

e

Wasserstein distance

(X, d) metric space, $p > 1$, $\mu_0 \in X$

$$\mathcal{P}_p(X) = \left\{ \mu \in \mathcal{P}(X), \int (d(x, \mu_0))^p \mu(dx) < +\infty \right\}$$

$\forall \mu, \mu' \in \mathcal{P}_p(X)$,

$$d_{\mathcal{P}}(\mu, \mu') = \inf_{\substack{\pi \in \mathcal{P}(X \times X) \\ (\star) \quad \pi_1 = \mu, \quad \pi_2 = \mu'}} \left(\int_{X \times X} d(x, y) \pi(dx, dy) \right)^p$$

where $\int_X \varphi(x) \pi_1(x) := \int_{X \times X} \varphi(x) \pi(dx, dy) \quad \forall \varphi \in C_b(X)$

$$\int_X \varphi(y) \pi_2(y) := \int_X \varphi(y) \pi(dy, dy)$$

| Theo: $d_{\mathcal{P}}$ is a distance on $\mathcal{P}_p(X)$.

Rh: If $\mu = \mu'$, then $\pi = \mu \otimes \mu'$ satisfies (\star) , as well as the measure

$$\tilde{\pi} \text{ def by } \int_{X \times X} \varphi(x, y) \tilde{\pi}(dx, dy) = \int_X \varphi(x, x) \mu(dx) \quad \forall \varphi \in C_b(X \times X)$$

In terms of random variables:

Fix $(\Omega, \mathcal{F}, \mathbb{P})$

$$\begin{cases} \text{Prop:} & d_{\mathcal{P}}(\mu, \mu') = \inf_{\substack{\mathcal{L}(X) = \mu \\ \mathcal{L}(Y) = \mu'}} \mathbb{E} \left[d^p(X, Y) \right]^{1/p}. \end{cases}$$

Proof: exercise.

- Theo (Kantorovich duality)

For $p=1$, the definitions of d_1 are consistent.

I₂₀₀ (when \mathbb{X} compact).

• Based on Sion's min max theorem:

$f: A \times B \rightarrow \mathbb{R}$ with A, B are closed and convex, A compact, and
 $f(\cdot, y)$ is convex lsc,
 $f(x, \cdot)$ is concave and u

Then $\min_{a \in A} \sup_{b \in B} f(a, b) = \sup_{b \in B} \min_{a \in A} f(a, b)$.

$$\begin{aligned} d_1(m, m') &= \inf_{\pi_1=m, \pi_2=m'} \int d(x, y) \pi(dx, dy) \\ &= \inf_{\pi \in \mathcal{M}_+(\mathbb{X} \times \mathbb{X})} \sup_{\varphi, \psi \in C(\mathbb{X})} \int d(x, y) \pi(dx, dy) \\ &\quad - \int \varphi(x) (\pi(dx, y) - m(dy)) - \int \psi(y) (\pi(dx, dy) - m'(dy)) \\ &= \inf_{\pi} \sup_{\varphi, \psi} \int (d(x, y) - \varphi(x) - \psi(y)) \pi(dx, dy) \\ &\quad + \int \varphi(x) m(dx) + \int \psi(y) m'(dy) \\ &= \sup_{\varphi, \psi} \inf_{\pi} \int \varphi m + \int \psi m' \\ &\quad d(x, y) - \varphi(x) - \psi(y) \geq 0 \end{aligned}$$

Given φ , the best choice for ψ

$$\text{is } \psi^*(y) = \inf_{x \in X} d(x, y) - \varphi(x)$$

1-lipsh.

Same arg. $\Rightarrow \varphi$ must be 1-lipsh.

$$\Rightarrow \psi^*(y) = -\varphi(y).$$

II) Differential calculus in $\mathcal{P}_2(\mathbb{R}^d)$

$\cdot \mathcal{P}_2 := \mathcal{P}_2(\mathbb{R}^d)$. Rd: Camomia-Delano

\cdot differentiability = \mathcal{P}_2 not a vector space

① L^2 -derivative

Def: let $U: \mathcal{P}_2 \rightarrow \mathbb{R}$. We say that U

has an L^2 -derivative if there exists a continuous and bounded map $\frac{\delta U}{\delta m}: \mathcal{P}_2 \times \mathbb{R} \rightarrow \mathbb{R}$ s.t

$$U(m') - U(m) = \int_0^1 \int_{\mathbb{R}^d} \frac{\delta U}{\delta m} ((1-t)m + tm', y) (m' - m)(dy) dt$$

$\forall m, m' \in \mathcal{P}_2$.

