Stochastic modelling and numerical tools around the physics of complex flows

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Abstract

Time: Week 32, August 8-12, 2022

Lectures: 10 x 45 min

Coordinator: Stefan Geiss

Credits: 2 ETCP

Evaluation: Pass/fail based on a homework returned after the course.

Description of the course

Contents of the course

(0) Introduction, motivation

(1) A short introduction to stochastic differential equations (SDE). Existence and uniqueness (strong & weak solutions). Fokker-Planck equation. Old and new examples from transport models.

(2) A visit to the mean field approximation. An introduction to McKean-Vlasov SDEs, motivated by turbulent flow models.

(3) Numerical approximation for SDEs. Time integration schemes for SDEs; Sampling algorithms for McKean-Vlasov SDEs. An introduction to the main numerical analysis tools and results.

If time permits:

(4) Modelling strong interactions between objects immersed in a flow, particles and walls. Introduction to SDEs with boundaries : reflected & confined SDEs.

(5) Introduction to the ergodic theory for SDEs. Stationary phenomena and equilibrium. Fast and slow variables.

Learning outcomes The main objective of these lectures is to give a concise overview of the theory of stochastic differential equations (SDE), as modelling and numerical tools. Starting form the basic properties of SDEs, the lectures will present different aspects of the theory, motivated and illustrated by their use in turbulent transport and its simulation. Stochastic differential equation are used in physics of fluids and in many related engineering approaches for industrial and environmental applications. SDEs' theory and turbulent transport have a long common history. Yet the design of predictive simulation tools for pollutant dispersion, sedimentation in the ocean, or many energy production processes, greatly challenge researches in the field of SDEs and their simulation, combining together all the aspects presented in this course.

Prerequisites Usual notions on measures, integration and probability theory. Brownian motion, Itô's formula, notion on continuous time martingales and Markov processes.

Contents

Notation

- r.v. random variable
- $\|\|$ Euclidean norm of $\mathbb{R}^d$.
- We denote by C_T the space $C([0,T];\mathbb{R}^d)$.
- $C_b^2(\mathbb{R}^d)$ is the space of bounded continuous functions with bounded continuous derivatives uo to the order 2.
- Let $(E,\mathcal{E})$ be a measurable space with its sigma algebra $\mathcal{B}$; $\mathcal{M}^1(E)$ is the set of probability measures on E.
- We will use several bracket symbols which, depending on the context, may denote.
  - a statistical mean $\langle X\rangle$.
  - a duality bracket $\langle f,p\rangle$ in $C_c^\infty,(C_c^\infty)'$, for a distribution p in $(C_c^\infty)'$ and f a test function.
  - For p in L^1_{loc}, the duality bracket L^1_{loc},L^∞;
  - For p in L^2, the scalar product in L^2, Banach space;
  - For p a probability measure on a space E, f bounded and measurable on E, $\langle f,p\rangle = \mathbb{E}[f(X)]$.
- ∂A refers to the boundary of the set A

Chapter I

Introduction and motivation: fluid mechanics and stochastic processes

I.1 Some basic notions of fluid mechanics and the Navier Stokes equations

A substances that flows is called as fluid.

Fluid mechanics is the branch of science that deals with the study of fluids (liquids and gases) in a state of rest or motion.

Fluid mechanics is an important subject of civil engineering, Mechanical and Chemical engineering, Material engineering.

Its various branches are fluid statics, fluid kinematics and fluid dynamics.

Consider an open volume $\mathcal{D}$ in $\mathbb{R}^d$, $d = 2, 3$ generally.

The motion of a fluid in $\mathcal{D}$ is describe by its velocity (as in solid mechanics). But contrary to solid mechanics where velocity have to be known just in few points, in fluid mechanics, predict the behaviour of a flow requires to compute the velocity field $(t, x) \mapsto v(t, x)$ in all points in a given time interval $[0, T]$ and space domain $\mathcal{D}$ occupied by the flow. The way the acceleration at a given point

$$\frac{D}{Dt}v(t, x)$$

depends to other flow quantities is describes by the Navier Stokes equations. Here $\frac{D}{Dt}v$ is the material derivative of the velocity:

$$\frac{D}{Dt}v(t, x) := \frac{\partial}{\partial t}v(t, x) + (v(t, x) \cdot \nabla)v(t, x).$$

I.1.1 The conservation of mass

The first equation that constitute the Navier Stokes system is the conservation of mass. We denote by $(t,x) \mapsto \rho(t,x)$ the mass density of fluid, that can varies in time and space with the flow motion.

Let Vol be an arbitrary open volume in the domain of the flow $\mathcal{D}$ with sufficiently[1] smooth boundary surface ∂Vol to admit an outward pointing unit normal vector field $\partial\text{Vol} \ni s \mapsto n(s)$. Let fix this arbitrary volume constant in time. Then the mass inside this volume is

$$m(t) = \int_{\text{Vol}} \rho(t,x)dx.$$

Th fluid is moving freely across Vol. So, if mass in this small Vol is conserved, the instantaneous rate of change of mass in Vol must be equal to the flux of mass $(\rho v)(t,x)$ across the surface boundary ∂Vol:

$$\frac{d}{dt}m(t) = \frac{d}{dt}\int_{\text{Vol}} \rho(t,x)dx = -\int_{\partial\text{Vol}} \big((\rho v)(t,s)\cdot n(s)\big)ds.$$

Assuming ρ, v sufficiently smooth, assuming n the outward normal to Vol sufficiently smooth, we apply the Divergence Theorem I.B.1, an integration by parts formula that gives :

$$\int_{\text{Vol}} \nabla\cdot(\rho v)(t,x)dx = \int_{\partial\text{Vol}} (\rho v)(t,s)\cdot n(s)ds. \tag{I.1}$$

Inserting this in the conservation identity leads to

$$\int_{\text{Vol}} (\partial_t\rho + \nabla\cdot(\rho v)(t,x))dx = 0.$$

Since Vol is an arbitrary volume, it follows

$$\partial_t\rho + \nabla\cdot(\rho v)(t,x) = 0, \text{for all } (t,x) \in [0,T]\times\mathcal{D}. \tag{I.2}$$

This is the first equation of mathematical fluid dynamics, which is called Continuity equation.

Incompressibility. Assume now, that the flow is incompressible. Incompressible flow refers to a flow in which the material density is constant in time within a fluid parcel (an infinitesimal volume that moves with the flow velocity). This means that

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{\partial\rho}{\partial t} + v\cdot\nabla\rho = 0, \tag{I.3}$$

Introducing this last relation in the Continuity equation leads to

$$\partial_t\rho + \nabla\cdot(\rho v)(t,x) = \partial_t\rho + (v\cdot\nabla\rho)(t,x) + \rho(\nabla\cdot v)(t,x) = \rho(\nabla\cdot v)(t,x) = 0 \tag{I.4}$$

[1] We need to define a normal to the boundary that requires generally C^1 regularity of the surface manifold, see also I.B

In many applications, flows can be considered as incompressible. This hypothesise could vanish depending on (large velocity) or (large) pressure. Let say, that for most environment application, medical application, and a large set of industrial applications flow can be considered incompressible.

Assume also that the flow is homogeneous (composed of one fluid only) and then

$$\rho(t,x) = \rho_0 > 0, \text{for all } (t,x) \in [0,T] \times \mathcal{D}.$$

and the incompressibility condition reduces to

$$(\nabla \cdot v) = \partial_{x_1} v_1 + \partial_{x_2} v_2 + \partial_{x_3} v_3 = 0, \text{for all } (t,x) \in [0,T] \times \mathcal{D}$$

An equivalent statement that implies incompressibility is that the divergence of the flow velocity is zero. Thus, the conservation of mass for an incompressible, homogeneous fluid imposes a constraint on the velocity only.

I.1.2 The momentum equation

Conservation of momentum refers to the formulation of the Newton's second law of motion:

$$\text{net force } = \text{mass } \times \text{ acceleration}$$

Again, this law, easy to apply in solid mechanics is more intricate for continuum mechanics.

Acceleration. is the time derivative of the velocity. But here again, material derivative have to be applied. So acceleration is

$$\frac{\mathrm{D}v}{\mathrm{D}t} \equiv \frac{\partial v}{\partial t} + v \cdot \nabla v.$$

Internal forces. The foundation of fluid mechanics, or continuum mechanics, is the stress principle of Cauchy. The idea of Cauchy on internal contact forces: Internal forces are forces which a fluid exerts on itself in trying to get out of its own way. These include pressure and viscous drag that a fluid element exerts on the adjacent element. The internal forces of a fluid are contact forces, i.e., they act on the surface of the fluid element Vol. Let τ denote this internal force vector, which is called Cauchy stress vector.

Thus, the equation for the conservation of linear momentum is, for an arbitrary constant-in-time volume Vol:

$$\int_{\text{Vol}} \rho_0(\partial_t v + (v\nabla v)(t,x)dx = \int_{\text{Vol}} F_{\text{ext}}(t,x)dx + \int_{\partial\text{Vol}} \boldsymbol{\tau}(t,s)ds. \tag{I.5}$$

Writing $\boldsymbol{\tau} = \mathbb{S}n$, by divergence theorem

$$\int_{\text{Vol}} \rho_0(\partial_t v + (v\nabla v)(t,x)dx = \int_{\text{Vol}} F_{\text{ext}}(t,x)dx + \int_{\text{Vol}} (\nabla \cdot \mathbb{S})(t,x)dx. \tag{I.6}$$

It then turns that the stress tensor $\mathbb{S}$ is decomposing in viscous stress and pressure effect :

$$\mathbb{S} = \mathbb{V} - p\mathbb{I}$$

where $p(t, x)$ is the (scalar) pressure field, and $\mathbb{V}$ is the viscous stress tensor. The minus sign in front of p comes from the fact that the pressure acts in the normal inward direction to the surface, whereas the divergence theorem operates with the outward unit normal.

Friction between fluid particles can only occur if the particles move with different velocities. For this reason, the viscous stress tensor depends on the gradient of the velocity. For reason of symmetry, it depends on the symmetric part of the gradient, the so-called velocity deformation tensor

$$\mathbb{D}(v) = \frac{1}{2}\big(\nabla v + (\nabla v)^t\big).$$

The antisymmetric part

$$\mathbb{O}(v) = \frac{1}{2}\big(\nabla v - (\nabla v)^t\big)$$

summarises the locally rotative motion part.

If the velocity gradients are not too large, one can assume that the dependency $\mathbb{V}$ depends linearly on the gradient, leading to the model :

$$\mathbb{V} = 2\mu\mathbb{D}(v) + (\zeta - \frac{2}{3}\mu)(\nabla \cdot v).$$

where μ and ζ are the first and second order viscosities of the fluid. The viscosity μ is also called dynamic or shear viscosity.

With the incompressibility condition, this reduce to

$$\mathbb{V} = 2\mu\mathbb{D}(v).$$

Moreover

$$\nabla \cdot (\nabla v) = \Delta v + (\nabla v)^t$$

where Δ is the Laplacian operator

$$\Delta = \partial^2_{x_1^2} + \partial^2_{x_3^2} + \partial^2_{x_3^2}.$$

And

$$\nabla \cdot (\nabla v)^t = \nabla(\nabla \cdot v) = 0,$$

by the incompressibility hypothesis. This leads to the

$$\int_{\text{Vol}} \rho_0(\partial_t v + (v\nabla v)(t, x)dx = \int_{\text{Vol}} F_{\text{ext}}(t, x)dx + \int_{\text{Vol}} \nabla p + \mu\Delta v(t, x)dx. \tag{I.7}$$

, we end up with this PDE involving the two state variables, the velocity field $v(t,x)$ as a 3D vector field and the pressure $p(t,x)$

$$
\begin{aligned}
\partial_t v(t,x) + (v\cdot\nabla)v(t,x) + \frac{1}{\rho_0}\nabla p(t,x) &= \nu\Delta v(t,x), \quad (t,x)\in(0.T]\times\mathcal{D},\\
(\nabla\cdot v)(t,x) &= 0, \quad (t,x)\in[0,T]\times\mathcal{D}\\
v(0,x) &= v_0(x), \quad x\in\mathcal{D}.
\end{aligned}
\tag{I.8}
$$

from a purely mathematical point of view, the very important problem of existence and smoothness of the solutions to Equation (1) remains largely unsolved to this day.

visit The Clay Mathematics Institute's Navier–Stokes equation prize

However, theoretical understanding of the solutions to these equations is incomplete. In particular, solutions of the Navier–Stokes equations often include turbulence, which remains one of the greatest unsolved problems in physics, despite its immense importance in science and engineering.

I.2 Complex flows : from laminar flows to turbulent flows

Turbulence is the time-dependent chaotic behaviour observed in many fluid flows, as illustrated in Figures I.1 and I.2

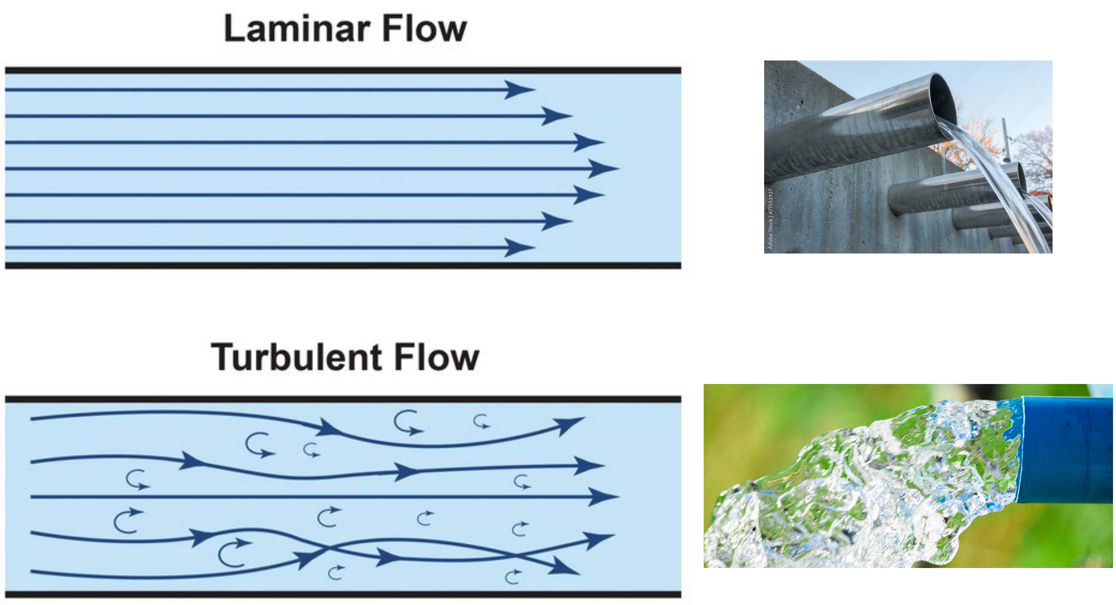


Figure I.1: Typical streamlines in laminar (top) or turbulent flows (bottom).

Figure I.2: The smoke of a candle, laminar at the bottom, turbulent at the top.

To analyse this phenomena, from a physical, mathematical, even numerical point of view, it is important to consider the Navier Stokes equation in a generic, universal way, without referring to a given application problem.

This is done with the help of characteristic quantities of a flow problem, typically

- L [m] – a characteristic length scale of the flow problem,
- U [m/s] – a characteristic velocity scale of the flow problem,
- T^* [s] – a characteristic time scale of the flow problem,

A change of variable is made in equation (I.8). Considering $L = U = T^* = 1$, we stay with (I.8) and we call $Re = \frac{1}{\nu}$ the Reynolds number (or more precisely $Re = \frac{UL}{\nu}$, with the change of variables $x = \frac{\tilde{x}}{L}$, $v = \frac{\tilde{v}}{U}$ and $t = \frac{\tilde{t}}{T^*}$, where the tilded variables are the old ones).

When the Reynolds number is hight turbulence may occur, eventually after a transition.

Reynolds decomposition.

Reynolds decomposition is a mathematical technique used to separate the mean value of a quantity from its fluctuations.

As an example of turbulent phenomena, wind near the surface is a turbulent phenomena. It is a continuous time phenomena, but velocity observation $(U_t^{(i),\text{obs}}, i = 1, 2, 3, t \geq 0)$, are measured at discrete time, as reported in Figure I.3. Here t is incremented each $\frac{1}{10}$ seconds.

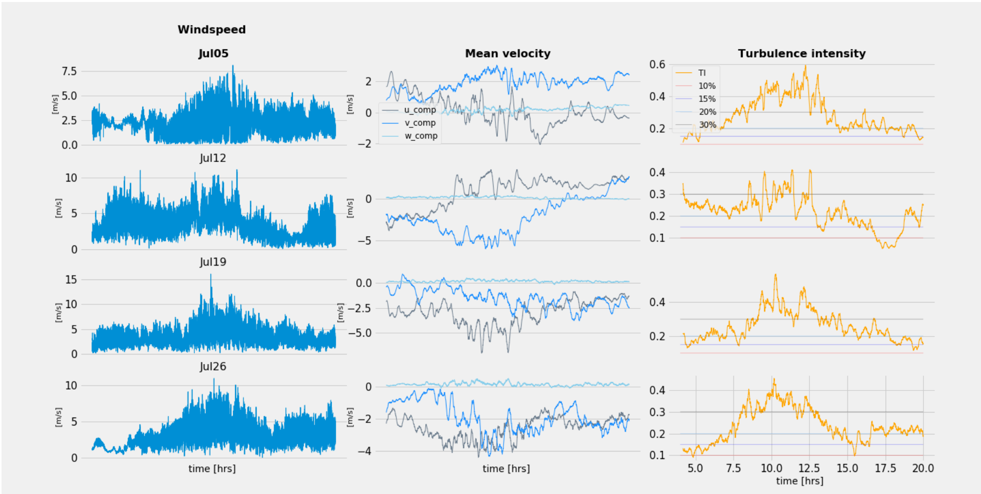


Figure I.3: Time series of the measurements of wind velocity vector, taken at a mast of 30 meters height with a sonic anemometer, obtained from the open observation platform of SIRTA (Site Instrumental de Recherche par Télédétection Atmosphérique. (Haeffelin et al. [Haeffelin et al., 2005])).

In first column, $\|U_t^{\text{obs}}\|$ is the Instantaneous wind speed.

In the middle column $\left\langle U_t^{\text{obs},i} \right\rangle = \frac{1}{\zeta} \sum_{t-\zeta \leq s < t} U_s^{\text{obs}}$ is the mean velocity computed for the same four days and for each wind velocity components.

The last column presents a characteristic quantity to qualify the strongness of turbulence phenomena. It is not necessarily constant in time.

$I_t = \dfrac{\sqrt{\left\langle \| U_t^{\text{obs}} - \left\langle U_t^{\text{obs}} \right\rangle \|^2 \right\rangle}}{\sqrt{3} \| \left\langle U_{(d)}^{\text{obs}} \right\rangle \|}$ is the Turbulent intensity.

In practice, it is very common to compute $\left\langle U_t^{\text{obs}} \right\rangle$ by an average in time over an interval of 10 minutes to 60 minutes, corresponding to a minimum in the wind power spectral density.
Here we choose with the time-window $\zeta = 40$ minutes.

If we consider the solution of the Navier Stokes equation (I.8), the chaos observed in the flow field may be interpreted as randomness. What we interpret as a statistical mean on observation, can be seen on the velocity field $(t,x) \mapsto v(t,x)$ as the introduction of the hidden variable ω, that transforms $(t,x) \mapsto v(t,x)$ to a random field $(t,x) \mapsto v(t,x,w)$, where implicitly we have introduced a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (even if it not well identified). And now the ensemble average operator $\langle \rangle$ is assimilated to the expectation operator under $\mathbb{P}$:

$$\langle v \rangle (t,x) := \int_\Omega v(t,x,\omega) d\mathbb{P}(\omega) = \mathbb{E}[v(t,x)].$$

Reynolds decomposition introduces means and fluctuations:

$$\begin{aligned} v(t,x,\omega) &= \langle v\rangle(t,x) + v'(t,x,\omega), \\ p(t,x,\omega) &= \langle p\rangle(t,x) + p'(t,x,\omega). \end{aligned}$$

The random field $v'(t,x,\omega)$ is named the turbulent velocity, same for the pressure decomposition.

