Jyväskylä Summer School, 2021

Take-Home problem for the course
"Differential calculus on the Wasserstein space and mean field games"
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In this problem, $\mathcal{P}_2 = \mathcal{P}_2(\mathbb{R})$ is the space of probability measures on $\mathbb{R}$ endowed with the 2−Wasserstein distance $\mathbf{d}_2$. We say that a map $U : \mathcal{P}_2 \to \mathbb{R}$ is C^2 if

- U has an L^2−derivative $\frac{\delta U}{\delta m}$ which has itself an L^2−derivative $\frac{\delta^2 U}{\delta m^2} : \mathcal{P}_2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with
$$\frac{\delta^2 U}{\delta m^2}(m, y, y') := \frac{\delta}{\delta m}\left(\frac{\delta U(\cdot, y)}{\delta m}\right)(m, y'),$$
- $\frac{\delta^2 U}{\delta m^2}$ is continuously differentiable in the last two variables with $D_m U(m, y) := D_y \frac{\delta U}{\delta m}(m, y)$ and $D^2_{mm}U(m, y, y') := D_{y'} D_y \frac{\delta^2 U}{\delta m^2}(m, y, y')$ continuous and bounded in all variables.

1. (Examples) Let
$$U_1(m) = \int_{\mathbb{R}} \phi(x) m(dx) \text{ and } U_2(m) = \int_{\mathbb{R}\times\mathbb{R}} \psi(x, y) m(dx) m(dy) \qquad \forall m \in \mathcal{P}_2,$$
where $\phi : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R} \times \mathbb{R}$ are Borel measurable and bounded maps. Give conditions on ϕ and ψ ensuring that U_1 and U_2 are C^2 and compute $D_m U_1$, $D_m U_2$, $D^2_{mm} U_1$ and $D^2_{mm} U_2$.

2. (Finite dimensional projections) From now on we assume that $U : \mathcal{P}_2 \to \mathbb{R}$ is C^2 and set
$$U^N(\mathbf{x}) = U^N(x_1, \dots, x_N) := U(m^N_{\mathbf{x}}) \qquad \forall \mathbf{x} = (x_1, \dots, x_N) \in \mathbb{R}^N$$
where $m^N_{\mathbf{x}} = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$. We have seen in the course that U^N is of class C^1 on $\mathbb{R}^N$ with
$$D_{x_i} U^N(\mathbf{x}) = \frac{1}{N} D_m U(m^N_{\mathbf{x}}, x_i) \qquad \forall \mathbf{x} \in \mathbb{R}^N,\ i \in \{1, \dots, N\}.$$
Show that U^N is of class C^2 on $\mathbb{R}^N$ with
$$D^2_{x_i x_j} U^N(\mathbf{x}) = \frac{1}{N^2} D^2_{mm} U(m^N_{\mathbf{x}}, x_i, x_j) \qquad \forall \mathbf{x} \in \mathbb{R}^N,\ i, j \in \{1, \dots, N\},\ i \neq j$$
and $$D^2_{x_i x_i} U^N(\mathbf{x}) = \frac{1}{N^2} D^2_{mm} U(m^N_{\mathbf{x}}, x_i, x_i) + \frac{1}{N} D_y D_m U(m^N_{\mathbf{x}}, x_i) \qquad \forall \mathbf{x} \in \mathbb{R}^N,\ i \in \{1, \dots, N\}.$$

3. Let (B^i) be a a family of independent 1−dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $(\mathcal{F}_t)$ satisfying the usual conditions, let (Z^i) be i.i.d. random variables on $\mathbb{R}$ with $\mathbb{E}[|Z^1|^2] < +\infty$ and independent of the (B^i) and let (α^i_t) be i.i.d random processes adapted to $(\mathcal{F}_t)$ such that $\mathbb{E}\left[\int_0^T |\alpha^1_t|^2 dt\right] < +\infty$ $\forall T > 0$. We consider the processes
$$X^i_t = Z^i + \int_0^t \alpha^i_s ds + B^i_t \qquad i \in \{1, \dots, N\},\ t \geq 0$$
and we define the random measure $m^N_{\mathbf{X_t}}$ by $m^N_{\mathbf{X_t}} := \frac{1}{N}\sum_{i=1}^N \delta_{X^i_t}$.
Write Itô's formula between $U^N(Z^1, \dots, Z^N)$ and $U^N(X^1_t, \cdots, X^N_t)$ in term of N, of $m^N_{\mathbf{X_t}}$, of the X^i, α^i, B^i and of the derivatives $D_m U$ and $D^2_{mm} U$.

4. We assume for simplicity that $D_m U$ and $D^2_{mm} U$ are Lipschitz continuous in all variables. With the notation of the previous question, let $m(t)$ be the law of X^1_t. Admitting that
$$\mathbb{E}\left[\sup_{t\in[0,T]} \mathbf{d}_2(m^N_{\mathbf{X_t}}, m(t))\right] = 0 \qquad \forall T > 0,$$
(which holds thanks to the law of large numbers), take expectation and pass to the limit as $N \to +\infty$ in the Itô's formula obtained in the previous question to recover the equality
$$U(m(t)) = U(m(0)) + \mathbb{E}\left[\int_0^t (\alpha^1_s \cdot D_m U(m(s), X^1_s) + \frac{1}{2} D_y D_m U(m(s), X^1_s)) ds\right].$$