stopping problems originated in Wald's sequential analysis [216] representing a method of statistical inference (sequential probability ratio test) where the number of observations is not determined in advance of the experiment (see pp. 1–4 in the book for a historical account). Snell [206] formulated a general optimal stopping problem for discrete-time stochastic processes (sequences), and using the methods suggested in the papers of Wald & Wolfowitz [219] and Arrow, Blackwell & Girshick [5], he characterized the solution by means of the smallest supermartingale (called *Snell's envelope)* dominating the gain sequence. Studies in this direction (often referred to as *martingale methods*) are summarized in [31].

The key equation $V(x) = \max(G(x), \mathsf{E}_x V(X_1))$ was first stated explicitly in [5, p. 219] (see also the footnote on page 214 in [5] and the book [18, p. 253]) but was already characterized implicitly by Wald [216]. It is the simplest equation of "dynamic programming" developed by Bellman (cf. [15], [16]). This equation is often referred to as the *Wald–Bellman equation* (the term which we use too) and it was derived in the text above by a dynamic programming principle of "backward induction". For more details on optimal stopping problems in the discrete-time case see [196, pp. 111–112].

Following initial findings by Wald, Wolfowitz, Arrow, Blackwell and Girshick in discrete time, studies of sequential testing problems for *continuous-time* processes (including Wiener and Poisson processes) was initiated by Dvoretzky, Kiefer & Wolfowitz [51], however, with no advance to optimal stopping theory.

A transparent connection between optimal stopping and free-boundary problems first appeared in the papers by Mikhalevich [135] and [136] where he used the "principle of smooth fit" in an ad hoc manner. In the beginning of the 1960's several authors independently (from each other and from Mikhalevich) also considered free-boundary problems (with "smooth-fit" conditions) for solving various problems in sequential analysis, optimal stopping, and optimal stochastic control. Among them we mention Chernoff [29], Lindley [126], Shiryaev [187], [188], [190], Bather [10], Whittle [222], Breakwell & Chernoff [22] and McKean [133]. While in the papers from the 1940's and 50's the 'stopped' processes were either sums of independent random variables or processes with independent increments, the 'stopped' processes in these papers had a more general Markovian structure.

Dynkin [52] formulated a general optimal stopping problem for Markov processes and characterized the solution by means of the smallest superharmonic function dominating the gain function. Dynkin treated the case of discrete time in detail and indicated that the analogous results also hold in the case of continuous time. (For a connection of these results with Snell's results [206] see the corresponding remark in [52].)

The 1960's and 70's were years of an intensive development of the general theory of optimal stopping both in the "Markovian" and "martingale" setting as well as both in the discrete and continuous time. Among references dealing mainly with continuous time we mention [191], [88], [87], [193], [202], [210], [194], [184], [117], [62], [63], [211], [59], [60], [61], [141]. The book by Shiryaev [196] (see also [195]) provides a detailed presentation of the general theory of optimal stopping in the "Markovian" setting both for discrete and continuous time. The book by Chow, Robbins & Siegmund [31] gives a detailed treatment of optimal stopping problems for general stochastic processes in discrete time using the "martingale" approach. The present Chapter I is largely based on results exposed in these books and other papers quoted above. Further developments of optimal stopping following the 1970's and extending to more recent times will be addressed in the present monograph. Among those not mentioned explicitly below we refer to [105] and [153] for optimal stopping of diffusions, [171] and [139] for diffusions with jumps, [120] and [41] for passage from discrete to continuous time, and [147] for optimal stopping with delayed information. The facts of dual problem (2.2.88)–(2.2.92) were used by a number of authors in a more or less disguised form (see [36], [13], [14], [176], [91], [95]).

Remark on terminology. In general theory of *Markov processes* the term 'stopping time' is less common and one usually prefers the term '*Markov time*' (see e.g. [53]) originating from the fact that the strong Markov property remains preserved for such times. Nevertheless in general theory of stochastic processes, where the strong Markov property is not primary, one mostly uses the term '*stopping*' (or 'optional') time allowing it to take either finite or infinite values. In the present monograph we deal with both Markov processes and processes of general structure, and we are mainly interested in optimal stopping problems for which the *finite* stopping times are of central interest. This led us to use the "combined" terminology reserving the term 'Markov' for all and 'stopping' for finite times (the latter corresponding to "real stopping" before the "end of time").

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