Kolmogorov scales. In his 1941 theory (called K41), Andrey Kolmogorov introduced the idea that the smallest scales of turbulence are universal (similar for every turbulent flow) and that they depend only on ν the already introduced kinematic viscosity, and on the dissipation ε :

$$\begin{aligned} \partial_t\langle v\rangle(t,x) + \langle v\rangle\cdot\nabla\langle v\rangle + \sum_j \partial_{x_j}\langle v'\, v'_j\rangle + \frac{1}{\rho_0}\nabla\langle p\rangle(t,x) = 0, \quad (t,x)\in(0,T]\times\mathcal{D}, \\ \nabla\cdot\langle v\rangle(t,x) = 0, \quad (t,x)\in[0,T]\times\mathcal{D} \\ \langle v\rangle(0,x) = \bar{v}_0(x), \quad x\in\mathcal{D}. \end{aligned}$$

Typically

- Kolmogorov length scale $\eta = \left(\dfrac{\nu^3}{\varepsilon}\right)^{1/4}$.
- Kolmogorov time scale $\tau_\eta = \left(\dfrac{\nu}{\varepsilon}\right)^{1/2}$.
- Kolmogorov velocity scale $u_\eta = (\nu\varepsilon)^{1/4}$.

The dissipation ε is a complex quantity to estimate precisely, but for most of applications, a rough estimation of the Kolmogorov length scale by

$$\varepsilon \simeq \frac{U^3}{L}$$

in engineering application, leads to

$$\eta \simeq \left(\frac{\nu^3 L}{U^3}\right)^{1/4}$$

good enough to get an order of magnitude. This has the following consequence:

For most of industrial or environmental flows, typical values of η are in $[50\mu\mathrm{m}, 1\mathrm{mm}]$.

This has some dramatic implication on the numerical computation of the solution of the Navier Stokes equation.

In a turbulent situation, a numerical method solving the Navier Stokes equation should have a resolution in space and time below the Kolmogorov scales. For $L = 1$ m, a cube of volume $1\ \mathrm{m}^3$

full of air, with $\nu = 1.610^{-5}\mathrm{m}^2/\mathrm{s}$. For a typical velocity of 1 m/s, we need more than $10^{15/4}$ grid points in each 3 directions, so an apparently simple case like that needs 10^{10} grid points! The same story is repeated for the time resolution.

Note that with $Re = \frac{UL}{\nu}$,

$$\eta \simeq \frac{L}{Re^{3/4}}.$$

This is far beyond our computational capabilities. Any engineering problem involving turbulence simply cannot be solved directly with the Navier Stokes equations. One needs to use models to deal with practical problems.

The direct simulation of turbulence by solving the Navier Stokes equations is reserved for a few ideal and simple problems, which today allow to refine the physical understanding of turbulence and thus to refine the models. This is a very active area of research that complements experimental measurements. This called the Direct numerical Simulation approach (DNS).

I.2.1 The Reynolds Navier Stokes equations.

In engineering, a turbulent model delivers statistical information on the flow fields (velocity, pressure, additional scalar like temperature). The computation of the two first moments (mean and variance) f these quantities are already sufficient to manage a lot of engineering issues (aircraft, turbine design, engine, many steps in the energy production processes, civil engineering in general).

Most of the approaches use (directly of indirectly) the Reynolds decomposition operator with a specific mean $\langle\rangle$ choice.

If we assume that the permutation of the mean operator $\langle\rangle$ with de derivatives of the flow fields is allowable, we can apply it to the Navier Stokes equation (I.8), to get the Reynolds Averaged Navier Stokes equations (or RANS equations).

Assuming again incompressibility and constant mass density ρ_0,

$$\begin{aligned}
\partial_t\langle v\rangle(t,x) + \langle v\cdot\nabla v\rangle(t,x) + \frac{1}{\rho_0}\nabla\langle p\rangle(t,x) = \nu\Delta\langle v\rangle(t,x), \quad (t,x)\in(0,T]\times\mathcal{D},\\
\nabla\cdot\langle v\rangle(t,x) = 0, \quad (t,x)\in[0,T]\times\mathcal{D}\\
\langle v\rangle(0,x) = \bar{v}_0(x), \quad x\in\mathcal{D}.
\end{aligned} \tag{I.9}$$

Writing it by component:

$$\begin{aligned}
\partial_t\langle v_i\rangle(t,x) + \langle v\cdot\nabla v_i\rangle(t,x) + \frac{1}{\rho_0}\partial_{x_i}\langle p\rangle(t,x) = \nu\Delta\langle v_i\rangle(t,x), \quad (t,x)\in(0.T]\times\mathcal{D},\\
\sum_i\partial_{x_i}\langle v_i\rangle(t,x) = 0, \quad (t,x)\in[0,T]\times\mathcal{D}\\
\langle v\rangle(0,x) = \bar{v}_0(x), \quad x\in\mathcal{D}.
\end{aligned} \tag{I.10}$$

But also, from the Navier Stokes mass equation we have

$$\sum_i \partial_{x_i}\langle v_i\rangle + \sum_i \partial_{x_i} v'_i = 0$$

which allows to write that the divergence of the fluctuation is also zero:

$$\nabla \cdot v' = \sum_i \partial_{x_i} v'_i = 0.$$

So, since the mean of the fluctuation is zero also, we get

$$\begin{aligned}
\langle v\cdot\nabla v_i\rangle &\overset{\text{by def of div}}{=} \left\langle \sum_j v_j \partial_{x_j} v_i \right\rangle \overset{\text{by } \langle v'\rangle = 0}{=} \sum_j \langle v_j\rangle \partial_{x_j}\langle v_i\rangle + \left\langle \sum_j v'_j \partial_{x_j} v'_i \right\rangle \\
&\overset{\text{added last term is zero by div free}}{=} \sum_j \langle v_j\rangle \partial_{x_j}\langle v_i\rangle + \left\langle \sum_j v'_j \partial_{x_j} v'_i \right\rangle + \left\langle \sum_j \partial_{x_j} v'_j v' i \right\rangle \\
&\overset{\text{making appear the R. Tens.}}{=} \sum_j \langle v_j\rangle \partial_{x_j}\langle v_i\rangle + \sum_j \partial_{x_j}\langle v'_i v'_j\rangle \\
\langle v\cdot\nabla v\rangle &= \langle v\rangle \cdot \langle \nabla v\rangle + \sum_j \partial_{x_j}\langle v'\ v'_j\rangle
\end{aligned}$$

The last term in this equality involves the correlation tensor of the velocity fluctuation

$$\mathcal{R}_{ij} = \langle v'_i v'_j\rangle$$

named the **Reynolds tensor**. This Reynolds tensor constitutes a new set of unknown variables, and the manipulation of the NS equations allows to have also an equation of the **Reynolds stress tensor**

$$\begin{aligned}
&\partial_t\langle v'_i v'_j\rangle + (\langle v\rangle \cdot \nabla\langle v'_i v'_j\rangle) \\
&= -\sum_{k=1}^{3} \partial_{x_k}\langle v'_i v'_j v'_k\rangle \quad \underbrace{-\frac{1}{\rho_0}\langle v'_j \partial_{x_i} p' + v'_i \partial_{x_j} p'\rangle}_{\text{velocity pressure gradient tensor } \Pi_{ij}} \quad + \nu \triangle_x \langle v'_i v'_j\rangle + \tfrac{p'}{\rho_0}\left(\partial_{x_j} v'_i + \partial_{x_i} v'_j\right) \\
&\quad + 2\ \underbrace{\nu \sum_{k=1}^{3} \langle \partial_{x_k} v'_i \partial_{x_k} v'_j\rangle}_{\text{dissipation tensor } \varepsilon_{ij}} \\
&\quad \underbrace{-\sum_{k=1}^{3}\left(\langle v'_i v'_k\rangle \partial_{x_k}\langle v_i\rangle + \langle v'_j v'_k\rangle \partial_{x_k}\langle v_j\rangle\right)}_{\text{turbulence production tensor } \mathcal{P}_{ij}}
\end{aligned}$$

(the second line are the transport terms.)

Most often, models that complete and close the RANS equations work on reduced equation form (typically taking the trace on the tensors) :

The half trace of the Reynold tenor is called the :

$$\text{turbulent kinetic energy } \mathit{tke}(t,x) := \frac{1}{2}\sum_{i=1}^{3}\langle v_i' v_i' \rangle(t,x).$$

Together with the dissipation that we have already introduced

$$\text{pseudo-dissipation} \quad \varepsilon(t,x) := \nu \sum_{i,j=1}^{3} \big\langle \partial_{x_j} v_i' \ \ \partial_{x_j} v_i' \big\rangle(t,x).$$

This quantity is almost always confusingly (and quantitatively) similar to the turbulent dissipation that involves the symmetric part of the turbulent velocity gradient tensor instead of the turbulent velocity gradient tensor itself.

$$\text{dissipation} \quad \varepsilon(t,x) = \tfrac{1}{2}\nu \sum_{i,j=1}^{3} \big\langle (\partial_{x_j} v_i' + \partial_{x_i} v_j')^2 \big\rangle(t,x).$$

Nevertheless there are still new unknown and we need to introduce parametrization (a relation between ε and *tke* to simplify this equation, eventually to eliminate it.

But we will come back to this question with stochastic models framework.

I.3 When stochasticity comes in the story : Lagrangian fluctuation

I.3.1 Fluid particle

A Fluid particle or a fluid parcel is a very small amount of fluid, identifiable throughout its dynamic history while moving with the fluid flow. It is also what is called a Lagrangian tracer.

If we consider that we can solve exactly/precisely the Navier Stokes equation, we can follow the trajectory of a tracer (a particle without mass) carried by the stream. It is a continuous time series $(X_t, V_t, t \in [0,T])$, following the equation :

$$\begin{aligned} dX_t &= V_t\,dt, \quad X_0 \text{ given} \\ V_t &= v(t, X_t), \quad V_0 = v(0, X_0) \end{aligned}$$

Is that possible to describe a process (a_t) such that

$$dV_t = a_t\,dt?$$

The answer to this question depends strongly on the model level of description.

I.3.1.1 Models at the microsclale (DNS) view point.

At this micro level, dV_t should be represented by the Lagrangian acceleration, as material derivative

$$dV_t = \big(\partial_t v(t, X_t) + v(t, X_t) \cdot \nabla v(t, X_t)\big)\, dt.$$

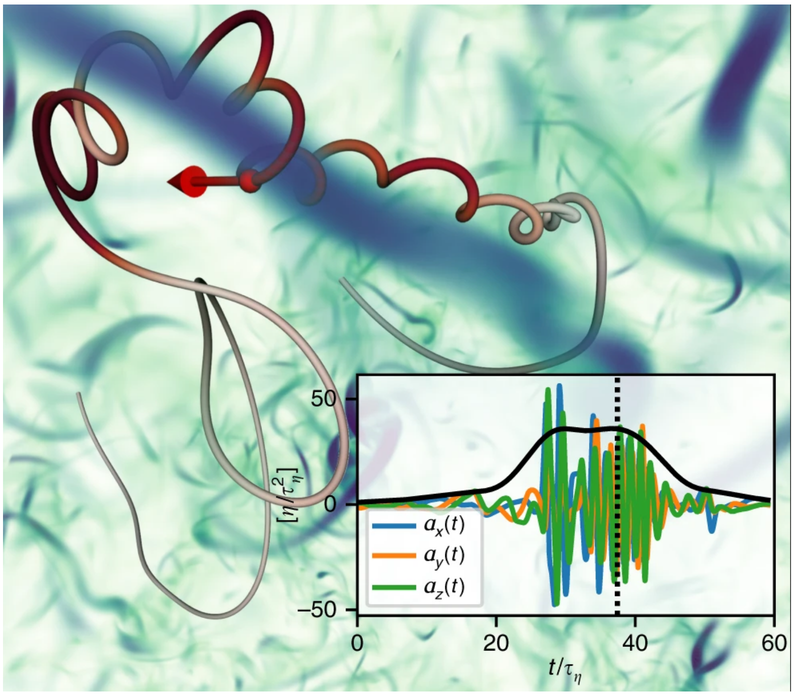


Figure I.4: Tracer particle encountering a vortex filament in turbulence. The tracer trajectory is coloured according to its instantaneous acceleration magnitude, and the blue-green volume-rendering corresponds to the intensity of the vorticity field. The particle acceleration components oscillate strongly in time (inset, in Kolmogorov units) when encountering the intense vortex filament. The root of the squared acceleration, coarse-grained over a few Kolmogorov time scales, varies only weakly during such an event (inset, black curve). The dashed line indicates the instant in time at which the vorticity field is visualized and the tracer is rendered as a sphere. (picture borrowed from [Bentkamp et al., 2019]).

Studying the statistical property of this times series, physicists measure the autocorrelation process of the Lagrangian velocity:

Depending on the complexity of models (on what we can measure), on the hypothesis we can make onto the turbulence phenomena :

- Isotropic turbulence : measured statistics are the same in the three spatial directions; when we consider a flow far from boundaries for example; versus non isotropic turbulence where typically we are close to boundaries that "generate" the chaos in the flow.

  Isotropic turbulence = invariance by rotation of all the statistics

- Homogeneous turbulence = invariance by translation of all the statistics

Assume that we are near to the idealized situation of HIT. Assume that the observed velocity process is stationary in time with mean velocity μ[2]. Then we can define the (temporal) Lagrangian autocorrelation function of the velocity as

$$\tau \mapsto \rho^L(\tau) = \frac{\langle (V_{t+\tau} - \mu) \cdot (V_t - \mu) \rangle}{\langle (V_t - \mu)^2 \rangle}.$$

Remark I.3.1. *Note that in non-isotropic situation, we may measure*

$$\tau \mapsto \rho^L(\tau, ij) = \frac{\left\langle (V^i_{t+\tau} - \mu_i)(V^j_t - \mu_j) \right\rangle}{\sqrt{\left\langle (V^i_{t+\tau} - \mu_i)^2 \right\rangle \left\langle (V^j_{t+\tau} - \mu_j)^2 \right\rangle}}$$

We may assume that the autocorrelation function is going to zero with τ increasing, and that the integral time

$$T_L = \int_0^{+\infty} \rho^L(\tau)\, d\tau$$

is well defined.

Gaussian processes with autocorrelation of the form $\rho^L(\tau) = \exp(-\tau/T_L)$ are well known examples of stochastic Itô diffusion processes:

Considering a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, endowed with a Brownian motion (W_t)

Brownian motion and Ornstein-Uhlenbeck processes. Brownian motion is named after the botanist Robert Brown who in 1827 described the motion of fine particles (pollen) suspended in a fluid. Between Brown's description and the current definition of Brownian motion, this object attracted the attention of physicists such as Einstein and Smoluchowski and mathematicians Wiener, Levy and Itô.

Definition I.3.2. *A real (standard) Brownian motion on $\mathbb{R}^+$ is a process $(W) = (W_t, t \geq 0)$ valued in $\mathbb{R}$ and with continuous trajectories, such that*

- *W_0 = 0.*
- *Any increment $W_t - W_s$ where $0 \leq s < t$, follows a Gaussian distribution of variance $t - s$.*
- *For all $0 = t_0 < t_1 < t_2 < \ldots < t_n$, the increments $\big(W_{t_{i+1}} - W_{t_i}; 0 \leq i \leq n\big)$ are independent.*

At $t \geq 0$ fixed, W_t is a Gaussian distribution v.a. $\mathcal{N}(0, t)$ and

$$\mathbb{P}(W_t \in [x, x + dx]) = p(x)dx = \frac{1}{\sqrt{2\pi t}} \exp(-\frac{x^2}{2t})\, dx.$$

[2] A stationary process is a stochastic process whose joint probability distribution does not change when shifted in time or space.

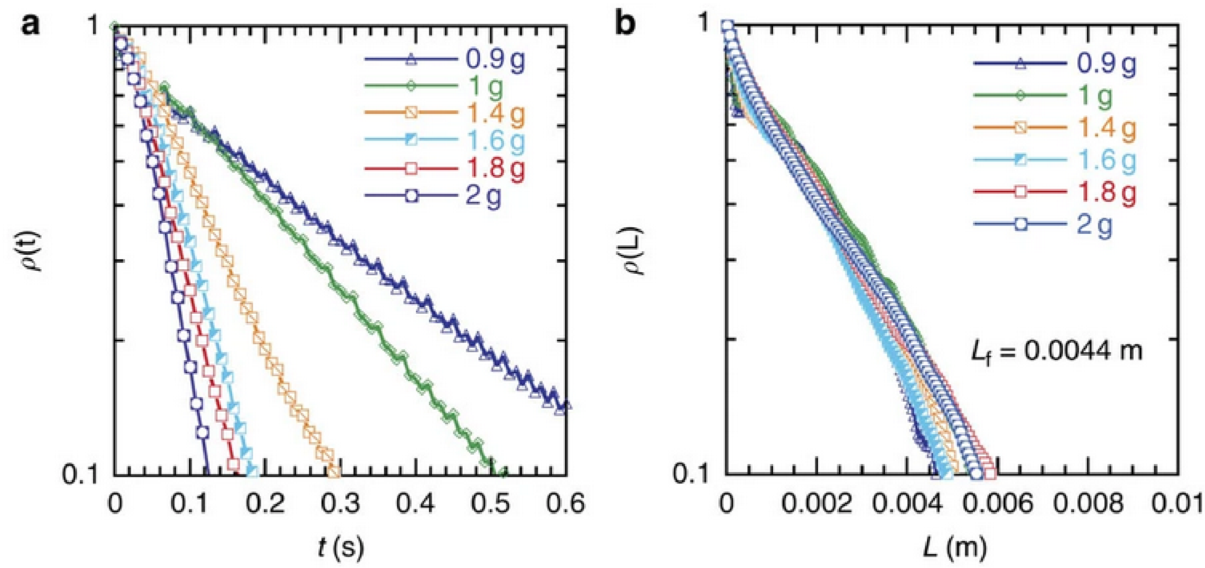


Figure I.5: *(a) Temporal Lagrangian autocorrelation function $\rho^L(\tau)$, computed for a range of forcing levels in the FWT. This function is decaying approximately exponentially, $\rho^L(\tau) \simeq \exp(-\tau/T_L)$, where T_L is Lagrangian integral time. (b) Spatial Lagrangian autocorrelation function $r \mapsto \rho^L(r) = \frac{\left\langle \left(V_t(r_0+r)-\mu\right)\cdot\left(V_t(r_0)-\mu\right)\right\rangle}{\langle (V_t(r_0)-\mu)^2\rangle}$ computed for the same conditions. As the turbulence energy is increased, Lagrangian integral length remains roughly constant in the broad range of forcing levels.*

These are autocorrelation measured for tracers submitted to different levels of (mean) acceleration ”measured in g”, corresponding to levels of energy introduced in the system. The system is assumed at equilibrium (stationary process), with μ denoting $\langle V_t \rangle$.

Picture borrowed from [Francois et al., 2013].

In particular, $W_t \in [-1.96\sqrt{t}, 1.96\sqrt{t}]$ with a probability of 95% (because $\mathbb{P}(|Z| \leq 1.96) \simeq 0.95$ when Z is of Gaussian distribution $\mathcal{N}(0,1)$). It can be shown that this property holds for the whole Brownian trajectory[3].

The existence of a Brownian motion, in the sense of the previous definition, is not obvious at all. See for example Wiener's construction in [Karatzas and Shreve, 1988], or [Revuz and Yor, 1991].

