

- III) 2) The mean field limit in the control prob

$$\bullet I_N := \inf_{(\alpha^i)} \mathbb{E} \left[\int_0^T \left(\frac{1}{n} \sum_{i=1}^n |\alpha_s^i|^2 + F(s, X_s^i, m_s^n) \right) ds \right]$$

s.t. $\begin{cases} dX_s^i = \alpha_s^i dt + dB_s^i \\ X_0^i = z^i \end{cases}$

$$\bullet I_\infty := \inf_{\alpha'} \mathbb{E} \left[\int_0^T \left(\frac{1}{2} |\alpha_s'|^2 + F(s, X_s', \bar{\alpha}(X_s')) \right) ds \right]$$

s.t. $\begin{cases} dX_s' = \alpha_s' dt + dB_s' \\ X_0' = z' \end{cases}$

Theo (Lacker, '17)

$$I_N \rightarrow I_\infty$$

$F: [0, T] \times \mathbb{R} \times \mathcal{P}_1(\mathbb{R}^d) \rightarrow \mathbb{R}^d$ continuous & bd

$z' \in L^2([0, T], \mathbb{R}^d)$

III 3) Existence of optimal sol to I_∞
and optimality conditions

Rewrite I_∞ by setting $m(t, dx) = \overline{\mathcal{L}(X_t')}$

$$I_\infty = \inf_{(m, \alpha)} \int_0^T \int_{\mathbb{R}} \left(\frac{1}{2} |\alpha(t, x)|^2 + F(t, x, m(t)) \right) m(t, dx) dt.$$

s.t. $\begin{cases} \partial_t m - \frac{1}{2} \Delta m + \operatorname{div}(\alpha m) = 0 \\ m(0) = m_0 \quad (\vdash \mathcal{L}(z')) \end{cases}$

Explanation of (*) (Kolmogorov eq)

if $\begin{cases} dX_t' = \alpha(t, X_t) dt + dB_t' \\ X_0' = z' \end{cases} \Rightarrow \mathcal{L}(X_t') = m(t)$

↑

ex. (Itô formula)

to be true in a
weak sense.
(distribution)



$\forall \varphi \in C_c^\infty((0, +\infty) \times \mathbb{R}^d),$

$$\int_{\mathbb{R}^d} \left(\partial_t \varphi(t, u) + \frac{1}{2} \Delta \varphi(t, u) - \alpha(t, u) \cdot \nabla \varphi(t, u) \right) m(t, du) dt \\ + \int_{\mathbb{R}^d} \varphi(0, u) m_0(du) = 0$$

- The inf is taken over

$$m \in C^0([0, T], \mathcal{P}_2(\mathbb{R}^d)),$$

$\alpha : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ Borel meas. with

$$\int_0^T \int_{\mathbb{R}^d} |\alpha|^2 m < +\infty.$$

- Set $E = \alpha m \in M((0, T) \times \mathbb{R}^d, \mathbb{R}^d)$

and note that $E << m$, $\frac{dE}{dm} = \alpha$

$$\Rightarrow I_\infty = \inf_{(m, E)} \left[\int_0^T \int_{\mathbb{R}^d} \frac{1}{2} \frac{|E|^2}{m} + \int_0^T \int_{\mathbb{R}^d} F(t, x, m(t)) m(t, dx) \right]$$

$(m, \infty) \rightarrow |\alpha|^2 m$ not $\in V$

$$(m, E) \rightarrow \underbrace{\frac{|E|^2}{m}}_{\sup} \in V$$

$\boxed{\text{Theorem: } \exists \text{ a solution to } I_\infty.}$

Sketch of proof: Taken (m^n, E^n) a minimizing sequence. Set $\alpha^n = \frac{dE^n}{dm^n}$

- Then $\exists C > 0$ st. $\int_0^T \int_{\mathbb{R}^d} |x^n|^2 m^n \leq C$
- $\Rightarrow dH_2(m^n(t), m^n(s)) \leq C(t-s)^{1/2} \quad \forall t > s.$
- $\sup_{0 \leq t \leq T} \int_{\mathbb{R}^d} |n|^2 m^n(t, dx) \leq C$
- \Rightarrow precompactness of (m^n) in $C^0([0, T], \mathcal{P}(\mathbb{R}^d))$

