

III) Mean field control $N \in N^*$

$$\min_{(\alpha^i)} \mathbb{E} \left[\int_0^T \frac{1}{N} \sum_{i=1}^N \left(|\alpha_r^i|^2 + F(t, X_r^i, \pi_{X_r}^N) \right) dt \right].$$

where $\begin{cases} dX_r^i = \alpha_r^i dt + dB_r^i \\ X_0^i = z^i \end{cases}$ iid $i \in \{1, \dots, N\}$

$N \rightarrow +\infty$?

If assumptions: $\begin{cases} \text{assume } (\alpha^i) \text{ iid} \\ (X^i) \text{ iid} \end{cases}$

(without W) \Rightarrow

Then by the LLN, $\pi_{X_r}^N \approx \mathcal{L}(X_r^i)$

at the limit; the pb becomes

$$\min_{\alpha^i} \mathbb{E} \left[\int_0^T \left(\frac{1}{2} |\alpha_r^i|^2 + F(t, X_r^i, \varphi(X_r^i)) \right) dt \right]$$

where $\begin{cases} dX_r^i = \alpha_r^i dt + dB_r^i \\ X_0^i = z^i \end{cases}$

Instead of (α^i) indep, one expects that

$$\alpha_r^i = \bar{\alpha}^i(t, X_r^1, \dots, X_r^N) \quad (\text{feedback form})$$

symmetry $\rightarrow \bar{\alpha}^i = \bar{\alpha}^i(t, X_r^i, \pi_{X_r}^N) \quad (\nabla \bar{\alpha} = \bar{\alpha}')$

III.1) Classical mean field limit

• Standard SDE (stochastic differential eq)

$$(SDE) \cdot \begin{cases} dX_t = b(t, X_t) dt + \sigma(t, X_t) dB_t \\ X_0 = x_0 \in \mathbb{R}^d \end{cases} \quad d \dim B \cap$$

- A sol of SDE is a process (X_t) with continuous path, adapted to (\mathcal{F}_t) , s.t.

$$\mathbb{E} \left[\int_0^T |X_t|^2 dt \right] < +\infty \quad \forall T > 0$$

and $X_t = x_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s$.

- Hyp: $b: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ Lipschitz, $\sigma: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ Lipschitz

Theo (Cauchy - Lipschitz)
 $\exists!$ sol to SDE and it is unique
 in law.

Exo (geometric BR) ($d=1$)

$$\begin{cases} dX_t = \mu X_t dt + \sigma X_t dB_t \\ X_0 = x_0 \end{cases}$$

Then $X_t = x_0 \exp \left\{ \left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma B_t \right\}$

Particle System: N particles, which $1, \dots, N$ (position of particle i at time t PS) $\begin{cases} dX_t^i = b(t, X_t^i, m_{X_t}^N) dt + \cancel{\sigma(X_t)} \underbrace{dB_t^i}_{\text{iid}} \\ X_0^i = z^i \end{cases}$

Assume $\begin{cases} \cdot b: [0, T] \times \mathbb{R}^d \times \mathbb{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}^d \\ \text{Lipschitz} \end{cases}$

$$\cdot z^i \in \mathbb{L}^2.$$

Exo: Show that there exists a unique sol. to (PS).

(use continuity-lipschitz).

$$\text{Exo: } d_2^2(m_x^n, m_y^n) \quad \forall x = (x_1, \dots, x_n) \in (\mathbb{R}^d)^n$$

Show $\leq \frac{1}{N} \sum_{i=1}^N |x_i - y_i|^2.$

Q: $N \rightarrow +\infty$?

Expected limit: McKean-Vlasov eq

$$(MKV) \quad \begin{cases} dX_t = b(t, X_t, \mathcal{L}(X_t)) dt + dB_t \\ X_0 = z \in \mathbb{L}^2 \end{cases}$$

[Theo: Equation (MKV) has a unique solution. (also in law)

Sketch of proof: on $[0, T]$, $T > 0$ small

$$\phi, Y_t \rightarrow Z_t$$

$$X \rightarrow \Phi(X)_t = z + \int_0^t b(s, X_s, \mathcal{L}(X_s)) ds + B_t.$$

For $t \in (0, T]$,

$$\mathbb{E} \left[|\Phi(X_s)_t - \Phi(Y_s)_t|^2 \right] = \mathbb{E} \left(\left| \int_s^t b(s, X_s, \mathcal{L}(X_s)) - b(s, Y_s, \mathcal{L}(Y_s)) ds \right|^2 \right)$$

$$\stackrel{\text{Lip}}{\rightarrow} \leq C T \mathbb{E} \left(\int_0^t (|X_s - Y_s|^2 + d_2^2(\mathcal{L}(X_s), \mathcal{L}(Y_s))) ds \right)$$

$$\mathbb{E} \left[\int_0^T |\Phi(X_s) - \Phi(Y_s)|^2 \right] \leq 2C T^2 \mathbb{E} \left(\int_0^T |X_s - Y_s|^2 ds \right) \stackrel{\mathbb{E}(|X_s - Y_s|^2)}{\leq} \mathbb{E}(|X_0 - Y_0|^2)$$

$$\leq C' T^2 \|X - Y\|_p^2 \Rightarrow \text{contraction P.} \\ T \text{ small.}$$