Given the Brownian motion (W), the Ornstein-Uhlenbeck (OU) process on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, is constructed as the unique solution of the following stochastic differential equation :

$$d\xi_t = -\frac{1}{T_L}\xi_t dt + \frac{\sigma}{T_L} dW_t,$$

whose solution writes (using the variation of constants method, applying the Itô Lemma to $f(t,\xi_t) = \xi_t \exp(t/T_L)$)

$$\xi_t = \xi_0 \exp(-\frac{t}{T_L}) + \int_0^t \frac{\sigma}{T_L} \exp(-(t-s)/T_L) dW_s.$$

Using the Itô integral of a deterministic function tip, we deduce that (ξ) is a Gaussian process and

$$\xi_t = \mathcal{N}\left(\xi_0 \exp(-\frac{t}{T_L}), \int_0^t \frac{\sigma^2}{T_L^2} \exp(2(t-s)/T_L) ds = \frac{1}{2}\frac{\sigma^2}{T_L}(1 - \exp(-2\frac{t}{T_L})\right).$$

Assuming ξ_0 independent of (W) and distributed according to

$$\xi_0 \simeq \mathcal{N}(0, \frac{1}{2}\frac{\sigma^2}{T_L}),$$

we immediately get that the process is stationary with

$$\forall t \geq 0, \quad \xi_t \simeq \mathcal{N}(0, \frac{1}{2}\frac{\sigma^2}{T_L}),$$

and

$$\forall t, s \geq 0, \quad \mathbb{E}[\xi_t \, \xi_s] = \frac{1}{2}\frac{\sigma^2}{T_L} \exp\left(-\frac{1}{T_L}(t \vee s - t \wedge s)\right).$$

Finally, we have that $(\xi_\cdot)$ is a good candidate to model velocity autocorrelation with

$$\rho_\xi(\tau) = \mathbb{E}[\xi_\tau \, \xi_0](\frac{1}{2}\frac{\sigma^2}{T_L})^{-1} = \exp\left(-\frac{\tau}{T_L}\right)$$

[3] $\forall t \geq 0$,

$$\mathbb{P}(\sup_{s \leq t} |W_s| \leq 2\sqrt{t}) \simeq 90\%.$$

and a model form (for a scalar V_t the speed velocity, or for one component of the velocity)

$$\begin{aligned} dX_t &= V_t dt \\ V_t &= \langle v\rangle(t, X_t) + V'_t \\ dV'_t &= d\xi_t \end{aligned}$$

Other information are available from the Kolmogorov 1941 Theory: for example, one expects the Lagrangian second-order structure function to behave linearly in both the time scale τ and the mean dissipation rate ε :

$$\left\langle (V'_{t+\tau} - V'_t)^2 \right\rangle = C_0\varepsilon\,\tau.$$

But

$$\begin{aligned} \left\langle (V'_{t+\tau} - V'_t)^2 \right\rangle &= \sigma^2\left\langle (\xi_{t+\tau} - \xi_t)^2 \right\rangle = \sigma(2\left\langle \xi^2 \right\rangle - 2\rho_\xi(\tau)) \\ &= \frac{\sigma^2}{T_L}(1 - \exp(-\frac{\tau}{T_L})) \\ &\simeq \frac{\sigma^2}{T_L^2}\tau + \mathcal{O}((\frac{\tau}{T_L})^2) \end{aligned}$$

So we identify

$$\sigma = \sqrt{C_0\varepsilon}\,T_L$$

to fit the first and second moments of measured statistics.

Behaviour as T_L going to zero. On figure I.5, we can observe that the Lagrangian integral time is decreasing with the "energy/acceleration" introduced in the turbulent system, leading to a more and more decorrelation of the velocity fluctuation.

Considering T_L as a parameter in the equivalent Langevin SDE

$$\begin{aligned} dX_t &= \langle v\rangle(t, X_t)dt + V'_t dt, \quad X_0 \\ dV'_t &= -\frac{1}{T_L}V'_t dt + \sqrt{C_0\varepsilon}dW_t, \quad V'_0 \end{aligned} \tag{I.11}$$

Consider the over-damped Langevin model

$$dY_t = \langle v\rangle(t, Y_t)dt + \sqrt{C_0\varepsilon}T_L dW_t, \quad X_0 \tag{I.12}$$

$$\begin{aligned} V'_t &= \exp(-\frac{t}{T_L})\Big(V'_0 + \int_0^t \sqrt{C_0\varepsilon}\exp(s/T_L)dW_s\Big) \\ X_t &= X_0 + \int_0^t \langle v\rangle(s, X_s)ds + \int_0^t \exp(-\frac{s}{T_L})\Big(V'_0 + \int_0^s \sqrt{C_0\varepsilon}\exp(\theta/T_L)dW_\theta\Big)ds \end{aligned}$$

For convenience, assume some uniform Lipschitz regularity (also uniform in time) on the mean field $x \mapsto \langle v\rangle(t,x)$: there exists a constant $L > 0$, for all y, y, for all $t \geq 0$,

$$\|\langle v\rangle(t,x) - \langle v\rangle(t,y)\| \leq L_{\langle v\rangle}\|x-y\|.$$

This is requiring some regularity from the solution of the Reynolds Navier Stokes equation. It a little bit strong requirement, but such hypotheses can obviously be weakened to only local regularity.

Considering $L^p(\Omega)$ the space of random variables X such that $\mathbb{E}[|X|]^p < +\infty$, we denote $\|X\|_{L^p(\Omega)} = \|X\|_p = (\mathbb{E}[|X|^p])^{\frac{1}{p}}$. Based on simple computations, the following lemma give a first idea of the expected damping convergence.

Lemma I.3.3. *Assume $\langle v\rangle$ Lipschitz uniformly in T_L, with Lipschitz constant $L_{\langle v\rangle}$. Then, for any $\beta \geq 1$, for any $p > 1$,*

$$\left\|\sup_{s\in[0,t]}|X_s - Y_s|\right\|_{2p} \leq T_L\big(\|V_0'\|_{2p} + C(p)\,\sqrt{C_0\varepsilon T_L}\,\big)\,\exp(L_{\langle v\rangle}t). \tag{I.13}$$

This non-asymptotic bound quantifies the approximation of the dynamics (X_t, V_t) by the Brownian one Y_t. The approximation is of first order in T_L as soon as εT_L is bounded.

Proof. Let's consider the difference

$$\begin{aligned}X_t - Y_t &= \int_0^t \big(\langle v\rangle(s,X_s) - \langle v\rangle(s,Y_s)\big)ds\\ &\quad + \int_0^t \exp(-\frac{s}{T_L})\Big(V_0' + \int_0^s \sqrt{C_0\varepsilon}\exp(\theta/T_L)dW_\theta\Big)ds + \sqrt{T_L}\sqrt{C_0\varepsilon}W_t\end{aligned}$$

From Itô formula applied to $(X_t - Y_t)^{2p}$,

$$\begin{aligned}(X_t - Y_t)^{2p} &= \int_0^t (2p)(X_s - Y_s)^{2p-1}\big(\langle v\rangle(s,X_s) - \langle v\rangle(s,Y_s)\big)ds\\ &\quad + \int_0^t (2p)(X_s - Y_s)^{2p-1}\exp(-\frac{s}{T_L})\Big(V_0' + \int_0^s \sqrt{C_0\varepsilon}\exp(\theta/T_L)dW_\theta\Big)ds\\ &\quad + \sqrt{T_L C_0\varepsilon}\int_0^t (2p)(X_s - Y_s)^{2p-1}dW_s\\ &\quad + \frac{1}{2}T_L C_0\varepsilon\int_0^t (2p)(2p-1)(X_s - Y_s)^{2p-2}ds\end{aligned}$$

Taking the $\sup$ in both sides, and applying the Lipschitz condition on $\langle v\rangle$:

$$\begin{aligned}
\sup_{s\le t}(X_s-Y_s)^{2p} \le &\int_0^t (2p)L_{\langle v\rangle}\sup_{\theta\le s}(X_\theta-Y_\theta)^{2p}ds\\
&+\frac{1}{2}C_0\varepsilon\int_0^t(2p)(2p-1)\left(\mathbb{E}[\sup_{r\le s}|X_r-Y_r|^{2p}]\right)^{\frac{2p-2}{2p}}dr\\
&+\int_0^t(2p)\sup_{r\le s}(X_r-Y_r)^{2p-1}\Big(|V_0'|\exp(-\frac{s}{T_L})+\sup_{r\le s}\int_0^r\sqrt{C_0\varepsilon}\exp((\theta-s)/T_L)dW_\theta\Big)ds\\
&+\sqrt{C_0\varepsilon}\sqrt{T_L}\sup_{s\le t}\int_0^s(2p)(X_r-Y_r)^{2p-1}dW_r\\
&+\frac{1}{2}T_LC_0\varepsilon\int_0^t(2p)(2p-1)\sup_{r\le s}|X_r-Y_r|^{2p-2}dr
\end{aligned}$$

We apply now Young inequality[4] in the second term:

$$\begin{aligned}
&\sup_{r\le s}(X_r-Y_r)^{2p-1}\Big(|V_0'|\exp(-\frac{s}{T_L})+\sup_{r\le s}\int_0^r\sqrt{C_0\varepsilon}\exp((\theta-s)/T_L)dW_\theta\Big)\\
&\le\frac{2p-1}{2p}\sup_{r\le s}(X_r-Y_r)^{2p}+\frac{1}{2p}\Big(|V_0'|\exp(-\frac{s}{T_L})+\sup_{r\le s}\int_0^r\sqrt{C_0\varepsilon}\exp((\theta-s)/T_L)dW_\theta\Big)^{2p}\\
&\le\frac{2p-1}{2p}\sup_{r\le s}(X_r-Y_r)^{2p}+\frac{2^{2p-1}}{2p}|V_0'|^{2p}\exp(-\frac{2p\ s}{T_L})+\frac{2^{2p-1}}{2p}\left(\sup_{r\le s}\int_0^r\sqrt{C_0\varepsilon}\exp((\theta-s)/T_L)dW_\theta\right)^{2p}
\end{aligned}$$

We take the expectation in the both sides, and in the last term, we apply Jensen inequality

$$\begin{aligned}
\mathbb{E}[\sup_{s\le t}(X_s-Y_s)^{2p}] = &\int_0^t(2p)\left(L_{\langle v\rangle}+(2p-1)\right)\mathbb{E}[\sup_{\theta\le s}(X_\theta-Y_\theta)^{2p}]ds\\
&+\mathbb{E}[|V_0'|^{2p}]\int_0^t 2^{2p-1}\exp(-\frac{2p\ s}{T_L})\\
&+2^{2p-1}\mathbb{E}[\int_0^t\left(\sup_{r\le s}\int_0^r\sqrt{C_0\varepsilon}\exp((\theta-s)/T_L)dW_\theta\right)^{2p}]\\
&+\sqrt{T_LC_0\varepsilon}\,\mathbb{E}[\sup_{s\le t}\int_0^s(2p)(X_r-Y_r)^{2p-1}dW_r]\\
&+\frac{1}{2}T_LC_0\varepsilon\int_0^t(2p)(2p-1)\left(\mathbb{E}[\sup_{r\le s}|X_r-Y_r|^{2p}\right)^{\frac{2p-2}{2p}}dr
\end{aligned}$$

[4] If $a\ge 0$ and $b\ge 0$ are nonnegative real numbers and if $p>1$ and $q>1$ are conjugate such that $\frac{1}{p}+\frac{1}{q}=1$m, then

$$ab\le\frac{a^p}{p}+\frac{b^q}{q}.$$

In the last term:

$$\left(\mathbb{E}[\sup_{r\leq s}|X_r - Y_r|^{2p}\right)^{\frac{2p-2}{2p}} = \frac{\mathbb{E}[\sup_{r\leq s}|X_r - Y_r|^{2p}}{\|\sup_{r\leq s}|X_r - Y_r|\|_{2p}^2}.$$

In the second term

$$\int_0^t 2^{2p-1}\exp(-\frac{2p\,s}{T_L}) = 2^{2p-1}\frac{T_L}{2p}(1-\exp(-\frac{2p\,t}{T_L}).$$

By the Doob's maximal inequality (see I.C.6),

$$\mathbb{E}\left[\left(\sup_{s\in[0,t]}\left|\int_0^s e^{-(s-r)/T_L}\,dW_r\right|\right)^{2p}\right] \leq (C(2p))^{2p}\mathbb{E}\left[\left(\int_0^t e^{-(t-r)/T_L}\,dW_r\right)^{2p}\right]$$

Let's call $Z_t = \int_0^t e^{-(t-r)/T_L}\,dW_r$. From Itô formula

$$\begin{aligned}\mathbb{E}[Z_t^p] &= \mathbb{E}[\frac{p(p-1)}{2}\int_0^t e^{-2(t-r)/T_L}Z_s^{p-2}ds]\\ &\leq \mathbb{E}[(\sup_{s\leq t}|Z_s|)^{p-2}]\frac{p(p-1)}{2}\frac{T_L}{2}(1-e^{-2t/T_L})\end{aligned}$$

So we obtained that

$$\mathbb{E}[(\sup_{s\leq t}|Z_s|)^p] \leq (C_p)^p\mathbb{E}[(\sup_{s\leq t}|Z_s|)^{p-2}]\frac{p(p-1)}{2}\frac{T_L}{2}(1-e^{-2t/T_L})$$

Applying Jensen inequality (see I.C.1) in the right-hand side

$$\mathbb{E}[(\sup_{s\leq t}|Z_s|)^p] \leq (C_p)^p\left(\mathbb{E}[(\sup_{s\leq t}|Z_s|)^p]\right)^{\frac{p-2}{p}}\frac{p(p-1)}{2}\frac{T_L}{2}(1-e^{-2t/T_L})$$

and thus

$$\begin{aligned}\left(\mathbb{E}[(\sup_{s\leq t}|Z_s|)^p]\right)^{\frac{2}{p}} &\leq (C_p)^p\frac{p(p-1)}{2}\frac{T_L}{2}(1-e^{-2t/T_L})\\ \left\|\sup_{s\leq t}|Z_s|\right\|_{2p} &\leq C_p'\sqrt{T_L}.\end{aligned}$$

Putting all together:

$$\begin{aligned}\mathbb{E}[\sup_{s\leq t}(X_s - Y_s)^{2p}] = &\int_0^t (2p)\left(L_{\langle v\rangle} + (2p-1) + \frac{(2p-1)C_0\varepsilon T_L}{2\|\sup_{r\leq s}|X_r - Y_r|\|_{2p}^2}\right)\mathbb{E}[\sup_{\theta\leq s}(X_\theta - Y_\theta)^{2p}]ds\\ &+ \mathbb{E}[|V_0'|^{2p}]2^{2p-1}\frac{T_L}{2p}(1-\exp(-\frac{2p\,t}{T_L})\\ &+ 2^{2p-1}(\sqrt{C_0\varepsilon})^{2p}C_p'\sqrt{T_L}\\ &+ (2p)\sqrt{T_L C_0\varepsilon}\,\mathbb{E}[\sup_{s\leq t}\int_0^s (X_r - Y_r)^{2p-1}dW_r]\end{aligned}$$

For the last term, we need to use the Burkholder-Davis-Gundy inequality (see (I.C.7)):

$$\mathbb{E}[\sup_{s\leq t}\int_0^s (X_r-Y_r)^{2p-1}dW_r] \leq C_p\mathbb{E}[\sqrt{\int_0^s (X_r-Y_r)^{4p-2}dr}] \leq \sqrt{s}C_p\mathbb{E}[\sup_{r\leq s}(X_r-Y_r)^{2p-1}]$$

So

$$\begin{aligned}
&\mathbb{E}[\sup_{s\leq t}(X_s-Y_s)^{2p}]\\
&=\int_0^t (2p)\left(L_{\langle v\rangle}+(2p-1)+\frac{(2p-1)C_0\varepsilon T_L}{2\|\sup_{r\leq s}|X_r-Y_r|\|_{2p}^2}+\sqrt{s}C_p\frac{(2p)\sqrt{T_L C_0\varepsilon}}{\|\sup_{r\leq s}|X_r-Y_r|\|_{2p}}\right)\\
&\qquad\times\mathbb{E}[\sup_{\theta\leq s}(X_\theta-Y_\theta)^{2p}]ds\\
&\quad+\mathbb{E}[|V_0'|^{2p}]2^{2p-1}\frac{T_L}{2p}(1-\exp(-\frac{2p\,t}{T_L})\\
&\quad+2^{2p-1}(\sqrt{C_0\varepsilon})^{2p}C_p'\sqrt{T_L}.
\end{aligned}$$

We notice that $s\mapsto\|\sup_{r\leq s}|X_r-Y_r|\|_{2p}$ is increasing in time from 0. Thus, there is a deterministic time s_0 such that $\|\sup_{r\leq s}|X_r-Y_r|\|_{2p}>\sqrt{T_L}$, for $s\geq s_0$. From that s_0 to T, we apply the Gronwall's inequality to end the proof.[5] □

Exercise I.1. *Assume that the mean velocity field $(t,x)\mapsto\langle v\rangle(t,x)$ is conserved along the Lagrangian trajectories $\frac{d}{dt}\langle v\rangle(t,X_t)=0$. Show then that equation (I.11) is equivalent to*

$$\begin{aligned}
dX_t &= V_t\,dt, \quad X_0\\
dV_t &= \frac{1}{T_L}\left(\langle v\rangle(t,X_t)-V_t\right)dt+\sqrt{C_0\varepsilon}dW_t, \quad V_0 \qquad \text{(I.14)}\\
\textit{with } & \textit{law}(X_0,V_0)=\mu
\end{aligned}$$

5

Lemma I.3.4. *Gronwall inequality. Let $t\mapsto g(t)$ a continuous function on $[0,T]$ with T arbitrary. Let us assume that*

$$0\leq g(t)\leq\alpha(t)+\beta\int_0^t g(s)ds$$

with $\beta\geq 0$ and $t\mapsto\alpha(t)$ in $L^1(0,T)$. Then for all $t\in[0,T]$

$$g(t)\leq\alpha(t)+\beta\int_0^t\alpha(s)\exp((t-s)\beta)ds.$$

When $\alpha(\cdot)\in C_b^1([0,T])$, we also have

$$g(t)\leq\alpha(0)\exp(\beta t)+\int_0^t\alpha'(s)\exp((t-s)\beta)ds.$$

Not just jargon. Analysis/modelling tools borrow a lot to signal theory. Autocorrelation function explores the process behaviour in time, while its power spectral density (PSD) explores the process in term of frequency.

$$S_\xi(\omega) = \int_{-\infty}^{+\infty} \langle \xi_\tau \xi_0 \rangle \exp(-i2\pi\omega\tau)\, d\tau$$

A white noise is a random stationary process (ξ_t) such that

$$\langle \xi_t \xi_s \rangle = D\delta_{|t-s|}$$

In particular, its spectral density is constant

$$S_\xi(\omega) = 2\int_{0}^{+\infty} \langle \xi_\tau \xi_0 \rangle \exp(-i2\pi\omega\tau)\, d\tau = 2D$$

A coloured noise is a random stationary process (ξ_t) such that

$$\langle \xi_t \xi_s \rangle = \frac{D}{T}\exp(-|t-s|/T)$$

with noise correlation time T, and with PSD

$$S_\xi(\omega) = \frac{2D}{T^2\omega^2 + 1}.$$

In particular coloured noise tends to white when T tends to zero.

We just manipulate the OU process

$$\xi_t = \mathcal{N}\left(\xi_0 \exp(-\frac{t}{T_L}), \int_0^t \frac{\sigma^2}{T_L^2}\exp(2(t-s)/T_L)ds = \frac{1}{2}\frac{\sigma^2}{T_L}(1-\exp(-2\frac{t}{T_L})\right).$$

with $\xi_0 \simeq \mathcal{N}(0, \frac{1}{2}\frac{\sigma^2}{T_L})$.

We just proved in the above lemma that the primitive $\int_0^t \xi_s ds$ converges strongly to σW_t. From this one can say that a Gaussian white noise process can be constructed as the time "derivative" of Brownian motion. However, from the mathematical point of view, it does not exist in the usual sense since with the probability one the trajectories of Brownian motion are nowhere differentiable.