- Explanation of $\iint |\underline{E}|^2$.
- Note that $\sup_{C \in \mathbb{R}^d} \left\{ \underbrace{\frac{a \cdot c}{|c|}}_{\substack{\in \mathbb{R}^d \\ \in \mathbb{R}}} - \underbrace{\frac{|c|^2 b}{2}}_{\substack{\in \mathbb{R} \\ \in \mathbb{R}}} \right\} = \begin{cases} \frac{|a|^2}{2b} & \text{if } b > 0 \\ 0 & \text{if } a = 0 \text{ and } b = 0 \\ +\infty & \text{otherwise} \end{cases}$

$$\iint |\underline{E}|^2 = \sup_{\varphi \in C^0([0, T] \times \mathbb{R}^d, \mathbb{R}^d)} \iint \varphi(t, x) \cdot E(dt, dx) - \iint |\varphi(t, x)|^2 m(t, dx) dt$$

- Moreover $\iint |\underline{E}^n|^2 \leq C$

because

$$\begin{aligned} \iint |\underline{E}^n|^2 &\leq \left(\iint \frac{|\underline{E}^n|^2}{m} \right)^{1/2} \left(\iint m^n \right)^{1/2} \\ &\leq C T^{1/2}. \end{aligned}$$

Theo (optimality condition)

If (m, α) is a minimum of I_∞ and if $G(m) := \int_{\mathbb{R}^d} F(t, x, m) m(dx)$ has L^2 -derivative with $\frac{\partial G}{\partial m}(m, \cdot)$ is smooth. Then $\exists v \in C^2([0, T] \times \mathbb{R}^d)$

$$\text{a.t. } \alpha = -Du \quad (m - a \cdot e)$$

and

$$\begin{cases} -\partial_t u - \frac{1}{2} \Delta u + \frac{1}{2} |Du|^2 = \frac{\int G}{S_m}(m(t), x) \\ \text{on } (0, T] \times \mathbb{R}^d \\ u(T, x) = 0 \quad \text{on } \mathbb{R}^d \end{cases}$$

Proof : See the CIME course.

IV) Mean field games.

In N-plays game : $i \in \{1, \dots, N\}$

$$J^i(\alpha^i) = \mathbb{E} \left[\int_0^T \frac{1}{2} |\alpha_t^i|^2 + g(t, X_t^i, m_{X_t^i}) dt \right]$$

s.t. $\begin{cases} dX_t^i = \alpha_t^i dt + dB_t^i \\ X_0^i = z^i \end{cases}$

Nash eq : $(\bar{\alpha}^i)_{1 \leq i \leq N}$ Nash eq

$$\text{if } \forall i, \forall \alpha^i \quad J^i(\bar{\alpha}^i, (\bar{\alpha}_j^i)_{j \neq i}) \leq J^i(\alpha^i, (\bar{\alpha}_j^i)_{j \neq i})$$

Mean field game eq : is a pair $(\bar{\alpha}, m)$

s.t. $\bar{\alpha}$ is a control and $m \in C^0([0, T], \mathcal{P}_2(\mathbb{R}^d))$

s.t. ① ~~not~~ minimizes

$$\text{If } \mathbb{E} \left[\int_0^T \frac{1}{2} |\alpha_t|^2 + g(t, X_t, m(t)) \right]$$

α where $dX_t = \alpha_t dt + dB_t$

$$X_0 = z$$

$\rightarrow \bar{X}$ associated to $\bar{\alpha}$

$$\textcircled{L} \quad \pi(t) = \mathcal{L}(\bar{x}_t)$$

Theo: Assume that $\hat{g} = \frac{SG}{Sm}$ and let (v, m, α) be s.t. (m, α) minimizes of I_∞ and v is given by the optimality conditions. Then (α, m) is a MFG eq.
 (Verification Theo)

