Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,1]})$ be a stochastic basis satisfying the usual conditions and let $(B_t)_{t\in[0,1]}$ be an $(\mathcal{F}_t)_{t\in[0,1]}$-Brownian motion as in the course.

First we introduce a 2-dimensional version of Itô's formula:

Theorem 0.1. *Let $B = (B_t)_{t\in[0,1]}$ a Brownian motion, and*

$$\begin{aligned} X_t &= x + \int_0^t a_s ds + \int_0^t L_s dB_s, \\ Y_t &= y + \int_0^t b_s ds + \int_0^t K_s dB_s \end{aligned}$$

be Itô-processes, and $f \in C^{1,2}([0,1] \times \mathbb{R}^2)$. Then

$$\begin{aligned} f(t, X_t, Y_t) &= f(0, X_0, Y_0) + \int_0^t \frac{\partial f}{\partial s}(s, X_s, Y_s) ds \\ &+ \int_0^t \frac{\partial f}{\partial x}(s, X_s, Y_s) dX_s + \int_0^t \frac{\partial f}{\partial y}(s, X_s, Y_s) dY_s \\ &+ \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_s, Y_s) L_s^2 ds + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial y^2}(s, X_s, Y_s) K_s^2 ds \\ &+ \int_0^t \frac{\partial^2 f}{\partial x \partial y}(s, X_s, Y_s) L_s K_s ds. \end{aligned}$$

(1) **Solving a linear SDE I:** Let $x, b \in \mathbb{R}$. Solve the SDE

$$dX_t = bX_t dt + dB_t \text{ with } X_0 = x$$

in the following way: define the process $Y = (Y_t)_{t\in[0,1]}$ by

$$Y_t = \frac{X_t}{e^{bt}}.$$

Then it holds $Y_t = f(t, X_t)$, where $f(t,x) = \frac{x}{e^{bt}}$, so that one can apply Itô's formula to write down Y_t explicitly. Multiplying Y_t by e^{bt} one can derive a representation for X_t.

(2) **Solving a linear SDE II:** Let $x, \sigma_1, \sigma_2 \in \mathbb{R}$. Solve the SDE

$$dX_t = (\sigma_1 X_t + \sigma_2) dB_t \text{ with } X_0 = x.$$

Hint: Define the stochastic exponential $\mathcal{E}(\sigma_1 B)_t := e^{\sigma_1 B_t - \frac{\sigma_1^2}{2} t}$, and define the process by $Y_t = \frac{X_t}{\mathcal{E}(\sigma_1 B)_t}$. Then $Y_t = f(t, X_t, B_t)$, where $f(t,x,b) = xe^{-\sigma_1 b + \frac{\sigma_1^2}{2} t}$. Proceed as before; apply Itô's formula to get a representation for Y_t, and multiply Y_t by $\mathcal{E}(\sigma_1 B)_t$ to get X_t.