A generalized notion of derivative have to be introduced, together with notion of generalized random process, in a way the fractional Brownian motion is defined (see e.g. the short notes [Zinde-Walsh and Phillips, 2003]).

A mathematical construction of white noise was given firstly by N. Wiener (1924).

Itô integrals versus Stratonovich integrals. Let's go around the nature of the noise entering the fluctuation models. From the point of view of the Navier Stokes equations, the time irregularity of the noise is not "necessary" in the discussion.

As we saw with the convergence of coloured noise to white noise, in such model discussion, one could quite easily work with a time regularised noise, such as any smoothing of the Brownian motion having a time continuous derivative (W_t^ε) such that for almost all $\omega \in \Omega$,

$$W_t^\varepsilon \longrightarrow W_t, \quad \text{in } [0,T] \text{ almost surely as } \varepsilon \text{ goes to zero.}$$

Then with classical Lipschitz conditions on real valued coefficients $x \mapsto b(x)$, $x \mapsto \sigma(x)$, and $x \mapsto \sigma'(x)$, the process $(X_t^\varepsilon, t \in [0,T])$ solution of

$$\frac{dX_t^\varepsilon}{dt} = b(X_t^\varepsilon) + \sigma(X_t^\varepsilon)\frac{dW_t^\varepsilon}{dt}$$

converges to $(X_t, t \in [0,T])$ almost surely as ε goes to zero, where

$$dX_t = b(X_t)dt + \frac{1}{2}\sigma(X_t)\sigma'(X_t)dt + \sigma(X_t^\varepsilon)dW_t$$

This is detailed in the famous theorem on the convergence of ordinary integrals to stochastic integrals first proved by [Wong and Zakai, 1965].

Another way of putting it is to say that $(X_t^\varepsilon, t \in [0,T])$ converges to $(X_t, t \in [0,T])$ almost surely as ε goes to zero, where

$$dX_t = b(X_t)dt + \sigma(X_t^\varepsilon) \circ dW_t$$

where $\int_0^t \sigma(X_s^\varepsilon) \circ dW_s$ (also denoted $\int_0^t \sigma(X_s^\varepsilon)\partial W_s$) is the Stratonovich integral (or Fisk-Stratonovich integral).

Stratonovich integral $\int_0^T \Phi_t \circ \mathrm{d}W_t$ is a random variable, defined as a limit in mean square of

$$\sum_{i=0}^{k-1} \frac{\Phi_{t_{i+1}} + \Phi_{t_i}}{2} \left(W_{t_{i+1}} - W_{t_i}\right)$$

as the partition $0 = t_0 < t_1 < \cdots < t_k = T$ of $[0,T]$ tends to 0 (in the style of a Riemann–Stieltjes integral).

Two important remarks on Stratonovich integral :

Remark I.3.5.

* *Stratonovich integral does not inherit of Brownian motion properties. In particular this integral loses the martingale property of the ito integral.*

* *On the other hand, the Stratonovich integral preserves the chain rule : for*

$$Y_t = f(X_t)$$

with f smooth enough

$$dY_t = f'(X_t) \circ dX_t.$$

In the previous discussion, the introduction of a noise model of a Brownian nature is therefore more natural with the Stratonovich integral, than with the Itô integral.

The hypothesis of a homogeneous isotropic turbulence (ε spatially homogeneous) allows to pass from one notion of integral to the other without worrying about the correction term, but this is no longer the case when the considered turbulence is less homogeneous (for example because of the presence of a wall boundary).

I.3.1.2 Model for turbulent closure : Macroscale view point and PDF approach, Fokker Planck equation

Fokker-Planck equation The Fokker–Planck (FK) equation is a partial differential equation that describes the time evolution of the probability density function of a Markovian process that possesses an infinitesimal operator.

FK equation is also called the Komogorov forward PDE, while the Kolmogorov backward PDE is related the Feymann Kac formula. These two connexion of Markov processes theory to PDE theory are very usefull tools for both analysis and numerical analysis.

On a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), \mathbb{P})$ endowed with a $\mathcal{F}_t$-adapted Brownian motion W, of dimension r (which we will always assume $r \leq d$ even if that means that we increase the state variables) and with a r.v X_0 $\mathcal{F}_0$-measurable, of law P_0, we consider the SDE

$$\begin{cases} X_t = X_0 + \int_0^t \sigma(t, X_s) dW_s + \int_0^t b(t, X_s) ds, \ 0 \leq t \leq T, \\ P_t = \mathbb{P} \circ X_t^{-1}. \end{cases} \tag{I.15}$$

Definition I.3.6. – Weak solution. *Let $b(t,x) : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^+ \times \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^r; \mathbb{R}^d)$ measurable. A weak solution to the SDE*

$$X_t = X_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s, \ 0 \leq t \leq T \tag{I.16}$$

is a triplet composed of

1) a filtered probability space $(\Omega, \mathcal{F}, \mathcal{P}, (\mathcal{F}_t))$ with a filtration $(\mathcal{F}_t)$ satisfying the usual conditions[a]

2) a $(\mathcal{F}_t)-$ Brownian motion valued in $\mathbb{R}^r$.

3) a process $(X_t, t \geq 0)$, $(\mathcal{F}_t)-$ adapted, continuous, valued in $\mathbb{R}^d$, such that

$$\int_0^T \left(|b| + |\sigma \sigma^t|^2 \right) (s, X_s) ds < +\infty, \ \mathbb{P} \ p.s.$$

and such that (I.16) is satisfied $\mathbb{P}$-p.s.

[a] $\mathcal{F}_0 \ni \mathcal{N}$ – the set of negliables of $\mathcal{F}$, and for all $t \geq 0$, $\mathcal{F}_t = \bigcap_{\epsilon > 0} \mathcal{F}_{t+\epsilon}$.

The martingale problem due to Stroock and Varadhan [Stroock and Varadhan, 1969] provides another way to define a solution of a stochastic differential equation. We denote by C_T the space $C([0,T];\mathbb{R}^d)$.

Let $(\mathcal{L}_t, t \geq 0)$ the infinitesimal generator associated to the SDE (I.16) with coefficients $(b(t,\cdot),\sigma(t,\cdot))$:

$$\mathcal{L}_t\phi(x) = \frac{1}{2}\sum_{i,j=1}^{d}(\sigma\sigma^t)_{ij}(t,x)\frac{\partial^2\phi}{\partial x_i\partial x_j}(x) + \sum_{i=1}^{d} b_i(t,x)\frac{\partial\phi}{\partial x_i}(x).$$

Definition I.3.7. – Martingale Problem. *The probability measure $Q \in \mathcal{M}^1(C_T)$ is solution of the martingale problem $((\mathcal{L}_t,)\mu)$ if*

(i) $Q_0 = \mu$.

(ii) For all $f \in C_b^2(\mathbb{R}^d)$, the $\mathbb{R}^d$-valued process $(\mathcal{M}_t)$, defined by ($x(\cdot)$ being the canonical variable in $\mathcal{M}^1(C_T)$)

$$\mathcal{M}_t^f = f(x(t)) - f(x(0)) - \int_0^t (\mathcal{L}_\theta f)(x(\theta))d\theta$$

is a Q-martingale.

The process $(\mathcal{M}_t^f, t \geq 0)$, defined above is a Q-martingale if

$$\mathbb{E}_Q\left[\mathcal{M}_t^f - \mathcal{M}_s^f \middle| (x(\theta), 0 \leq \theta \leq s)\right] = 0,\ \forall 0 \leq s \leq t,$$

or equivalently, if for all $g \in C_b(\mathbb{R}^n)$ and for all $0 \leq t_1 < \ldots < t_n < s$,

$$\mathbb{E}_Q\left[(\mathcal{M}_t^f - \mathcal{M}_s^f)g(x(t_1),\ldots x(t_n))\right] = 0.$$

From a solution of the MP, we can go back to a solution to the Equation (I.16) in the following manner. To simplify the discussion we assume that the diffusion coefficient is a constant $\sigma(t,x) \equiv \sigma$. Applying (ii) for $f(x) = x$, we call $(\sigma W_t, t \geq 0)$ the resulting martingale, with

$$W_t := \frac{1}{\sigma}\left(x(t) - x(0) - \int_0^t b(s,x(s))ds\right)$$

$$\text{and } dW_t = \frac{1}{\sigma}\left(dx(t) - b(t,x(t))dt\right)$$

Applying (ii) again for $f(x) = x^2$, we call $(\mathcal{M}_t^{x^2})_{t\geq 0}$ the resulting martingale.

$$\mathcal{M}_t^{x^2} = x^2(t) - x^2(0) - \int_0^t 2x(s)b(s,x(s))ds - \sigma^2 t.$$

But

$$\begin{aligned}\int_0^t 2x(s)b(s,x(s))ds &= 2x(0)(\int_0^t b(s,x(s))ds) + 2\int_0^t \left(\int_0^s b(r,x(r)dr\right) b(s,x(s)ds + \int_0^t \sigma W_s b(s,x(s))ds\\ &= 2x(0)(\int_0^t b(s,x(s))ds) + (\int_0^t b(s,x(s))ds)^2\\ &\qquad + 2\sigma W_t(\int_0^t b(s,x(s))ds) - 2\int_0^t (\int_0^s b(r,x(r))dr)dW_s.\end{aligned}$$

Developing $x^2(t)$ in the definition of $\mathcal{M}_t^{x^2}$ and subtracting the above line. we get

$$\mathcal{M}_t^{x^2} = \sigma^2 W_t^2 + 2\sigma x(0)W_t + 2\int_0^t (\int_0^s b(r,x(r)dr)dW_s - \sigma^2 t.$$

An so

$$(W_t^2 - t) = \frac{1}{\sigma^2}\left(\mathcal{M}_t^{x^2} - 2\sigma x(0)W_t - 2\int_0^t \left(\int_0^s b(\theta,x(\theta))d\theta\right) dW_s\right).$$

As the stochastic integral $(\int_0^t (\int_0^s B(\theta,X_\theta)d\theta)\, dW_s, t \geq 0)$ is a (local) martingale. This last equality identifies W as a Brownian motion, according to the Lévy martingale characterization of Brownian motion (see Theorem I.C.8). Moreover, taking the expectation in (ii), we get

$$\mathbb{E}f(X_t) = \mathbb{E}f(X_0) + \int_0^t \mathbb{E}\left[\frac{1}{2}\sigma^2 f''(X_s) + B(s,X_s)f'(X_s)\right] ds. \tag{I.17}$$

Weak solution of SDE and solution to martingale problem are two equivalent notions

Indeed, the law on C_T of a weak solution to the SDE (I.16) provides immediately a solution $Q = \mathbb{P}\circ X^{-1}$ to the martingale problem $(\mathcal{L},\mu)$, but this requires some assumption on the data (b,σ) and an adapted set of test functions for f that allow to qualify the apriori local martingale $\mathcal{M}^f$ as a martingale under the law of the process X. If (b,σ) are unbounded, we can reduce the set of test functions to $C_c^2(\mathbb{R}^d)$ rather than $C_b^2(\mathbb{R}^d)$.

Conversely, if Q is a solution to the martingale problem, the space C_T, measured by Q give a probability space, the Lévy characterisation allows to identify a Brownian motion on that space such that the canonical process $x(\,\cdot\,)$ satisfies (I.16) Q p.s. (see e.g. [Karatzas and Shreve, 1988] for more details on weak solution for SDEs.

Theorem I.3.8. – Weak existence of to SDE (I.16), via martingale problem [Stroock and Varadhan, 1969, Stroock and Varadhan, 1979]. *Assume (b,σ) bounded and continuous on $\mathbb{R}^+\times\mathbb{R}^d$, and μ a probability measure on $\mathbb{R}^d$ admitting at least a finite moment of order $\gamma > 0$. Then there exists a solution to the martingale problem $(\mathcal{L},\mu)$.*

Remark I.3.9. *In Theorem I.3.8, one can replace the assumption* (b, σ) *bounded by* (b, σ) *increasing at most linearly. This requires from* μ *to have at least a moment of order* $2m$ *with* $m \geq 1$*. Indeed, it is then possible to recover some tightness on the time marginal laws of the process* X*, showing*

$$\mathbb{E}|X_t - X_s|^{2m} \leq C(1 + \mathbb{E}\|X_0\|^{2m})(t-s)^m,$$

But the $2m$*-order moment control of the* $\|X_t\|$ *require the control of its initial condition.*

I.3.1.3 About uniqueness result for martingale problem

1. The uniqueness of the solution to the MP for $(\mathcal{L}, \mu)$ is equivalent to the uniqueness of the weak solution of (I.16).

2. **Uniqueness from the regularity of the backward Kolmogorov PDE.**

   Following [Stroock and Varadhan, 1969, Stroock and Varadhan, 1979].

   Consider the MP associated to $(\mathcal{L}, \delta_y)$, with solution P^y. For all $f \in C_b(\mathbb{R}^d)$, for $t \leq T$, consider the Kolmogorov PDE

$$\begin{cases} \partial_s u_f + \mathcal{L}_s u_f = 0, \ 0 \leq s < t \\ u_f(t, x) = f(x) \end{cases} \tag{I.18}$$

   Assume that the Cauchy problem (I.18) possesses a solution in $C([0, +\infty) \times \mathbb{R}^d) \cap C^{1,2}((0, +\infty) \times \mathbb{R}^d)$, bounded on $[0, T] \times \mathbb{R}^d$, for arbitrary $T < +\infty$. Then, for all $t \in [0, T]$,

$$u_f(0, y) = \mathbb{E}_{P^y} f(x(t))$$

   with $x(t)$ satisfying (I.16) under P^y.

   Now, consider two solutions P^y et Q^y of the MP $(\mathcal{L}, \delta_y)$. With the Itô formula,

$$\mathbb{E}_{P^y} f(x(t)) = u_f(0, y) = \mathbb{E}_{Q^y} f(x(t).$$

   f being chosen in a set of determinant functions for measures of $\mathbb{R}^d$, we can conclude on the equality of the time-marginales $P^y_t = Q^y_t$, for all $0 \leq t \leq T$, and by extension to all $t \geq 0$.

   To get $P^x = Q^x$, the previous reasoning must be extended to any finite dimensional distribution. This is done by induction on the n times to consider (see the details in the proof of Proposition 4.27 (Chapter 6) of [Karatzas and Shreve, 1988]).

   The existence of a smooth solution to the Kolmogorov PDE requires some assumption on (b, σ). For example, b, σ Hölder and $a = \sigma\sigma^t$ strongly elliptic are classical assumptions, but other set of hypotheses are possible.

3. **Uniqueness from the drift removing argument.**

   If b is bounded measurable, if $a = \sigma\sigma^t$ is continuous in x, uniformly in time, and strongly elliptic uniformly in time, then the martingale problem is well posed

   This result from [Stroock and Varadhan, 1979] uses such argument.

   For simplicity, assume $r = d = 1$. If (b, σ) are measurable, and b bounded, σ^{-1} bounded and σ uniformly Lipschitz in x, well-posedness can be deduce form a Girsanov argument.

   Indeed, assumption on σ allow to consider the unique strong solution on a given probability space $(\Omega, \mathbb{P}, W)$, of the SDE
$$dY_t = \sigma(t, Y_t)dW_t.$$

   Moreover,

$$Z_t = \exp\left(-\int_0^t (\sigma^{-1}b)(s, Y_s)dW_s - \frac{1}{2}\int_0^t (\sigma^{-1}b)^2(s, Y_s)ds \right)$$

   is an exponential martingale. So under $\mathbb{Q} = Z_T\mathbb{P}$ restricted to the sigma-algebra $\mathcal{F}_T$ generated from W,
$$B_t = W_t - \int_0^t (\sigma^{-1}b)(s, Y_s)ds$$
   is a $\mathbb{Q}$ - Brownian motion (Girsanov Theorem (see e.g [Karatzas and Shreve, 1988], and so $\mathbb{Q}$ is solution to MP.

4. **Uniqueness from the FK equation.**

   Assume the existence of a solution to (I.16), Consider the time marginal $P_t = \mathbb{P} \circ X_t^- 1$.

   For all $\phi \in C_c^2(\mathbb{R}^d)$, we define the integration bracket:

$$\langle P_t, \phi \rangle = \int_{\mathbb{R}^d} \phi(x)P_t(dx) = \mathbb{E}[\phi(X_t)].$$

   Then from the Itô formula,

$$\frac{d}{dt}\langle P_t, \phi \rangle = \langle P_t, \mathcal{L}_t\phi \rangle. \tag{I.19}$$

   With two integrations by part, for all $\phi, \psi \in C_c^\infty(\mathbb{R}^d)$,

$$\langle \psi, \mathcal{L}_t\phi \rangle = \langle \mathcal{L}_t^*, \phi \rangle$$

   with

$$\mathcal{L}_t^*\psi(x) = \frac{1}{2}\sum_{i,j=1}^d \frac{\partial^2}{\partial x_i \partial_{x_j}} \big((\sigma\sigma^t)_{ij}(t,x)\psi(x) \big) - \sum_{i=1}^d \frac{\partial}{\partial x_i} \left(b_i(t,x)\psi(x) \right).$$

This means that the set of time -marginal (P_t) is solution, in the sense of distributions, to the Fokker Planck PDE

$$\begin{cases} \dfrac{\partial P_t}{\partial t} = \mathcal{L}_t^* P_t, \text{ on } [0,T]\times\mathbb{R}^d \\ P_0 = \mu \text{ given.} \end{cases}$$

Any set of hypotheses from PDE theory that grants the uniqueness of the FP PDE allows to conclude on the uniqueness for the MP.

Exercise I.2.

1). Write the Fokker-Planck PDE for the time-marginal laws of (X_t, V_t) and (Y_t) solution of (I.14) and (I.12).

2). Let $(\widetilde{x}_t, \widetilde{u}_t)_{t\geq 0}$ and $(\widetilde{Y}_t)_{t\geq 0}$ be the solutions to the equations (I.14) and (I.12) in the case where $\langle v\rangle(t,x) \equiv 0$.
We define $\Gamma_{OU}\colon \mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}^2 \mapsto \mathbb{R}$ as the transition density of $(\widetilde{x}_t, \widetilde{u}_t)_{t\geq 0}$ meaning that $\Gamma_{OU}(t; y, v; x, u) = \mathcal{P}_{y,v}\left((\widetilde{x}_t, \widetilde{u}_t) \in (dx, du)\right)/dxdu$ and $\Gamma_B\colon \mathbb{R}^+\times\mathbb{R}\times\mathbb{R} \mapsto \mathbb{R}$ as the transition density of $(\widetilde{Y}_t)_{t\geq 0}$ such that $\Gamma_B(t; y; x) = \dfrac{1}{\sqrt{2\pi\sigma^2 t}}\exp\left(-\frac{1}{2\sigma^2 t}(x-y)^2\right)$.

3). Write the Fokker-Planck PDEs for $(t,x,u) \mapsto \Gamma_{OU}(t; y, v; x, u)$ and $(t,x)\mapsto \Gamma_B(t; y; x)$.

4). Show that the mild equation

$$\rho(t,x,u) = \int_{\mathbb{R}^2} \Gamma_{OU}(t,y,v;x,u)\mu(dy,dv) \tag{I.20}$$

$$+ \int_0^t \int_{\mathbb{R}^2} \frac{\partial}{\partial v}\Gamma_{OU}(t-s,y,v;x,u)\frac{1}{T_L}\langle v\rangle(s,y)\rho(s,y,v)dydv\,ds \tag{I.21}$$

is solution to the Fokker Planck PDE associates to SDE (I.14).

5). Write the mild equation for $p(t,x)$ the density solution of the Fokker-Planck equation for SDE (I.12).

