

• Let (m_n) be a sequence in $\mathcal{P}(X)$

$$m_n \xrightarrow{*} m \text{ if } \forall \varphi \in C_b(X), \quad \int_X \varphi m_n \rightarrow \int_X \varphi m.$$

• Monge-Kantorovich distance:

Fix $n_0 \in X$.

$$\mathcal{P}_1(X) = \left\{ m \in \mathcal{P}(X) \mid \int_X d(n, n_0) m(dx) < +\infty \right\}$$

(indep of n_0)

$$\text{M-K distance: } d_1(m, m') = \sup_{\substack{\varphi \text{ 1-lip}}} \int_X \varphi(u) (m - m')(du)$$

$(m, m') \in \mathcal{P}(X)$

where the sup is taken over $\varphi: X \rightarrow \mathbb{R}$

$$\text{s.t. } |\varphi(x) - \varphi(y)| \leq d(x, y) \quad \forall x, y \in X.$$

[Prop: d_1 is a distance on $\mathcal{P}_1(X)$

Sketch of proof:

$$- d_1 \geq 0$$

$$- d_1(m, m) = 0$$

$$- \forall m, m', m'',$$

$$d_1(m, m'') = \sup_{\varphi \text{ 1-lip}} \int_X \varphi(u) (m - m' + m' - m'') (du)$$

$$\leq \sup_{\varphi \text{ 1-lip}} \int_X \varphi(m - m') + \sup_{\varphi \text{ 1-lip}} \int_X \varphi(m' - m'')$$

$$= d_1(m, m') + d_1(m', m'').$$

$$\cdot d_{\mathcal{P}_1}(m, m') = 0 \Rightarrow m = m'.$$

i.e. $\int \varphi m = \int \varphi m' \quad \forall \varphi \in C_b(X)$

Easy: $\int_X \varphi m = \int_X \varphi m' \quad \forall \varphi \text{ Lipsch.}$

If $\varphi \in C_b(X)$, then $\varphi_b(u) = \sup_{y \in X} |\varphi(y) - b|d(u, y)$

then φ_b is b -Lipschitz, bounded

& $\varphi_b \xrightarrow[b \rightarrow 0]{} \varphi$ locally unif.

Theo: Let (m_n) be a sequence in $\mathcal{P}_1(X)$
and $m \in \mathcal{P}_1(X)$. Then

$$m_n \xrightarrow{d} m \quad (\Rightarrow) \quad \left\{ \begin{array}{l} m_n \xrightarrow{*} m \\ \forall x_0 \in X, \end{array} \right.$$

$$\int_X d(u, x_0) m_n(dx) \rightarrow \int d(u, x_0) m(dx)$$

Ref: • Ambrosio-Gigli-Savaré
• Santambrogio
• Notes in my course.

• Functions depending on a lot of variable

Let $U_N: X^N \rightarrow \mathbb{R}$ ($N \in \mathbb{N}^+$ large)

Theo (Lion) Assume X is compact,

• U_N symmetric: $\forall (x_1, \dots, x_N) \in X^N, \forall \sigma \in \Omega_N$

$$U_N(x_{\sigma(1)}, \dots, x_{\sigma(N)}) = U_N(x_1, \dots, x_N)$$

• U_N unif. bd, ($\exists M, |U_N| \leq M \forall N$)

$$\exists \varepsilon > 0, |U_N(x_1, \dots, x_n) - U_N(y_1, \dots, y_n)| \leq \sum_{i=1}^N d(x_i, y_i).$$

Then $\exists n_k$ and $U: \mathcal{P}(X) \rightarrow \mathbb{R}$ s.t.

$$\sup_{(x_1, \dots, x_n) \in X^{N_k}} |U_{N_k}(x_1, \dots, x_n) - U(m_x^{N_k})| \xrightarrow{k \rightarrow +\infty} 0$$

Ex: $X = [0, 1]$, $\varphi \in C([0, 1]) \rightarrow \frac{1}{n} \sum_{i=1}^n \varphi(x_i)$

$$U_N(x_1, \dots, x_n) = \frac{1}{n} \sum_{i=1}^n \varphi(x_i)$$

$$\text{Then } U(m) = \int_0^1 \varphi(x) m(dx)$$

$$\text{and } U_N(x_1, \dots, x_n) = U(m_x^n).$$

Sketch of proof:

$$\text{let } \tilde{U}_N(m) = \sup_{n=(n_1, \dots, n_n) \in X^n} U_n(m) - \varepsilon d(m, m_x^n) \quad \forall m \in \mathcal{P}(X) = \mathcal{P}(X)$$

Then one can prove that

- \tilde{U}_N is ε -Lipsh. on $\mathcal{P}(X)$ { use
- \tilde{U}_N is unif. bd { Arzeli-Angel
- $\mathcal{P}(X)$ is compact
- $U_N(x_1, \dots, x_n) = \tilde{U}_N(m_x^n) \circ$
- Law of large numbers

• Fix $(\mathcal{S}, \mathcal{F}, \mathbb{P})$ an atomless proba
space
(ex: $\mathcal{S} = [0, 1]$, \mathcal{F} Borel σ -field, \mathbb{P} Lebesgue measure)

Then $\forall m \in \mathbb{R}, (\mathbb{X}) \exists X \in L^1(\mathcal{S}, \mathbb{X})$ s.t.

$$\mathcal{L}(X) = \text{law of } X = m.$$

i.e. $\forall \varphi \in C_b(\mathbb{X}), \mathbb{E}[\varphi(X)] = \int_{\mathbb{X}} \varphi \, m.$

Prop: $d_1(m, m') =$

$$\inf_{\substack{X, X'}} \mathbb{E}[d(X, X')],$$

$\mathcal{L}(X) = m, \mathcal{L}(X') = m'$

Proof: exercise.

Fix $\mathbb{X} = \mathbb{R}^d$, ($d \in \mathbb{N}^+$), endowed
with the euclidean distance.

• L.L.N = if (X_n) i.i.d random variables
taken values in \mathbb{R}^d with $\mathbb{E}[|X_n|] < +\infty$

then $\bar{X}_n = \frac{1}{N} \sum_{i=1}^N X_i \xrightarrow{\substack{\text{a.s} \\ \text{and in } L^1}} \mathbb{E}[X_n]$

• Set $m_X^N = \frac{1}{N} \sum_{i=1}^N \delta_{X_i}$ random measure
of \mathbb{R}^d .

[Theo: $m^N X \xrightarrow{d_1} L(X_0)$ a.s and in L'
(Glim \rightarrow L.L.N.)]