Examples:

$$\textcircled{1} \quad \text{If } U(m) = \int_{\mathbb{R}^d} \varphi(x) m(dx) \quad (\varphi \in C_b(\mathbb{R}^d))$$

$$\text{then } \frac{\delta U}{\delta m}(m, y) = \varphi(y).$$

$$\textcircled{2} \quad \text{If } U(m) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(x, y) m(dx) m(dy) \quad (\varphi \in C_b(\mathbb{R}^d \times \mathbb{R}^d))$$

$$\frac{\delta U}{\delta m}(m, y) = \int_{\mathbb{R}^d} (\varphi(x, y) + \varphi(y, x)) m(dx).$$

(exercise).

② Intrinsic derivative

Def.: Let $U: \mathcal{P}_2 \rightarrow \mathbb{R}$. Assume that U has an L^2 -derivative s.t. $\int_m U: \mathcal{P}_2 \times \mathbb{R}^d \rightarrow \mathbb{R}$ will $D_y \int_m U: \mathcal{P}_2 \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ continuous and bounded.

The intrinsic derivative of U is

$$D_m U(m, y) = D_y \int_m U(m, y).$$

Examples:

$$\textcircled{1} \quad U(m) = \int_{\mathbb{R}^d} \varphi(u) m(dx) \quad (\varphi \in C_b^1(\mathbb{R}^d))$$

$$\Rightarrow D_m U(m, y) = D\varphi(y)$$

$$\textcircled{2} \quad U(m) = \int_{\mathbb{R}^{2d}} \varphi(u, y) m(du) m(dy)$$

$$D_m U(m, y) = \int_{\mathbb{R}^d} (D_1 \varphi(u, y) + D_2 \varphi(y, u)) m(dy).$$

Let $\phi: \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a ~~bounded~~ measurable vector field with at most a linear growth.

If $m \in \mathcal{P}_2(\mathbb{R}^d)$, then $\phi \# m \in \mathcal{P}_2(\mathbb{R}^d)$

$$\text{where } \int_{\mathbb{R}^d} \varphi(x) \phi \# m(dx) := \int_{\mathbb{R}^d} \varphi(\phi(y)) m(dy) \quad \forall \varphi \in C_b^1(\mathbb{R}^d)$$

Prop: Assume $U: \mathcal{P}_2 \rightarrow \mathbb{R}$ has an intrinsic derivative. Then

$$\lim_{h \rightarrow 0} \frac{U((id + h\phi) \# m) - U(m)}{h} = \int_{\mathbb{R}^d} D_m U(m, y) \cdot \phi(y) m(dy)$$

Proof:

$$\begin{aligned}
 U((id + h\phi)\#_m) - U(m) &= \\
 &\int_0^1 \int_{\mathbb{R}^d} \frac{\partial U}{\partial m}(m_{t,h}, y) ((id + h\phi)\#_m - m)(dy) dt \\
 &= \int_0^1 \int_{\mathbb{R}^d} \left(\frac{\partial U}{\partial m}(m_{t,h}, y) + h\phi(y) - \frac{\partial U}{\partial m}(m_{t,h}, y) \right) m(dy) \\
 &\approx h \int_0^1 \int_{\mathbb{R}^d} D_y \frac{\partial U}{\partial m}(m, y) \cdot \phi(y) m(dy) + o_h(1)
 \end{aligned}$$

- Projection over finite dim. spaces:

let $U : \mathbb{P}_2 \rightarrow \mathbb{R}$ and $N \in \mathbb{N}^*$.

$$\begin{aligned}
 U_N : (\mathbb{R}^d)^N &\rightarrow \mathbb{R} \\
 x = (x_1, \dots, x_N) &\mapsto \underbrace{U(m_x^N)}_{\rightarrow = \frac{1}{N} \sum^n s_{x_i}}
 \end{aligned}$$

Prop: Assume that U has an infinitesimal derivative. Then U_N is C^1 with

$$D_{x_i} U_N(x) = \frac{1}{N} D_m U(m_x^N, x_i)$$