Stochastic Lagrangian approach. We come back to the computational fluid dynamics (CFD) view point that deals with the numerical computation of the RANS momentum equation

$$\partial_t\langle v\rangle(t,x) + \langle v\rangle\cdot\nabla\langle v\rangle + \sum_j \partial_{x_j}\langle v'\, v'_j\rangle + \frac{1}{\rho_0}\nabla\langle p\rangle(t,x) = \frac{1}{Re}\Delta\langle v\rangle(t,x), \quad (t,x)\in(0,T]\times\mathcal{D},$$
$$\nabla\cdot\langle v\rangle(t,x) = 0, \quad (t,x)\in[0,T]\times\mathcal{D}$$
$$\langle v\rangle(0,x) = \bar{v}_0(x), \quad x\in\mathcal{D}.$$

together with a given parametrisation for the Reynolds tensor $\mathcal{R}_{ij} = \langle v'_i v'_j\rangle$; a turbulent closure. We assume that the Reynolds number is very large so we can neglect $\frac{1}{Re}\Delta\langle v\rangle$ in front of all the other terms

$$\partial_t \langle v \rangle(t,x) + \langle v \rangle \cdot \nabla \langle v \rangle + \sum_j \partial_{x_j} \langle v' \, v'_j \rangle + \frac{1}{\rho_0} \nabla \langle p \rangle(t,x) = 0, \quad (t,x) \in (0,T] \times \mathcal{D},$$
$$\nabla \cdot \langle v \rangle(t,x) = 0, \quad (t,x) \in [0,T] \times \mathcal{D}$$
$$\langle v \rangle(0,x) = \bar{v}_0(x), \quad x \in \mathcal{D}.$$

Eulerian approaches numerically solve RANS + closure with a dedicated method. They are differentiated by the definition that is implemented from the average operator, with some consequence on the modelling accuracy of the solver.

For example RANS method defines $\langle \cdot \rangle$ as time-averaging operator, while Large Eddy simulation (LES) use a convolution filter to define $\langle \cdot \rangle$ with a kernel defined to cut some small scales. Differences are in the closure equations, a some hierarchy exists (as shown in Figure I.3.1.3). By contrast to the fully-resolved direct simulation, models that support CFD methods are referred as sub-grid models.

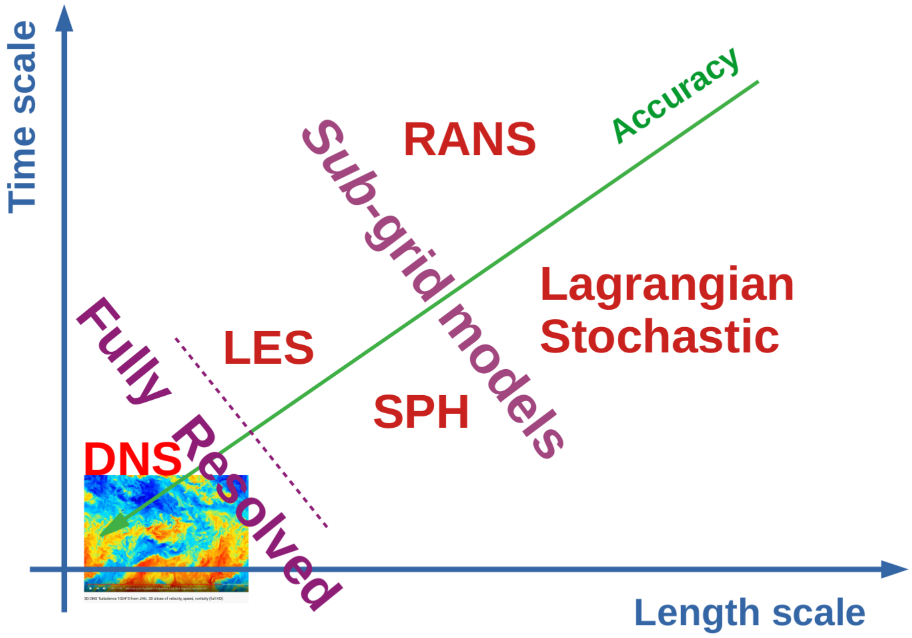


Figure I.6: A schematic hierarchy of numerical approaches for turbulent flows.

PDF approaches / Stochastic Lagrangian approaches. PDF stands for Probability density function. PDF or Stochastic Lagrangian approaches are probabilistic model and numerical approaches based on the reintroduction, at a larger scale, of a probabilistic meaning of the average $\langle \cdot \rangle$.

In CFD, turbulence modelling gives access mainly to the averaged Eulerian velocity and other second moments according to the model. Stochastic Lagrangian approach focus on describing the dynamics of a fluid-particle -or virtual fluid parcel- and its characteristic position and instantaneous velocity

$$(X_t, \;\; U_t),$$

dynamics characterized by a SDE, which suitably approximate the motion of

$$\left(X^+(t) = X_0 + \int_0^t v(t, X^+(s) ds, \;\; v(t, X^+(t)) \right).$$

This SDE is constructed on the basis of a Fokker Planck equation for the density function relative to the position and the velocity of the fluid particle. This joint probability density of the process $((X_t, U_t); 0 \leq t \leq T)$, denoted below by ϱ, allows to interpret the Reynolds operator $\langle \cdot \rangle$, the expectation symbol $\mathbb{E}$ being notably associated to the probability measure $\mathbb{P}$, under which the Brownian motion (B_t) driving the SDE is defined.

Eulerian PDF approach Let $\rho_{\text{Euler}}(t, x; V)$ be the probability density function of the (Eulerian) random field $v(t, x, \omega)$, then

$$\begin{aligned} \langle v^{(i)} \rangle (t, x) &= \int_{\mathbb{R}^3} u^i \rho_{\text{Euler}}(t, x; u) du, \\ \langle v^{(i)} v^{(j)} \rangle (t, x) &= \int_{\mathbb{R}^3} u^i u^i \rho_{\text{Euler}}(t, x; u) du. \end{aligned}$$

The closure problem is reported on the PDE satisfied by the probability density function ρ_{Euler}. This is the so-called PDF method.

In his seminal work, [Pope, 1994b] proposes to model the PDF ρ_{Euler} with the Lagrangian probability density function, or equivalently with a Lagrangian description of the flow.

Let $\rho_{\text{Lagrangian}}(t; x, v)$ be the probability density function of fluid-particle with state (X_t, V_t). In the case of incompressible flow, with constant mass of particles the relationship between ρ_E and ρ_L is

$$\rho_{\text{Euler}}(t, x; v) dv = \frac{\rho_{\text{Lagrangian}}(t; x, v)}{\int_{\mathbb{R}^3} \rho_{\text{Lagrangian}}(t; x, u) du} \, dv$$

For any random variable $g(V_t)$, with finite moment,

$$\langle g(v(t, x)) \rangle = \int_{\mathbb{R}^3} g(v) \rho_{\text{Euler}}(t, x; v) dv = \mathbb{E} \left[g(V_t) \middle| X_t = x \right]$$

We then redefine the $\langle \; \rangle$ with the Lagrangian density:

$$\langle g(v(t, x)) \rangle = \frac{\int_{\mathbb{R}^3} g(v) \rho_{\text{Lagrangian}}(t, x, v) \, dv}{\int_{\mathbb{R}^3} \rho(t, x, v) \, dv} = \frac{1}{\rho(t, x)} \int_{\mathbb{R}^3} g(v) \rho_{\text{Lagrangian}}(t, x, v) \, dv \tag{I.22}$$

where we set

$$\rho(t, x) = \int_{\mathbb{R}^3} \rho_{\text{Lagrangian}}(t, x, v) \, dv$$

.

A reference stochastic Lagrangian model is the *Generalized Langevin Model* (GLM),

$$\begin{cases} dX_t^{(i)} = U_t^{(i)}dt,\ 1 \le i \le 3, \\ dU_t^{(i)} = -\dfrac{1}{\rho}\dfrac{\partial\langle p\rangle}{\partial x_i}(t,X_t)dt \\ \qquad\qquad + \displaystyle\sum_{j=1}^{3} G_{ij}(t,X_t)(U_t^{(j)} - \langle v^j\rangle(t,X_t))dt + \sqrt{C_0(t,X_t)\varepsilon(t,X_t)}dB_t^{(i)}, \\ \qquad (B_t = (B_t^{(i)}),\ t \ge 0) \text{ is a standard 3D-Brownian motion,} \end{cases} \tag{I.23}$$

Let's write the FK equation associated to this generic GLM: $\mathbb{P}\left((X_t, V_t) \in dxdv\right) := \rho_{\text{Lagr.}}(t;x,v)dxdv$.

$$\begin{aligned} \partial_t\rho_{\text{Lagr.}} + (v\cdot\nabla_x\rho_{\text{Lagr.}}) &= \frac{1}{\rho}\left(\nabla_x\langle p\rangle(t,x)\cdot\nabla_v\rho_{\text{Lagr.}}\right) \\ &\quad - \nabla_v\cdot\left(G(t,x)(\langle v\rangle(t,x) - v)\rho_{\text{Lagr.}}\right) + \frac{1}{2}C_0(t,x)\varepsilon(t,x)\triangle_v\rho_{\text{Lagr.}} \end{aligned}$$

The SDE (I.23) is designed to be consistent with the Navier-Stokes equations through formal developments on the Fokker-Planck equation above.

$$\begin{aligned} &\partial_t\int_{\mathbb{R}^3\times\mathbb{R}} g(v)\rho_{\text{Lagr.}}(t,x,v)\,dv + \left(\nabla_x\cdot\int_{\mathbb{R}^3}\left(v\cdot g(v)\rho_{\text{Lagr.}}(t,x,v)\right)\,dv\right) \\ &= G_{ij}(t,x)\int_{\mathbb{R}^3}\partial_{v_i}g(v)\left(v_j - \langle v^{(j)}\rangle(t,x)\right)\rho_{\text{Lagr.}}(t,x,v)dv \\ &\quad + \frac{1}{2}C_0(t,x)\varepsilon(t,x)\int_{\mathbb{R}^3}\triangle_v g(v)\rho_{\text{Lagr.}}(t,x,v)\,dv. \end{aligned} \tag{I.24}$$

With $g(v) = 1$, integrating w.r.t. dv gives the equation for the conservation of mass,

$$\rho(t,x) = \int \rho_{\text{Lagr.}}(t;x,v)dv$$

$$\partial_t\int\rho_{\text{Lagr.}}dv + \nabla_x\cdot\left(\frac{\int v\rho_{\text{Lagr.}}dv}{\int\rho_{\text{Lagr.}}dv}\int\rho_{\text{Lagr.}}dv\right) = 0 \Longleftrightarrow \partial_t\rho + \nabla_x\cdot(\rho\langle v\rangle) = 0.$$

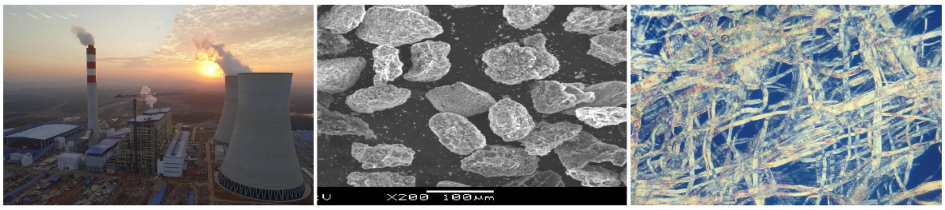
With $g(v) = v^i$, integrating w.r.t. vdv gives the RANS momentum equation.

$$\partial_t\langle v^{(i)}\rangle(t,x) + \left(\nabla_x\cdot\langle v^{(i)}v\rangle(t,x)\right) = -\frac{1}{\rho}\partial_{x_i}\langle p\rangle(t,x). \tag{I.25}$$

With $g(v) = v_iv_j$ we recover the Reynolds stress model equation, according to the choice C_0 and G.

The SDE (I.23) is of McKean Vlasov type.

I.3.2 Particle-laden flows



Particles in turbulence are important for many industrial and biological applications.
What is important:

- Passive *vs* Active particle
- Size, **Shape**, Inertia, Rheology
- Particle-particle interactions (collisions, agglomerations)
- Additional forces (electrostatics, Brownian motion)

I.3.2.1 Particle dynamics

- About the flow the phase :

  The Kolmogorov length scale η_K represents the smallest length scale for fluid motions in a turbulent flow. For most of industrial or environmental flows, $\eta_K \in [50\mu\mathrm{m}, 1\mathrm{mm}]$.

  The governing equations for the fluid are the Navier-Stokes equations, complemented by transport equations for a set of scalar fields (temperature,...) when needed.

  The flow field $U_f(t,x)$ is computed :
  • below the scale η_K (Direct Numerical Simulation).⚠ From very expensive to totally prohibitive computation time.

  • above the scale η_K (Engineering application, CFD). Required turbulence modelling: only mean and second order moments velocity are computed $U_f(t,x) = \langle U_f \rangle(t,x) + \text{noise}$.

- About the inclusion phase :

  • Small (spherical) particles, with diameters $d_p \ll \eta_K(\sim 30\mu\mathrm{m})$ so that point-wise approximation is reasonable : description is given by Lagrangian equation on center of mass position and velocity $(x_p(t), U_p(t))$.

The Lagrangian equation obtained by applying the fundamental laws of classical mechanics for each particle carried by the flow (Hydrodynamical approach) :

Hydrodynamical forces on particles :

$$\frac{dx_p}{dt} = U_p(t), \qquad m_p \frac{dU_p}{dt} = F_{f\mapsto p} + F_{p\mapsto p} + F_{ext}, \qquad \left(\frac{d\Omega_p}{dt} = M_{f\mapsto p} + M_{p\mapsto p}\right)$$

translational and rotational velocity for non spherical part.

For particles heavier than the fluid (inertial particle: $\rho_p \gg \rho_f$, corresponding to typical size $d_p \geq 5 - 10\mu\mathrm{m}$) the drag force exerted by the fluid $F_{f\mapsto p}$ is dominant:

$$\frac{dU_p}{dt} = \frac{(U_s - U_p)}{\tau_p} + \mathbf{g}, \qquad \tau_p = \frac{\rho_p}{\rho_f} \frac{4d_p}{3C_D|U_s - U_p|}, \qquad \tau_p(d_p) \nearrow$$

The particle relaxation timescale τ_p measures the particle inertia, as the timescale over which particle velocities adjust to **the local fluid velocity seen** $U_s(t)$.

F_{ext}: gravity g can be neglected in front of drag force.

Interaction between particles $F_{p\mapsto p}$ are collision events, and produces agglomerations, fragmentations...

A complete picture of all of the hydrodynamical approach can be found in the following review [Minier, 2016].

The particle state vector is now $(x_p(t), U_p(t), U_s(t))$
The velocity seen: $U_s(t) = U_f(t, x_p(t))$.

I.3.2.2 Statistical descriptions of single-phase turbulence

Modelling the fluid velocity seen by a particle:

$$U_f(t,x) = \langle U_f \rangle(t,x) + \text{noise}(t,x)$$

Modelling coherency requires to adopt Lagrangian point of view also for the fluid :

$$\langle U_f \rangle(t,x) = \langle U_f(t) | x_f(t) = x) \rangle$$

with a General Langevin Model:

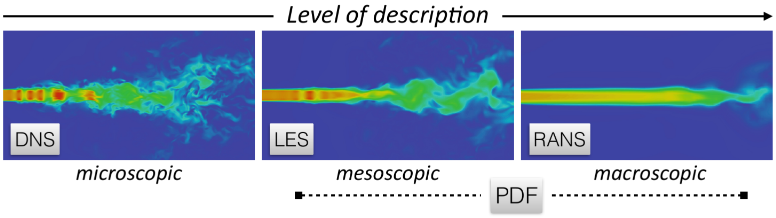
$$\begin{cases} dx_f(t) = U_f(t)dt, \\ dU_f^{(i)}(t) = -\partial_{x_i}\langle p\rangle(t, x_f(t))dt + \left(G_{ij}\left(U_f^{(j)} - \left\langle U_f^{(j)}\right\rangle\right)\right)(t, x_f(t))dt + \sigma_{i,j}(t, x_f(t))dB_t^{(i)} \end{cases}$$

B is a 3D-Brownian motion.

$$G_{ij} = -\frac{C_R}{2}\frac{\varepsilon}{tke}\delta_{ij} + C_2 \frac{\partial\langle U^{(i)}\rangle}{\partial x_j}, \qquad \sigma_{i,j} = \frac{2}{3}\left(C_R\varepsilon + C_2\mathcal{P} - \varepsilon\right)\delta_{ij},$$

Figure 1. (left) A jet of water directed downward into water with Reynolds number approximately 2300 (obtained in Van Dyke, M. and Van, D., An Album of Fluid Motion, 1982); (right) the same jet simulated in CFD using the RANS approach.

Figure I.7: top : Borrowed from ROCKY Blog : Turbulent dispersion model: discover how to account for the turbulence effect on particle trajectories.



- Mean field approximation for the fluid particle leads to a complex McKean Vlasov SDE, where $\langle \cdot \rangle(t,x) = \mathbb{E}[\cdot\,| x_f(t) = x]$.

  $u'(t)$ being the centred Lagrangian velocity $U_f(t) - \langle U_f\rangle(t, x_p(t)$.

  $\mathcal{P} = \frac{1}{2}\mathcal{P}_{ii}$, the turbulent production term $\mathcal{P}_{ij} := -(\langle u'_i u'_k\rangle \partial_{x_k}\left\langle U^{(i)}\right\rangle - \left\langle u'_j u'_k\right\rangle \partial_{x_k}\left\langle U^{(j)}\right\rangle$

  ε is closed with coherent parametrisation involving *tke* $= \frac{1}{2}\langle u'_i u'_i\rangle$.

I.3.2.3 Langevin model for dispersed particles embedded in a turbulent flows using a dynamic PDF model

- Inertial particle ($\rho_p \gg \rho_f$) :

$$
\begin{aligned}
&dx_p(t) = U_p(t)dt,\\
&dU_p(t) = \tfrac{1}{\tau_p}(U_s(t) - U_p(t))dt\\
&dU_s^i(t) = -\tfrac{1}{\rho_f}\partial_{x_i}\langle p\rangle(t, x_p(t))dt\\
&\qquad\qquad + (\langle U_p^j\rangle - \langle U_f^j\rangle)\partial_{x_i}\langle U_f^j\rangle dt + G^*_{ij}(U_s^j - \langle U_f^j\rangle(t, x_p(t))dt + \sigma_{ij}(t, x_p(t))dB_j
\end{aligned}
$$

$G^*_{ij} = \frac{T_L}{T^*_L}G_{ij}$ where $\frac{T_L}{T^*_L}$ is a model factor. T^*_L is correlation timescale of the velocity of the fluid seen / T_L is the Lagrangian correlation timescale of the velocity.

- Highly inertial particle limit : overdamped dynamics

  When $T_L^* \to 0$, and finite τ_p (high inertia), $G_{ij}^* \sim \frac{1}{T_L^*}$,

$$
\begin{aligned}
dx_p(t) &= U_p(t)dt,\\
dU_p^i(t) &= \tfrac{1}{\tau_p}(\langle U_f^i\rangle(t,x_p(t)) - U_p^i(t))dt\\
&\qquad + \tfrac{1}{\tau_p}G_{ij}^{-1}\tfrac{1}{\rho_f}\partial_{x_j}\langle p\rangle(t,x_p(t))dt + \tfrac{1}{\tau_p}(G^{*-1}(\sigma))_{ij}(t,x_p(t))dB_j
\end{aligned}
$$

- Small-enough particles (colloids, $\rho_p \ll \rho_f$, $d_p \leq 1-2\mu\mathrm{m}$). Drag force is complemented with molecular effects and Brownian motion:

$$
\begin{aligned}
dx_p(t) &= U_p(t)dt,\\
dU_p(t) &= \tfrac{1}{\tau_p}(U_s(t) - U_p(t))dt + K_{Brow}(\tau_p)dW(t)\\
dU_s^i(t) &= -\tfrac{1}{\rho_f}\partial_{x_i}\langle p\rangle(t,x_p(t))dt\\
&\qquad + (\langle U_p^j\rangle - \langle U_f^j\rangle)\partial_{x_i}\langle U_f^j\rangle dt + G_{ij}^*(U_s^j - \langle U_f^j\rangle(t,x_p(t))dt + \sigma_{ij}(t,x_p(t))dB_j
\end{aligned}
$$

  with K_{Brow} increasing with $1/(\tau_p)^{\frac{1}{2}}$. B and W are assumed independent.