• Convergence of the particle system:

Let $(\tilde{X}_t^{N,i})_{0 \leq t \leq N}$ of (PS).

Let $(\tilde{X}_t^{N,i})_{1 \leq i \leq N}$ be the ref to

$$\begin{cases} d\tilde{X}_t^{N,i} = b(t, \tilde{X}_t^{N,i}, \mathcal{Z}(\tilde{X}_t^{N,i}))dt + dB_t^i \\ \tilde{X}_0^{N,i} = Z^i (= X_0^{n,i}) \end{cases}$$

Then the $\tilde{X}_t^{N,i}$ are indep and have the same law.

Theo $\forall i \in \mathbb{N}, \quad \forall T > 0$

$$\lim_{N \rightarrow +\infty} \mathbb{E} \left[\sup_{0 \leq t \leq T} |\tilde{X}_t^{N,i} - X_t^{N,i}|^2 \right] = 0$$

Proof:

$$\begin{aligned} |\tilde{X}_t^{N,i} - X_t^{N,i}|^2 &= \left| \int_0^t b(s, \tilde{X}_s^{N,i}, \mathcal{Z}(\tilde{X}_s^{N,i})) \right. \\ &\quad \left. - b(s, X_s^{N,i}, \mathcal{Z}(X_s^{N,i})) ds \right|^2 \\ &\leq CT \int_0^t |\tilde{X}_s^{N,i} - X_s^{N,i}|^2 + \|b\|_2^2 \left(\tilde{m}_s^N, m_{X_s}^N \right) ds. \end{aligned}$$

$$\text{let } \tilde{m}_s^N = \mathcal{Z}(\tilde{X}_s^{N,i})$$

$$\leq CT \int_0^t |\tilde{X}_s^{N,i} - X_s^{N,i}|^2 + \|b\|_2^2 \left(\tilde{m}_s^N, m_{X_s}^N \right) +$$

$$\|b\|_2^2 \left(m_{\tilde{X}_s}^N, m_{X_s}^N \right) ds$$

$$|\tilde{X}_t^{n,i} - X_t^{n,i}|^2 \leq CT \int_0^t |\tilde{X}_s^{n,i} - X_s^{n,i}|^2 + d\|_{\mathbb{L}^2}^2(\tilde{\mu}_s^n, \mu_{X_s}^n) + \frac{1}{N} \sum_{j=1}^N |\tilde{X}_s^{n,j} - X_s^{n,i}|^2 \quad (\ast)_i$$

• Estimate of $\mathbb{E} \left[\frac{1}{N} \sum_{j=1}^N |\tilde{X}_s^{n,j} - X_s^{n,i}|^2 \right]$

Sum $(\ast)_i$ over i + $\mathbb{E}[]$

$$\mathbb{E} \left[\frac{1}{N} \sum_{i=1}^N |\tilde{X}_t^{n,i} - X_t^{n,i}|^2 \right] \leq C_T \mathbb{E} \left[\frac{1}{N} \int_0^t \sum_{i=1}^N |\tilde{X}_s^{n,i} - X_s^{n,i}|^2 + \|_{\mathbb{L}^2}^2(\tilde{\mu}_s^n, \mu_{X_s}^n) \right] = O_p(1)$$

• Gronwall inequality

$$\Rightarrow \mathbb{E} \left[\frac{1}{N} \sum_{i=1}^N |\tilde{X}_t^{n,i} - X_t^{n,i}|^2 \right] \leq \mathbb{E} \left[\int_0^t e^{2C_T(t-s)} \|_{\mathbb{L}^2}^2(\tilde{\mu}_s^n, \mu_{X_s}^n) ds \right] = O_p(1)$$

• Back to $(\ast)_i$:

$$\mathbb{E} \left[|\tilde{X}_t^{n,i} - X_t^{n,i}|^2 \right] \leq \mathbb{E} \left[\int_0^t |\tilde{X}_s^{n,i} - X_s^{n,i}|^2 ds \right] + O_p(1)$$

• Conclusion by Gronwall ineq.