- Generalized Langevin to Einstein limit $\tau_p \to 0$ overdamped dynamics

$$
\begin{aligned}
dx_p(t) &= U_s(t)dt + \tau_p K_{Brow}(\tau_p)dW(t)\\
dU_s^i(t) &= -\tfrac{1}{\rho_f}\partial_{x_i}\langle p\rangle(t,x_p(t))dt\\
&\qquad + G_{ij}^*(U_s^j - \langle U_f^j\rangle(t,x_p(t))dt + \sigma_{ij}(t,x_p(t))dB_j
\end{aligned}
$$

Appendix

I.A Material derivative

In continuum mechanics, the material derivative describes the time rate of change of some physical quantity (like heat or momentum) of a material element that is subjected to a space-and-time-dependent macroscopic velocity field.

The material derivative is defined for any tensor field y that depends only on position and time coordinates, $y = y(x,t)$:

$$\frac{\mathrm{D}y}{\mathrm{D}t} \equiv \frac{\partial y}{\partial t} + \mathbf{v} \cdot \nabla y, \tag{I.26}$$

where ∇y is the covariant derivative of the tensor. Generally the convective derivative of the field $v \cdot \nabla y$, the one that contains the covariant derivative of the field, can be interpreted both as involving the streamline tensor derivative of the field $v(\nabla y)$, or as involving the streamline directional derivative of the field $(v\nabla)y$, leading to the same result. Only this spatial term, containing the flow velocity, describes the transport of the field in the flow, while the other describes the intrinsic variation of the field, independent of the presence of any flow. Confusingly, sometimes the name ”convective derivative” is used for the whole material derivative D/Dt, instead for only the spatial term $v \cdot \nabla$ The effect of the time-independent terms in the definitions are for the scalar and tensor case respectively known as advection and convection.

I.B The normal to a 3D surface and Divergence Theorem

Given a three-dimensional surface defined implicitly by

$$S = \{(x,y,z);\ F(x,y,z) = 0\},$$

where F is piecewise-C^1 function, then the outward unit normal is defined as

$$n = \frac{\nabla F}{\|\nabla F\|}.$$

If the 3D-surface is defined parametrically in the form

$$S = \{(\phi, \psi) \mapsto (x(\phi, \psi), y(\phi, \psi), z(\phi, \psi)), \}$$

define the vectors

$$\begin{aligned} a &= (\partial_\phi x, \partial_\phi y, \partial_\phi z) \\ b &= (\partial_\psi x, \partial_\psi y, \partial_\psi z). \end{aligned}$$

Then the unit normal vector is

$$n = \frac{a \times b}{\|a \times b\|},$$

where $\times$ stands for the cross product or vector product [6].

Theorem I.B.1 (Ostrograsky Divergence Theorem (fluid mechanics) / Gauss Theorem (electromagnetism))**.**
Let $\mathbb{D}$ a compact domain in de $\mathbb{R}^d$, with a piecewise smooth boundary $\partial\mathbb{D}$. The closed manifold $\partial\mathbb{D}$ is oriented by outward-pointing normals, and $n(\cdot)$ is the outward pointing unit normal at each point on the boundary. If F is a continuously differentiable vector field defined on a neighbourhood of $\mathbb{D}$, then

$$\int_{\mathbb{D}} (\nabla \cdot F) dv = \int_{\partial\mathbb{D}} (F \cdot n) ds \tag{I.27}$$

I.C Selected reminders of probability theory and stochastic processes

Theorem I.C.1. Jensen's inequality. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex function. Let X be a real r.v. such that $\mathbb{E}[|X|] < +\infty$ and $\mathbb{E}[|\phi(X)|] < +\infty$. Then*

$$\phi(\mathbb{E}[X]) \leq \mathbb{E}[\phi(X)].$$

Proof. The proof is based on a property of convex functions: ϕ can be written as the supremum of affine functions family, which are all lower than it. In particular, there exists a countable set of real numbers $(a_n, b_n)_{n\in\mathbb{N}}$ such that $\phi(x) = \sup_{n\geq 1}(a_n x + b_n)$. Thus,

$$\mathbb{E}[\phi(X)] \geq \mathbb{E}[a_n X + b_n] = a_n \mathbb{E}[X] + b_n.$$

This inequality being true for all n, by passing to the sup we obtain that $\mathbb{E}\{\phi(X)\} \geq \sup_n(a_n\mathbb{E}(X) + b_n) = \phi(\mathbb{E}(X))$. □

[6] The cross product is defined for the Euclidean space $\mathbb{R}^3$. It is an antisymmetric product ($A \times B = -B \times A$) and $A \times B$ is defined as

$$A \times B = \|A\|\|B\| \sin(\theta_{AB})$$

where θ_{AB} is the angle formed by the two vectors. The result is orthogonal to the plan containing (A, B) and oriented according to the right-hand rule.

Theorem I.C.2.

a) Hölder's Inequality. Let X and Y, two real valued r.v. such that $\mathbb{E}[|X|^p + |Y|^p] < \infty$, for $p > 1$ and let q the conjugate of p s.t. $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\mathbb{E}[XY] \leq \mathbb{E}[|XY|] \leq \mathbb{E}[|X|^p]^{\frac{1}{p}}\, \mathbb{E}[|Y|^q]^{\frac{1}{q}}.$$

b) Minkowski's inequality. Let X and Y, two real valued r.v. such that $\mathbb{E}[|X|^p + |Y|^p] < \infty$, for $1 < p < \infty$. Then

$$\mathbb{E}[|X+Y|^p]^{\frac{1}{p}} \leq \mathbb{E}[X^p]^{\frac{1}{p}} + \mathbb{E}[Y^p]^{\frac{1}{p}}.$$

Proof. The function $x \mapsto e^x$ is strictly convex, i.e. that for all $x\alpha \in]0,1[$, $e^{\alpha x + (1-\alpha)y} \leq \alpha e^u + (1-\alpha)e^v$. Then for two r.v U and V such that $\mathbb{E}(e^U + e^V) < \infty$, taking $x = \log\left(\frac{e^U}{\mathbb{E}(e^U)}\right)$ and $y = \log\left(\frac{e^V}{\mathbb{E}(e^V)}\right)$, we have

$$\left(\frac{e^U}{\mathbb{E}(e^U)}\right)^{\alpha}\left(\frac{e^V}{\mathbb{E}(e^V)}\right)^{1-\alpha} \leq \alpha\frac{e^U}{\mathbb{E}(e^U)} + (1-\alpha)\frac{e^V}{\mathbb{E}(e^V)}.$$

Thus by taking the expectation of both sides of the above inequality, we obtain

$$\mathbb{E}\{e^{\alpha U}e^{(1-\alpha)V}\} \leq \left(\mathbb{E}(e^U)\right)^{\alpha}\left(\mathbb{E}(e^V)\right)^{(1-\alpha)}. \tag{I.28}$$

The Hölder inequality results from (I.28), taking $U = p\log|X|$, $V = q\log|Y|$ and $\alpha = \frac{1}{p}$.

It is sufficient to write that $(X+Y)^p = X(X+Y)^{p-1} + Y(X+Y)^{p-1}$, to deduce Minkowski's inequality from Hölder's inequality. □

I.C.1 Filtration and adaptation

Definition I.C.3. *A filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a increasing family (for the ensemble inclusion sense) of sub-sigma algebra of $\mathcal{F}$, denoted $(\mathcal{F}_t, t \geq 0)$:*

$$0 \leq s < t, \qquad \mathcal{F}_s \subset \mathcal{F}_t.$$

I.C.1.1 Example

Let denote $(\mathcal{F}_t^X, t \geq 0)$ the filtration generated by a given process $(X_t, t \geq 0)$, defined as follows: For all $t \geq 0$, $\mathcal{F}_t^X$ is the smallest sigma-algebra that makes all the applications $\omega \in \Omega \to X_\theta(\omega)$ measurable for all $\theta \leq t$.

$$\mathcal{F}_t^X := \sigma(X_\theta, 0 \leq \theta \leq t).$$

Usual conditions

All the filtrations in this document are assumed to fulfil the usual conditions: Let

$$\mathcal{N} = \{A \in \mathcal{F};\ \mathbb{P}(A) = 0\}.$$

a) If $A \in \mathcal{N}$, then for all $t \geq 0,\ A \in \mathcal{F}_t$.

b) $(\mathcal{F}_t, t \geq 0)$ est right-continuous:

$$\forall t \geq 0,\ \ \mathcal{F}_t = \mathcal{F}_{t^+} = \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}.$$

The filtration generated by a process (X) does not satisfies naturally the usual conditions. The filtration is them corrected to get what is called the natural filtration :

$$\mathcal{F}_t^X = \sigma\left(\bigcap_{\varepsilon > 0} \sigma(X_s, s \leq t + \varepsilon) \bigcup \mathcal{N}\right).$$

When a filtration $(\mathcal{F}_t)$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is imposed, one define the Brownian motion adapted to this filtration:

Definition I.C.4. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathcal{P})$ a filtered probability space. An $\mathcal{F}_t$–Brownian motion (or Wierner process) $(W_t, t \geq 0)$ is a real-valued process with continuous trajectories that satisfies*

$(W_t, t \geq 0)$ is $(\mathcal{F}_t)$-adapted.

$\forall 0 \leq s \leq t$, $W_t - W_s$ is independent of $\mathcal{F}_s$; $W_t - W_s$ has the same law than W_{t-s} and is a Gaussian $\mathcal{N}(0, t-s)$.

$W_0 = 0\ \mathbb{P}$ p.s.

If $(\mathcal{F}_t) = (\mathcal{F}_t^W)$, then the two definitions I.3.2 and I.C.4 coincide.

I.C.2 Martingales in continuous time

Consider a filtered probabilized space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$.

Definition I.C.5. *A process $(M_t, t \geq 0)$ is a $\mathcal{F}_t$–martingale if*

(i) *(M_t) is a $\mathcal{F}_t$–adapted process.*

(ii) *$\mathbb{E}|M_t| < +\infty$, for all $t \geq 0$, (i.e. the process (M_t) is integrable).*

(iii) *For all $s \leq t$, $\mathbb{E}[M_t/\mathcal{F}_s] = M_s$.*

- If we replace **(iii)** by :

  (iii') For any $s \leq t$, $\mathbb{E}(M_t/\mathcal{F}_s) \leq M_s$, we say that (M_t) is an supermartingale.

- If we replace **(iii)** by :

  (iii") For any $s \leq t$, $\mathbb{E}(M_t/\mathcal{F}_s) \geq M_s$, we say that (M_t) is a submartingale.

  - If a process (M_t) is a $\mathcal{F}_t$–martingale, then

  $$\forall t \geq 0, \ \ \mathbb{E}(M_t) = \mathbb{E}(M_0).$$

    The expectation of a martingale remains constant over time!

  - One of the great interests of the martingale property is that it allows to make quantitative computation on processes and SDEs.

Examples of martingales

The simplest example of a martingale: let X be a real, integrable v.a. real, integrable then, the process (M_t) defined for all $t \geq 0$ by

$$M_t = \mathbb{E}(X/\mathcal{F}_t)$$

is a martingale.

Now let $(W_t, t \geq 0)$ be a standard Brownian FF_t motion.

1. (W_t) is a $\mathcal{F}_t$–martingale.

2. $(W_t^2 - t, t \geq 0)$ is a $\mathcal{F}_t$–martingale.

3. $(\exp(\sigma W_t - \frac{\sigma^2}{2}t), t \geq 0)$ is a $\mathcal{F}_t$–martingale.

I.C.3 Doob's maximal inequality:

Martingales are objects which play a great role in the theory of stochastic theory of stochastic processes, in particular because one can apply to them the following maximum inequality:

Theorem I.C.6. (Doob's inequality). *Let (M_t) be a real* **continuous** *$\mathcal{F}_t$–martingale (i.e. with continuous trajectories). Then $(|M_t|)$ is a continuous submartigale. For any $p > 1$ such that $M_T \in L^p(\Omega)$ (i.e. $\mathbb{E}|M_T^p| < +\infty$), $\sup_{0 \leq t \leq T} |M_t|$ is in $L^p(\Omega)$ (i.e. $\mathbb{E}[\sup_{0 \leq t \leq T} |M_t|^p] < +\infty$).*

$$\left(\mathbb{E}\left[\sup_{0 \leq t \leq T} |M_t^p|\right]\right)^{\frac{1}{p}} \leq \frac{p}{p-1} \left\{\mathbb{E}\left(|M_T|^p\right)\right\}^{\frac{1}{p}}.$$

The constant $\frac{p}{p-1}$ is not optimal in p. For some specific cases of Brownian martingales/Brownian integrals, some authors gave sharpen constant (see e.g. [Peškir, 1998]).

However the constant $\frac{p}{p-1}$ is not so bad. Consider the case where (M) is a standard Brownian (W). Then, using the explicit maximum law of Brownian motion:

$$\begin{aligned}
\left\{\mathbb{E}\left(\sup_{0\le t\le T}|W_t|\right)^p\right\}^{\frac{1}{p}} &= \left\{\mathbb{E}\left(\max\left(\sup_{0\le t\le T}(W_t), -\inf_{0\le t\le T}(W_t)\right)\right)^p\right\}^{\frac{1}{p}} \\
&\le \left\{\mathbb{E}\left(\sup_{0\le t\le T}(W_t) - \inf_{0\le t\le T}(W_t)\right)^p\right\}^{\frac{1}{p}} \\
&\overset{\text{triangular inequality for the } L_p - norm}{\le} 2\left\{\mathbb{E}\left[\left(\sup_{0\le t\le T}(W_t)\right)^p\right]\right\}^{\frac{1}{p}} \\
&\overset{\text{maximun law for BM}}{\le} 2\left\{\mathbb{E}\left[|W_T|^p\right]\right\}^{\frac{1}{p}}.
\end{aligned}$$

For $p = 2$, we find above the Doob maximal inequality for Brownian motion. For $p > 2$, the above calculation, although exploiting a known identity on the maximum law of Brownian motion, becomes a coarser estimate than the Doob inequality (indeed, $2 > p/(p-1)$ as soon as $p > 2$).

I.C.4 GBD Inequality for Martingales

Theorem I.C.7 (Burkholder-Davis-Gundy). *For any $0 \le p < \infty$, there exist two positive constants c_p, C_p such that, for all continuous local martingales X with $X_0 = 0$ and for any stopping time τ, we have the following inequalities*

$$c_p\,\mathbb{E}\left[\langle X\rangle_\tau^p\right] \le \mathbb{E}\left[\left(\sup_{t\le\tau} X_t\right)^{2p}\right] \le C_p\,\mathbb{E}\left[\langle X\rangle_\tau^p\right].$$

I.C.5 Lévy characterisation of Brownian motion

A d-dimensional Brownian motion $X = (X^1, \ldots, X^d)$ on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P})$ is a continuous adapted process with $X_0 = 0$ such that, for any $t > s \ge 0$, $X_t - X_s$ is independent of $\mathcal{F}_s$ and multivariate normal with zero mean and covariance matrix $(t-s)I_d$.

Theorem I.C.8. *Let $X = (X^1, \ldots, X^d)$ be a d-dimensional (local) martingale with $X_0 = 0$. Then, the following are equivalent.*

- *X is a Brownian motion on the underlying filtered probability space.*
- *X is continuous and $(X^i_t X^j_t - \delta_{ij}t)$ is a (local) martingale for $1 \le i, j \le d$.*

I.C.6 Itô integral

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and W a $\mathcal{F}_t$–BM valued in $\mathbb{R}$. $T > 0$ finite but arbitrary.
We consider and elementary process in space

$$\mathcal{E}^b_{\mathcal{F}}(0,T) = \{\phi; \phi_t(\omega) = \sum_{i=0}^{n-1} \phi_i(\omega)\mathbb{1}_{]t_i,t_{i+1}]}(t),\ 0 = t_0 < t_1 < \ldots < t_n = T,$$
$$\text{the } \phi_i \text{ being r.v. } \mathcal{F}_{t_i}\text{-mesurables and bounded.}\}$$

- We set

$$I(\phi) = \sum_{i=0}^{n-1} \phi_i(W_{t_{i+1}} - W_{t_i})$$

- For all $t \in [0,T]$, we set

$$I_t(\phi) = \sum_{i=0}^{n-1} \phi_i \left(W_{t_{i+1}\wedge t} - W_{t_i \wedge t}\right)$$

and we denote it $I_t(\phi) = \displaystyle\int_0^t \phi_s dWs$. For all $s \leq t$, $I_t(\phi) - I_s(\phi) = \int_s^t \phi_\theta dW_\theta$.

- By construction, trajectories of $(I_t(\phi, t \in [0,T])$ are time continuous. $(I_t(\phi), t \in [0,T])$ is an $\mathcal{F}_t$-adapted process, and the application $I_t : \phi \to I_t(\phi)$ is linear in ϕ on the space $\mathcal{E}^b_{\mathcal{F}}(0,T)$.

So it is possible to extend this integral definition to a wider class of integrands as soon as the extended space admit the elementary space $\mathcal{E}^b_{\mathcal{F}}(0,T)$ to be dense in it (for a metric that makes the larger space a Hilbert space[7]).
For arbitrary fixed $T > 0$, on define

$$M^2_{\mathcal{F}}(0,T) = \Big\{(\phi_t, 0 \leq t \leq T) \text{ valued in } \mathbb{R}, \mathcal{F}_t\text{–adaptd}; \mathbb{E}\left(\int_0^T \phi_\theta^2 d\theta\right) < +\infty\Big\}.$$

[7] $M^2_{\mathcal{F}}(0,T)$ is an Hilbert space : meaning that the space has a scalar product, that generate a metric that make the space a Banach space. here the scalar product is $(\phi \cdot \psi) = \mathbb{E}\left[\int_0^T \phi_\theta \psi_\theta d\theta\right]$.

Theorem I.C.9. *For ϕ, ψ in $M^2_{\mathcal{F}}(0,T)$, it exists a continuous and adapted modification*[a] *of the process $(I_t(\phi), t \leq T)$, denoted $(\int_0^t \phi_\theta \, dW_\theta, t \leq T)$ such that $\forall s \leq t \leq T$,*

i) $\left(\int_0^t \phi_\theta dW_\theta, 0 \leq t \leq T\right)$ *is a $\mathcal{F}_t$–martingale.*

ii) $\mathbb{E}\left[\int_s^t \phi_\theta \, dW_\theta \int_s^t \psi_\theta dW_\theta \middle| \mathcal{F}_s\right] = \mathbb{E}\left[\int_s^t \phi_\theta \psi_\theta \, d\theta \middle| \mathcal{F}_s\right]$.
Itô isometry

$$\mathbb{E}\left[\left(\int_s^t \phi_\theta dW_\theta\right)^2\right] = \mathbb{E}\left(\int_s^t \phi_\theta^2 d\theta\right).$$

iii) *If $(W^1_t, W^2_t, t \geq 0)$ are two independent $\mathcal{F}_t$- Brownian motions,*

$$\mathbb{E}^{\mathcal{F}_s}\left(\int_s^t \phi_\theta dW^1_\theta \int_s^t \psi_\theta dW^2_\theta\right) = 0.$$

[a] A process $(Y_t, t \leq 0)$ is said to be a modification of $(X_t, t \geq 0)$ if for all $t \geq 0$ $\mathbb{P}(Y_t = X_t) = 1$.

Vademecum :

Let (W_t) $\mathcal{F}_t$–BM and (ϕ_t) $\mathcal{F}_t$–adapted. $\left(\int_0^t \phi_s dW_s\right)$ is defined as soon as $\int_0^T \phi_s^2 ds < +\infty$ $\mathbb{P}$ p.s., or as soon as $\mathbb{E}\left(\int_0^T \phi_s^2 ds\right) < +\infty$.
$\left(\int_0^t \phi_s dW_s\right)_{0 \leq t \leq T}$ is a martingale as soon as $\mathbb{E}\left(\int_0^T \phi_s^2 ds\right) < +\infty$,

I.C.7 Itô formula

Le $(W_t, t \geq 0)$ a $\mathcal{F}_t$–BM valued in $\mathbb{R}^r$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$.

Definition I.C.10. *An Itô process (X) is a continuous and $\mathcal{F}_t$-adapted process such that*

$$X_t = X_0 + \int_0^t b_\theta d\theta + \int_0^t \sigma_\theta dW_\theta \;\; \mathbb{P} \; p.s.$$

$$dX_t = b_t dt + \sigma_t dW_t, X(0) = X_0$$

where

1) X_0 *is a $\mathcal{F}_0$ measurable r.v.*

2) $(b_t)_{0 \leq t \leq T}$ *is a $\mathbb{R}^d$ valued process, $\mathcal{F}_t$–adapted and such that* $\int_0^T |b_s| ds < +\infty \;\; \mathbb{P} \; p.s.$

3) $(\sigma_t)_{0 \leq t \leq T}$ *is a matrix valued process such that $\forall t \in [0,T]$, $\sigma_t \in \mathcal{L}(\mathbb{R}^r, \mathbb{R}^d)$ and $i = 1, \ldots, d$, $j = 1, \ldots, r$ $\sigma^{i,j}$ is $\mathcal{F}_t$–adapted and* $\displaystyle\int_0^T |\sigma_s^{i,j}|^2 ds < +\infty \; \mathbb{P} \; p.s.).$

If $(\sigma^{i,j} \in M^2_{\mathcal{F}}(0,T))$, (X_t) is a martingale if and only if, for almost all $t \in [0,T]$, $\mathbb{P} - ps$, $b^i_t = 0$, $i = 1, \ldots, d$.

Let $u(t,x)$ on $[0,T] \times \mathbb{R}^d$, valued in $\mathbb{R}$, in $C^{1,2}([0,T] \times \mathbb{R}^d)$ (one continuous derivatice in time, two in space)

$$\begin{aligned}
&\text{gradient of } u \\
&\nabla u(t,x) \;=\; \left(\frac{\partial u}{\partial x_i}(t,x), 1 \le i \le d\right) \in \mathbb{R}^d \\
&\text{Hessian of } u \\
&u_{xx}(t,x) \;=\; \left(\frac{\partial^2 u}{\partial x_i \partial x_j}(t,x), 1 \le i,j \le d\right) \in \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d).
\end{aligned}$$

Proposition I.C.11. *$\mathbb{P}$- almost surely, for all $t \in [0,T]$,*

$$\begin{aligned}
u(t,X_t) =& u(0,X_0) + \int_0^t \frac{\partial u}{\partial s}(s,X_s)ds + \int_0^t \nabla u(s,X_s)\cdot dX_s \\
&+ \frac{1}{2}\int_0^t \mathrm{trace}\left(\sigma_s \sigma_s^t u_{xx}(s,X_s)\right) ds \\
=& u(0,X_0) + \int_0^t \frac{\partial u}{\partial s}(s,X_s)ds + \int_0^t \nabla u(s,X_s)\cdot dX_s \\
&+ \frac{1}{2}\int_0^t \sum_{i,j=1}^d \frac{\partial^2 u}{\partial x_i \partial x_j} \sum_{k=1}^r \sigma_s^{ik}\sigma_s^{jk} ds
\end{aligned}$$

$$\begin{aligned}
du(t,X_t) = &\left(\frac{\partial u}{\partial t}(t,X_t) + \nabla u(t,X_t)\cdot b_t\right) dt + \left(\frac{1}{2} trace(\sigma_t \sigma_t^t u_{xx}(t,X_t)\right) dt \\
&+ \nabla u(t,X_t)\cdot \sigma_t dW_t.
\end{aligned}$$

Proposition I.C.12. (Integration by part formula for deterministic integrand.) *$f : [0,T] \to \mathbb{R}$ in C^1 and W BM $\mathbb{R}$-valued*

$$\int_0^t f(s)dW_s = f(t)W_t - \int_0^t f'(s)W_s ds.$$

Applying Itô formula to $f(t)W_t$:

$$f(t)W_t = f(0)0 + \int_0^t f'(s)W_s ds + \int_0^t f(s)dW_s.$$

Proposition I.C.13. (Stochastic integration by part)). *Let W BM $\mathbb{R}$-valued. Consider two Itô processes valued in $\mathbb{R}$:*

$$\begin{cases} dX_t = b_t dt + \sigma_t dW_t, \\ dY_t = \beta_t dt + \gamma_t dW_t. \end{cases}$$

Then

$$d(X_tY_t) = X_t dY_t + Y_t dX_t + \sigma_t\gamma_t dt,$$

or

$$X_tY_t = X_0Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \int_0^t \sigma_s\gamma_s ds.$$

Proof. Apply the 2D Itô formula, for $u(t, x, y) = xy$ to the process (X_t, Y_t). □

I.C.8 Tightness criteria

Definition I.C.14. *Let (S, d) be a metric space and let Π be a family of probability measures on $(S, Borel(S))$. We say that Π is relatively compact if every sequence of elements of Π contains a weakly convergent subsequence. We say that Π is tight if for every $\epsilon > 0$, there exists a compact set $K \subseteq S$ such that $P(K) \leq 1 - \epsilon$, for every $P \in \Pi$.*

If $X_\alpha, \alpha \in A$ is a family of random variables, each one defined on a probability space $(1\Omega_\alpha, \mathcal{F}_\alpha, P_\alpha)$ and taking values in S, we say that this family is relatively compact or tight if the family of induced measures $\{P \circ X_\alpha^{-1}\}$ has the appropriate property.

Theorem I.C.15. *(Prohorov (1956)). Let Π be a family of probability measures on a complete, separable metric space (S, d). This family is relatively compact if and only if it is tight.*

The spaces $C([0, T]; \mathbb{R}^d)$, $T < +\infty$ and $C([0, +\infty); \mathbb{R}^d)$

We provide $C([0, T]; \mathbb{R}^d)$ with the topology of uniform convergence.

$$d_{\text{u}}(x, y) = \|x - y\|_\infty = \sup_{t\in[0,T]} |x(t) - y(t)|.$$

We provide $C([0, +\infty); \mathbb{R}^d)$ with the metric of local uniform convergence associated to the metric d_{ul} :

$$d_{\text{ul}} = \sum_{n\in\mathbb{N}} 2^{-n} \left(1 \wedge \sup_{0\leq t\leq n} |x(t) - y(t)|\right).$$

Theorem I.C.16. *The space $C([0, T]; \mathbb{R}^d)$, provided with the distance d_u is complete and separable. Moreover the Borelian sigma-algebra of $C([0, T]; \mathbb{R}^d)$ is equal to the sigma-algebra generated by the coordinates :*

$$\mathcal{B}(C([0, T]; \mathbb{R}^d)) = \sigma\left(\{x \in C([0, T]; \mathbb{R}^d), (x(t_1), \ldots, x(t_p)) \in A, p \geq 1, A \in \mathcal{B}((\mathbb{R}^d)^p)\}\right).$$

The space $C([0, +\infty); \mathbb{R}^d)$, provided with the distance d_{ul} verifies the same properties (see e.g. [Jacod and Shiryaev, 2013]).

The Kolmogorov-Chentsov criteria plays an important role in many construction of SDE solution by limit of approximations.

Theorem I.C.17 (Kolmogorov-Chentsov). *Let $(X^n_t, t \in [0,T], n = 1, 2, \cdots)$, a familly of p càdlàg processes.*
The sequence of laws of (X^n) is tight if there exists some positive constants, K, γ and α, uniform in n, such that

$$\begin{aligned}&(a)\ \ \mathbb{E}[|X^n_t - X^n_s|^\gamma] \leq K|t-s|^{1+\alpha},\ \ 0 \leq s,t \leq T,\\ &(b)\ \ \mathbb{E}[|X^n_t|^\gamma] \leq K,\ \ \ 0 \leq t \leq T\end{aligned}$$

(see e.g, [H. Kinita, 1986], [Karatzas and Shreve, 1988][Chapter 2. Problem 4.11]).

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Chapter II

Introduction to McKean-Vlasov SDEs

II.1 Mean field approximation

II.1.1 Systems of particles with pairwise interactions

Consider a system of N identical particles, interacting in pairs, (molecules, atoms, ions in a gas or solvent). We assume that these particles can be considered as points, so that describing the motion of the centre of gravity, the centre of charge, is sufficient (the volume of the particle is constant). We deal with the description of N state variables

$$(\overline{X},\overline{V}) = (X^{(1,N)}, V^{(1,N)}, \dots X^{(N,N)}, V^{(N,N)})$$

in the phase space $\mathbb{R}^d \times \mathbb{R}^d$.

Particle by particle, we can start from the following typical system :

$$\begin{cases} dX_t^{(i,N)} = \overline{V}_t^{(i,N)} dt \\ dV_t^{(i,N)} = -\nabla_x \Phi(X_t^{(i,N)})dt - \displaystyle\sum_{j=1;j\neq i}^{N} \nabla_x \beta(|X_t^{(i,N)} - X_t^{(j,N)}|)dt + \sigma dW_t^i \end{cases} \tag{II.1}$$

where (W) is a Brownian motion in $\mathbb{R}^{Nd}$.

Some examples for the potentials Φ and β:

- Coulomb potential (electric field), for a charged particle of charge q, of position x, which undergoes an ambient field from a source of charge Q at point x_Q:

$$\Phi_Q(x) = \frac{1}{4\pi\epsilon_0}\frac{qQ}{|x - x_Q|}$$

where $\epsilon_0 \simeq 8.854 \times 10^{12} F.m^{-1}$[1] is a universal constant called the dielectric constant, or permittivity

[1] 1 farad (F) is the capacitance of an insulated electrical conductor for which an addition of 6.241×10^{18} electrons causes an increase in its potential of 1 volt.

of the vacuum. And for two particles of identical charge q located at x and y

$$\beta_c(x,y) = \frac{1}{4\pi\epsilon_0}\frac{q^2}{|x-y|}.$$

- Gravitational field potential, for a particle of mass m, of position x, which undergoes an ambient field from a source of mass M at the point x_M :

$$\Phi_M(x) = G\frac{mM}{|x-x_M|}.$$

  where G is the gravitational constant.

- Lennard-Jones potential: strong interaction between (neutral) atoms in a molecule.

$$\beta_{LJ}(|x-y|) = P_0\left[\left(\frac{r_0}{|x-y|}\right)^{12} - 2\left(\frac{r_0}{|x-y|}\right)^{6}\right].$$

  The attractive term to the 6th power, dominant at long distances, is the Van der Waals interaction. The repulsive term of exponent 12, dominant at short distances, accounts for the electrostatic repulsion between electrons, which prevents the mutual interpenetration of the electron clouds of two atoms.

The initial state of the system is described by the distribution $P_0^N(d\overline{x}, d\overline{v})$ over $\mathbb{R}^{2dN}$; $P_t^N(d\overline{x}, d\overline{v})$ denotes the marginal distribution at time t.

Definition II.1.1 (Symmetric law)**.** *Let E be a separable metric space. Let P^N be a probability measure on E^N. P^N is symmetric on E^N if*

$$\forall B \in \mathcal{B}(E^N), \qquad P^N(B_1\times\cdots\times B_N) = P^N(B_{\tau(1)}\times\cdots\times B_{\tau(N)}),$$

for any permutation τ on $\mathbb{N}$, leaving invariant the complementary of a finite set in $\{1,\ldots,N\}$. If $P^N = \mathcal{L}(Y^i, i=1,\ldots,N)$, we say that the variables (Y^i) are exchangeable.

Remark II.1.2. • *If P^N is symmetric, then for $\Phi : E^k \to \mathbb{R}$ bounded measurable, $k \leq N$*

$$\begin{aligned}\left\langle \Phi, P^N(dy_1,\ldots,dy^N)\right\rangle &= \mathbb{E}\left[\Phi(Y_1,\ldots,Y_k)\right]\\ &= \frac{(N-k)!}{N!}\sum_{i_1\neq i_2,\ldots,\neq i_k}\mathbb{E}[\Phi(Y_{i_1},\ldots,Y_{i_k})].\end{aligned}$$

• Symmetry plays a significant role in the mean-field approximation and its mathematical justification, called the chaos propagation property.

• If the particles are not exchangeable, we cannot write a reduced dynamics for the system.

We assume that the initial distribution $P_0^N(\overline{X}, \overline{V})$ is symmetric.

Then the distribution $P_t^N = U_N \circ (\overline{X}_t, \overline{V}_t)^{-1}$ remains symmetric at all times, as the dynamics (II.1) is invariant by the permutations τ on $\{1,\ldots,N\}$.

Using the exchangeability of the system, we compute the mean energy of the system $\mathcal{E}$:

$$\mathcal{E} = N\mathbb{E}[(V_t^{(1,N)})^2] + N\mathbb{E}[\Phi(X_t^{(1,N)}) + N(N-1)\mathbb{E}\beta(|X_t^{(1,N)} - X_t^{(2,N)}|)$$

Even by distributing the energy over the N particles, the energy of a given particle in the system will tend towards infinity with N, if we do not renormalise the interaction kernel by N or $N-1$.

We are interested, in the asymptotic $N \to +\infty$, by the reduced dynamics of one particle among the others, in order not to have to follow the N physical particles at the same time.

Renormalized dynamics:

$$\begin{cases} dX_t^{(i,N)} = V_t^{(i,N)} dt \\ dV_t^{(i,N)} = -\nabla_x \Phi(X_t^{(i,N)}) - \dfrac{1}{N-1} \displaystyle\sum_{j=1;j\neq i}^{N} \nabla_x \beta(|X_t^{(i,N)} - X_t^{(j,N)}|) + \sigma dW_t^i \end{cases}$$

and its Fokker-Planck equation (also known in this context as Liouville equation)

$$\begin{cases} \dfrac{\partial}{\partial t} P^N + \displaystyle\sum_{i=1}^{N} v^i \frac{\partial}{\partial x^i} P^N = \sum_{i=1}^{N} \nabla_x \Phi(x^i) \frac{\partial}{\partial v^i} P^N + \frac{1}{N-1} \sum_{i,j=1;i\neq j}^{N} \nabla_x \beta(|x^i - x^j|) \frac{\partial}{\partial v^i} P^N + \frac{1}{2}\sigma^2 \Delta_{\bar{v}} P^N. \\ P_0^N \text{ given.} \end{cases}$$

Set $\Omega_i = \mathbb{R}^{2d}$, and define the marginals P_t^i on $\Omega_1 \times \cdots \times \Omega_i$ by

$$\begin{aligned} P_t^1 &= \int_{\Omega_2 \times \ldots \times \Omega_N} P_t^N(x^1, v^1, x^2, v^2, \ldots, x^N, v^N) dx^2 dv^2 \ldots dx^N dv^N, \\ P_t^2 &= \int_{\Omega_3 \times \ldots \times \Omega_N} P_t^N(x^1, v^1, x^2, v^2, x^3, v^3, \ldots, x^N, v^N) dx^3 dv^3 \ldots dx^N dv^N \\ \text{until} \qquad P_t^{N-1} &= \int_{\Omega_N} P_t^N(x^1, v^1, \ldots, x^{N-1}, v^{N-1}, x^N, v^N) dx^N dv^N. \end{aligned}$$

We assume that P_t^N vanishes on $\partial(\Omega_1 \times \ldots \times \Omega_N)$ and we integrate the Liouville equation over $\Omega_2 \times \ldots \times \Omega_N$:

$$\int_{\Omega_2 \times \ldots \times \Omega_N} \sum_{i=1}^{N} v^i \frac{\partial}{\partial x^i} P^N dx^2 dv^2 \ldots dx^N dv^N = v^1 \frac{\partial}{\partial x^1} P_t^1 + \sum_{i=2}^{N} \int_{\Omega_2 \times \ldots \times \Omega_N} v^i \frac{\partial}{\partial x^i} P^N dx^2 dv^2 \ldots dx^N dv^N$$

The second term vanishes. Same for

$$\int_{\Omega_2 \times \ldots \times \Omega_N} \frac{1}{N-1} \sum_{i,j=1;i\neq j}^{N} \nabla_x \beta(|x^i - x^j|) \frac{\partial}{\partial v^i} P^N dx^2 dv^2 \ldots dx^N dv^N,$$

where only the contribution $i = 1$ remains:

$$\int_{\Omega_2 \times \ldots \times \Omega_N} \frac{1}{N-1} \sum_{j=2}^{N} \nabla_x \beta(|x^1 - x^j|) \frac{\partial}{\partial v^1} P^N dx^2 dv^2 \ldots dx^N dv^N.$$

Laws being symmetric:

$$\int_{\Omega_2\times\ldots\times\Omega_N} \nabla_x\beta(|x^1-x^2|)\frac{\partial}{\partial v^1}P^N dx^2dv^2\ldots dx^Ndv^N$$
$$= \int_{\Omega_2} \nabla_x\beta(|x^1-x^2|)\frac{\partial}{\partial v^1}\int_{\Omega_3\times\ldots\times\Omega_N} P^N dx^3dv^3\ldots dx^Ndv^Ndx^2dv^2 = \int_{\Omega_2} \nabla_x\beta(|x^1-x^2|)\frac{\partial}{\partial v^1}P^2_t dx^2dv^2$$

And we end with

$$\begin{cases} \dfrac{\partial}{\partial t}P^1 + v^1\dfrac{\partial}{\partial x^1}P^1 = \nabla_x\Phi(x^1)\dfrac{\partial}{\partial v^1}P^1 + \displaystyle\int_{\Omega_2}\nabla_x\beta(|x^1-x^2|)\dfrac{\partial}{\partial v^1}P^2 + \dfrac{1}{2}\sigma^2 v^1 P^1. \\ P^1_0 \text{ given.} \end{cases}$$

This equation for P^1 is unclosed!. If we write the equation for P^2, following the same procedure:

$$\frac{\partial}{\partial t}P^2 + v^1\frac{\partial}{\partial x^1}P^2 + v^2\frac{\partial}{\partial x^2}P^2$$
$$= \nabla_x\Phi(x^1)\frac{\partial}{\partial v^1}P^2 + \nabla_x\Phi(x^2)\frac{\partial}{\partial v^2}P^2 + \int_{\Omega_3}\nabla_x\beta(|x^1-x^2|)\left(\frac{\partial}{\partial v^1}P^3 + \frac{\partial}{\partial v^2}P^3\right) + \frac{1}{2}\Delta_{v^1,v^2}P^2$$

and so on. We thus obtain a system of N chained equations, the last one being the Liouville equation itself for P^N.

This system of equations constitutes the BBGKY hierarchy, after the names of the physicists who established it independently, Bogolioubov (1946), Born (1946), Green (1946), Kirkwood (1946) and Yvon (1935)

II.1.2 Mean field approximation

This approximation consists in *neglecting* the correlations in phases, when the number of particles becomes large, and thus allows us to write that

$$P^2_t(x^1,v^1,x^2,v^2) = P^1_t(x^1,v^1)P^1_t(x^2,v^2).$$

The equation for P^1 then becomes closed: as N tends to infinity,

$$\begin{cases} \dfrac{\partial}{\partial t}P^1 + v^1\dfrac{\partial}{\partial x^1}P^1 \simeq \nabla_x\Phi(x^1)\dfrac{\partial}{\partial v^1}P^1 \\ \qquad\qquad + \dfrac{1}{2}\left(\displaystyle\int_{\Omega_2}\nabla_x\beta(|x^1-x^2|)P^1(dx^2,dv^2)\right)\dfrac{\partial}{\partial v^1}P^1 + \dfrac{1}{2}\sigma^2\Delta_{v^1}P^1. \\ P^1_0 \text{ given.} \end{cases}$$

This PDE is called the Vlasov equation after the physicist who first proposed this type of reduced dynamics in plasma physics (1936).

The mathematical justification of the mean field approximation is due to the pioneering work of Kac (1959) who was interested in another equation also resulting from a BBGKY type hierarchy, the Boltzmann equation.

Kac introduced the notion of chaos propagation; McKean took up this notion in 1967 to construct *McKean non-linear SDE*.

Indeed, from the form of the reduced PDE obtained, a reduced SDE is proposed for the position and the instantaneous velocity of a particle in kinetic form:

$$\begin{cases} dX_t = V_t dt \\ dV_t = -\nabla_x \Phi(X_t)dt - \nabla_x \beta[X_t; p_t]dt + \sigma dW_t \\ p_t = \mathbb{P} \circ X_t^{-1} \end{cases}$$

where

$$\nabla_x \beta[x, \mu] := \int_{\mathbb{R}^d} \nabla_x \beta(x, y)\mu(dy).$$

This reduced model obtained is called a one-point (or one-particle) model.

In physics nowadays, a growing number of second-order approximations are being proposed to take into account spatial correlation structures. The proposed reduced model is then called a two-point (or two-particle) model.

II.2 McKean-Vlasov SDEs (and PDEs)

We are interested in the following class of non linear SDEs (in the sense of McKean), having the following prototypical form. We introduce

$$\begin{aligned} &\sigma : \mathbb{R}^d \times \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^r, \mathbb{R}^d), \\ &b : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d, \qquad \text{mesurable and bounded.} \end{aligned}$$

Let $\mathcal{M}^1(\mathbb{R}^d)$ be the set of measure of probability one. We define, for all $\rho \in \mathcal{M}^1(\mathbb{R})$, $x \in \mathbb{R}^d$,

$$\begin{aligned} b[x, \rho] &:= \int_{\mathbb{R}} b(x, y)\rho(dy), \\ \sigma[x, \rho] &= \int_{\mathbb{R}^d} \sigma(x, y)\rho(dy) \quad \text{ and } \; a[x; p] = \sigma[x, p]^t \sigma[x, p]. \end{aligned}$$

For all $T > 0$, we want to construct a solution to the following equation

$$\begin{cases} X_t = X_0 + \int_0^t \sigma[X_s, P_s]dW_s + \int_0^t b[X_s, P_s]ds, \; \text{ for } 0 \leq t \leq T, \\ P_t = \mathbb{P} \circ X_t^{-1}, \end{cases} \tag{II.2}$$

on a propabilistic space $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), \mathbb{P})$, endowed with a Brownian motion W of dimension r and with a r.v X_0, $\mathcal{F}_0$-measurable, of law P_0.

• Even if b and σ are not time-dependent functions, the SDE (II.2) could not be considered as a time-homogeneous problem. In particular, (X) alone is not a Markov process, as the coefficients keep the memory of the past, via the law dependency

Example : the solution of

$$dX_t = C(\mathbb{E}(X_t) - X_t)dt + dW_t; X_0 \text{ given}$$

behaves differently from

$$dY_t = C(m - Y_t)dt + dW_t; X_0 \text{ given.}$$

Il particular for all $t > 0$, $\mathbb{E}[X_t] = \mathbb{E}[X_0]$, but $\mathbb{E}[Y_t] = \mathbb{E}[X_0]\exp(-Ct) + m(1 - \exp(-Ct))$. Both solutions only coincide for $m = \mathbb{E}[X_0]$.

• A solution(X) to (II.2) is also a solution to

$$X_t = X_0 + \int_0^t \widetilde{\mathbb{E}}\left[\sigma(X_s, Y_s)\right] dW_s + \int_0^t \widetilde{\mathbb{E}}\left[b(X_s, Y_s)\right] ds,\ 0 \leq t \leq T,$$

where Y is a copy of X, on another probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t), \widetilde{\mathbb{P}}))$.

The associated Vlasov-Fokker-Planck euqation

Given a solution to the SDE (II.2), the marginal laws $(P_t, t \in [0, T])$ are solution in the distribution sense of the parabolic PDE

$$\begin{cases} \dfrac{\partial P_t}{\partial t} = \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j}\left(a_{i,j}[x; P_t]P_t\right) - \sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left(b_i[x; P_t]P_t\right) & \text{in } \mathbb{R}^d, \\ P_0 \text{ given.} \end{cases} \tag{II.3}$$

(II.3) is a **Vlasov**-Fokker-Planck equation . In particular it conserves the total mass. Indeed, for $B^R = \{\|y\|_d \leq R\}$,

$$\frac{d}{dt}\int_{B^R} P_t(dx) = \int_{B^R} \text{div}\left(\partial_{x_j} a_{\cdot,j}[x; P_t]P_t - b[x, P_t]P_t\right) dx.$$

Using divergence Theorem I.B.1, assuming that the measures P_t do not charge mass at infinity, we get

$$\frac{d}{dt}\left(\int_{B^R} P_t(dx)\right) = \int_{\{y=R\}} \left((\partial_{x_j} a_{\cdot,j}[x; P_t]P_t - b[x, P_t]P_t) \cdot n(y)\right) \underset{R \to +\infty}{\longrightarrow} 0.$$

So, if P_0 is a of mass one, it is also the case for P_t, for $t \geq 0$.

For all finite $T > 0$ fini, $(P_t, t \in [0, T])$ is a weak solution of (II.3), if for all $\phi \in C_c^2(\mathbb{R}^d)$,

$$\frac{d}{dt}\langle P_t, \phi\rangle = \left\langle P_t, \frac{1}{2}\sum_{i,j} a_{ij}[x; P_t]\frac{\partial^2 \phi}{\partial_{x_i}\partial_{x_j}}(x) + \sum_{i=1}^{d} b_i[x; P_t]\frac{\partial \phi}{\partial_{x_i}}(x)\right\rangle. \tag{II.4}$$

This weak solution formulation for the PDE makes appear the differential operator

$$\mathcal{L}_{[p]}\phi(x) = \frac{1}{2}\sum_{i,j} a_{ij}[x;p]\frac{\partial^2\phi}{\partial_{x_i}\partial_{x_j}}(x) + \sum_{i=1}^{d} b_i[x;p]\frac{\partial\phi}{\partial_{x_i}}(x)$$

as the infinitesimal generator for the McKean SDE.

Theorem II.2.1. *We assume that*

(a) the kernel-coefficients $b(x,y)$ and $\sigma(x,y)$ are continuous and bounded functions on $\mathbb{R}^{2d}$.

(b) $\mathbb{E}|X_0|^\beta < \infty$, for a $\beta > 0$. We denote by P_0 the law of X_0.

Then for all $T < +\infty$, there exists a weak solution to the McKean SDE (II.2).

We come back later on the proof of this theorem. We first introduce the tools for the propagation of chaos method.

II.3 Propagation of Chaos

Let $(E, \mathcal{E})$ a measurable space, endowed with it sigma-algebra $\mathcal{B}$; $\mathcal{M}^1(E)$ is the set of probability measure on E.

The mathematical justification of the mean field approximation is due to the pioneering work of [Kac, 1954] who was interested in the BBGKY hierarchy, also produced by the Boltzmann equation (the integro-differential equation of the kinetic theory of a sparse non-equilibrium gas). Kac introduces the notion of chaos propagation, the modern presentation made here is due to [Sznitman, 1991b].

Definition II.3.1 (Weak convergence of measures)**.** *Let $(E, \mathcal{E})$ a probability space.*
A sequence of finite measures μ_n on E converge weakly to the measure μ, if for all functions f continuous and bounded on E, $f \in C_b(E)$,

$$\langle f, \mu_n\rangle = \int_E f(x)\mu_n(dx) \underset{n\to+\infty}{\longrightarrow} \langle f, \mu\rangle = \int_E f(x)\mu(dx).$$

Now let (E, d_E) be a separable metric space, endowed with its distance d_E. $\mathcal{B}(E)$ is the Borel sigma algebra of E. We denote by $(X^i, i = 1 \ldots N)$ the canonical coordinates on E^N, $N \in \mathbb{N}$.

Definition II.3.2 (Chaoticity). *Let $(U_N, N \in \mathbb{N})$ a sequence of symmetric probability measures on E^N.*
Let u a probability measure on E.
We say that the sequence $(U_N, N \in \mathbb{N})$ is u-chaotic if, for all $k \geq 1$, for all functions $\phi_1, \dots, \phi_k$ in $C_b(E)$.

$$\lim_{N \to +\infty} \left\langle U_N, \phi_1 \otimes \dots \otimes \phi_k \underbrace{\otimes 1 \dots \otimes 1}_{N-k\otimes} \right\rangle = \prod_{i=1}^{k} \langle u, \phi_i \rangle,$$

or equivalently

$$\lim_{N \to +\infty} \mathbb{E}_{U_N} \left[\phi_1(X_1) \times \dots \times \phi_k(X_k) \right] = \prod_{i=1}^{k} \mathbb{E}_u[\phi_i(X)]$$

Above, X is the canonical coordinate on E.
Thus, in the limit $N \to +\infty$, the k marginal laws of (U_N) are iid, of law u.

We denote

$$\mu_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{X^i}$$

the empirical measure associated to U_N.

- μ_N is a $r.v.$ on (E^N, U_N) valued in $\mathcal{M}^1(E)$.

Theorem II.3.3. *The sequence $(U_N, N \in \mathbb{N})$ is u-chaotic if and only if the sequence of variable (μ^N, N) valued in $\mathcal{M}^1(E)$ converges in law to u. In other words, we have the convergence of $\mathcal{L}(\mu_N)$ weakly in $\mathcal{M}^1(\mathcal{M}^1(E))$ toward δ_u when $N \to +\infty$.*

Definition II.3.4 (Tight family of probabilities). *Let (E, d) a metric space. A family $\{\mathbb{P}_\alpha\}$ of probabilities on $(E, \mathcal{B}(E))$ is tight, if for all $\varepsilon > 0$, there exists a compact set $K \subseteq E$ such that*

$$\sup_\alpha \mathbb{P}_\alpha(E - K) \leq \varepsilon.$$

The family of the $\mathbb{P}_\alpha$ support stay concentrated on a compact K.

Definition II.3.5 (Relative compactness). *$\{\mathbb{P}_\alpha\}$ is relatively compact is all sequences of $\{\mathbb{P}_\alpha\}$ contains a weak converging sub-sequence.*

Theorem II.3.6 (Prokhorov's Theorem). *Let (E,d) a complete[a] and separable[b] The family $\{\mathbb{P}_\alpha\}$ on $(E,\mathcal{B}(E))$ is tight if and only if $\{\mathbb{P}_\alpha\}$ is relatively compact.*

[a] E is said to be complete if every Cauchy sequence of E has a limit in E. The completeness property depends on the distance.

[b] A separable space is a topological space containing a finite or countable and dense subset, i.e. containing a finite or countable set of points whose closure is equal to the whole topological space.

Lemma II.3.7 (Tension criterion for propagation of chaos ([Sznitman, 1991b])). *When (E,d) is complete and separable, the random variables $(\mu_N, N\geq 1)$ with values in $\mathcal{M}^1(E)$ are tight (i.e. the laws $\mathcal{L}(\mu_N)$ are tight) if-and-only-if the laws on E of X^1 under U_N are tight.*

Proof of theorem II.2.1. The proof consists in construct a solution Q to the martingale problem associated to $(\mathcal{L}_{[\cdot]}, P_0)$ where, for all $p\in\mathcal{M}^1(\mathbb{R}^d)$,

$$\mathcal{L}_{[p]}\phi(x) = \frac{1}{2}\sum_{i,j} a_{ij}[x;p]\frac{\partial^2\phi}{\partial_{x_i}\partial_{x_j}}(x) + \sum_{i=1}^d b_i[x;p]\frac{\partial\phi}{\partial_{x_i}}(x).$$

Definition II.3.8. *$Q\in\mathcal{M}^1(C_T)$ is a solution to the martingale problem associated to $(\mathcal{L}_{[\cdot]},\mu)$ if*

(i) $Q_0=\mu$.

(ii) For all $f\in C_b^2(\mathbb{R}^d)$,

$$\mathcal{M}_t^f = f(x(t)) - f(x(0)) - \int_0^t \left(\mathcal{L}_{[Q_\theta]}f\right)(x(\theta))d\theta$$

is a Q-martingale.

• We consider the space $(\mathbb{R}^d\times C(\mathbb{R}^+;\mathbb{R}^d), \mathcal{B}(\mathbb{R}^d\times C(\mathbb{R}^+;\mathbb{R}^d))^{\mathbb{N}^*}$ endowed with the product measure $(P_0\otimes\mathcal{W})^{\otimes\mathbb{N}^*}$, where $P_0=\mathcal{L}(X_0)$ and $\mathcal{W}$ is the Wiener measure of dimension $d\wedge r$.

We consider the family of processes $((X^{i,n}, i=1,\ldots,n), n\in\mathbb{N}^*)$ solution to

$$X_t^{i,n} = X_0^i + \int_0^t \sigma[X_s^{i,n};\mu_s^n]dW_s^i + \int_0^t b[X_s^{i,n};\mu_s^n]ds \qquad t\in\mathbb{R}^+, i\in\{1,\ldots,n\}, \tag{II.5}$$

where $(X_0^i, W^i, i\in\mathbb{N}^*)$ are the canonical coordinates on $((\mathbb{R}^d\times C(\mathbb{R}^+;\mathbb{R}^d), \mathcal{B}(\mathbb{R}^d\times C(\mathbb{R}^+;\mathbb{R}^d)))^{\mathbb{N}^*}$ and where

$$\mu_\cdot^n = \frac{1}{n}\sum_{j=1}\delta_{X_\cdot^{j,n}}\in\mathcal{M}^1(C(\mathbb{R}^+,\mathbb{R}^d)).$$

Equivalently

$$X_t^{i,n} = X_0^i + \int_0^t \sigma[X_s^{i,n}; \mu_s^n] dW_s^i + \int_0^t b[X_s^{i,n}; \mu_s^n] ds, \quad t \in \mathbb{R}^+, i \in \{1, \ldots, n\},$$

or, in a vectorial form

$$\underline{X}_t = \underline{X}_0 + \int_0^t \Sigma(\underline{X}_s) d\underline{W}_s + \int_0^t B(\underline{X}_s) ds$$

with B and Σ continuous and bounded, and the initial condition $\underline{X}_0$ admitting a positive moment.
Applying theorem I.3.8, we get the existence of a solution to the linear system of size n.

• The next step consists in applying the tightness criterion in Lemma II.3.7.
We use the Kolmogorov Chentsov criterion I.C.17 for Itô processes :

Exercise II.1. *Applying the Itô formula, to prove that, for* $0 \le s \le t \le T$,

$$\|X_t^{1,n} - X_s^{1,n}\|^4 \le C|t-s|^2.$$

Applying Lemma II.3.7, we get the tightness of the sequence of the laws $(\mathcal{L}(X^{1,n}, n)$, and so that the laws of the (μ^n, n) r.v. are tight in $\mathcal{M}^1(\mathcal{M}^1(C_T))$.

Let denote by $(Q^n = \mathcal{L}(\mu^n), n)$ a converging sub sequence, and let Q^∞ the limit of the sub sequence, still indexed by n.

We consider the infinitesimal generator

$$\mathcal{L}(p)\phi(x) = \frac{1}{2} \sum_{i,j} a_{ij}[x; p] \frac{\partial^2 \phi}{\partial_{x_i} \partial_{x_j}}(x) + \sum_{i=1}^d b_i[x; p] \frac{\partial \phi}{\partial_{x_i}}(x).$$

And we recall the following characterization of the conditioning by $\sigma\{x_\theta; 0 \le \theta \le s\}$:

$$\mathbb{E}\left[M_t - M_s | \sigma\{x_\theta; 0 \le \theta \le s\}\right] = 0$$

$$\Leftrightarrow$$

$$\forall l \in \mathbb{N}^*, \forall g \in C_b(\mathbb{R}^{ld}), \forall 0 \le t_1 < \ldots < t_l < s; \mathbb{E}\left[(M_t - M_s) g(x(t_1), \ldots, g(x(t_l))\right] = 0.$$

For a given $f \in C_b^2(\mathbb{R}^d)$, and $0 \le s \le t \le T$.
We also fix an arbitrary $l \in \mathbb{N}^*$, a $g \in C_b(\mathbb{R}^{ld})$ and a l-partition, $0 \le t_1 < \ldots < t_l < s$.
We consider the application $m \mapsto F(m)$ on $\mathcal{M}^1(C_T)$, defined by

$$F(m) = \left\langle m, \left(f(x(t)) - f(x(s)) - \int_s^t \mathcal{L}(m_\theta) f(x_\theta) d\theta \right) \times g(x(t_1), \ldots, g(x(t_l)) \right\rangle$$

F **is continuous on** $\mathcal{M}^1(C_T)$, par the continuity of the coefficients b and σ, of f and derivates, of g, and of and of coordinates applications $m \to m_\theta$.

• **Let show that, for Q^∞ almost m, $F(m) = 0$,** then m is solution to the MP and II.2.1 is proven.

F being continuous,

$$\begin{aligned}\mathbb{E}_{Q^\infty}|F| &= \lim_{n\to+\infty} \mathbb{E}_{Q^n}|F| \\ &= \lim_{n\to+\infty} \mathbb{E}\left|\frac{1}{n}\sum_{i=1}^n \left\{ f(X_t^{i,n}) - f(X_s^{i,n}) - \int_s^t \mathcal{L}(\mu_\theta^n) f(X_\theta^{i,n}) d\theta \right\} g(X_{t_1}^{i,n}, \dots, X_{t_l}^{i,n})\right|.\end{aligned}$$

From the Itô formula,

$$\begin{aligned}\mathbb{E}_{Q^\infty}|F| &= \lim_{n\to+\infty} \mathbb{E}\left|\left(\frac{1}{n}\sum_{i=1}^n \int_s^t \sum_{j=1}^d \partial_{x_j} f(X_\theta^{i,n}) \sum_{k=1}^r \sigma^{j,k}[X_\theta^{i,n}, \mu_\theta^n] dW_\theta^{i,(k)}\right) g(X_{t_1}^{i,n}, \dots, X_{t_l}^{i,n})\right| \\ &\leq \|g\|_\infty \lim_{n\to+\infty} \mathbb{E}\left|\frac{1}{n}\sum_{i=1}^n \int_s^t \sum_{j=1}^d \partial_{x_j} f(X_\theta^{i,n}) \sum_{k=1}^r \sigma^{j,k}[X_\theta^{i,n}, \mu_\theta^n] dW_\theta^{i,(k)}\right|\end{aligned}$$

Now using the Itô isometry ,

$$\begin{aligned}&\mathbb{E}\left(\frac{1}{n}\sum_{i=1}^n \int_s^t \sum_{j=1}^d \partial_{x_j} f(X_\theta^{i,n}) \sum_{k=1}^r \sigma^{j,k}[X_\theta^{i,n}, \mu_\theta^n] dW_\theta^{i,(k)}\right)^2 \\ &= \mathbb{E}\left(\frac{1}{n^2}\sum_{i=1}^n \int_s^t \sum_{j=1}^d \left(\partial_{x_j} f(X_\theta^{i,n})\right)^2 \sum_{k=1}^r \left(\sigma^{j,k}[X_\theta^{i,n}, \mu_\theta^n]\right)^2 d\theta\right) \\ &\leq \|\nabla f\|^2 \|\sigma\|^2 C(t) \frac{1}{n}.\end{aligned}$$

So we get

$$\lim_{n\to+\infty} \mathbb{E}_{Q^n}|F| \leq \lim_{n\to+\infty} C(T)\frac{1}{\sqrt{n}} = 0.$$

We have just constructed a solution to the martingale problem, by converging the system of particles.

If we have uniqueness of the solution u to the martingale problem, then we can conclude that $Q^\infty = \delta_u$, which implies that the particle system is u-chaotic.

Without additional regularity assumptions on b and σ to run some contraction argument, to obtain the uniqueness of the solution of the martingale problem it is then difficult to do without the uniqueness of the distributional solution to the McKean Vlasov PDE (II.3).

□

II.4 Bibliography

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