Stochastic Processes 1

-7- (08.03.2010)

(1) Let $0 < p < \infty$, $\Omega = [0,1)$, and $\mathcal{F}_n := \sigma\Big(\big[\frac{k-1}{2^n}, \frac{k}{2^n}\big) : k = 1,...,2^n \Big)$ and λ the Lebesgue measure. Define $M_n(t) := 2^{\frac{n}{p}}$ for $t \in [0, 2^{-n})$ and $M_n(t) := 0$ for $t \in [2^{-n}, 1)$ for $n = 0, 1, ...$

- (a) Show that $(M_n)_{n=0}^\infty$ is a martingale for $p = 1$.
- (b) Is $(M_n)_{n=0}^\infty$ a super- or sub-martingale for $p \neq 1$?
- (c) For what p the sequence $(M_n)_{n=0}^\infty$ is a Cauchy-sequence in L_1?
- (d) Let $p = 1$. Is $(M_n)_{n=0}^\infty$ uniformly integrable, that means, does one have

$$\lim_{c \uparrow \infty} \sup_{n=0,1,...} \int_{|M_n| \geq c} |M_n(t)| dt = 0?$$

(2) Assume that $\varepsilon_1, \varepsilon_2, \varepsilon_3 : \Omega \to \{-1, 1\}$ are independent random variables such that $\mathbb{P}(\varepsilon_i = 1) = \mathbb{P}(\varepsilon_i = -1) = 1/2$ and that $f : \mathbb{R}^3 \to \mathbb{R}$ is a function. Prove that

$$\mathbb{E}(f(\varepsilon_1, \varepsilon_2, \varepsilon_3) | \sigma(\varepsilon_1, \varepsilon_2)) = g(\varepsilon_1, \varepsilon_2)$$

with $g(\varepsilon_1, \varepsilon_2) := (1/2)(f(\varepsilon_1, \varepsilon_2, -1) + f(\varepsilon_1, \varepsilon_2, 1))$.

(3) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\varepsilon_1, \varepsilon_2, ... : \Omega \to \mathbb{R}$ independent random variables such that $\mathbb{P}(\varepsilon_n = 1) = \mathbb{P}(\varepsilon_n = -1) = 1/2$. Let $v_0 \in \mathbb{R}$ and $v_n : \mathbb{R}^n \to \mathbb{R}$ be functions. Define $M_0 := 0$, $M_1(\omega) := \varepsilon_1(\omega) v_0$,

$$M_n(\omega) := \varepsilon_1(\omega) v_0 + \sum_{k=2}^n \varepsilon_k(\omega) v_{k-1}\Big(\varepsilon_1(\omega), ..., \varepsilon_{k-1}(\omega)\Big)$$

and $\mathcal{F}_n := \sigma(\varepsilon_0, ..., \varepsilon_n)$.

- (a) Is $(M_n)_{n=0}^\infty$ a martingale?
- (b) Let $Z_0 := 1$ and

$$Z_n(\omega) := e^{M_n(\omega) - \frac{1}{2} \sum_{k=1}^n |v_{k-1}(\varepsilon_1(\omega),...,\varepsilon_{k-1}(\omega))|^2}.$$

Show that $(Z_n)_{n=0}^\infty$ is a super-martingale.

(4) Assume that $\varepsilon_1, ..., \varepsilon_n : \Omega \to \mathbb{R}$ are independent random variables such that $\mathbb{P}(\varepsilon_i = 1) = p$ and $\mathbb{P}(\varepsilon_i = -1) = q$ for some $p, q \in (0,1)$ with $p + q = 1$. Define the stochastic process $X_k := e^{a(\varepsilon_1 + ... + \varepsilon_k) + bk}$ for $k = 1, ..., n$ and $X_0 := 1$ with $a > 0$ and $b \in \mathbb{R}$ and the filtration $(\mathcal{F}_k)_{k=0}^n$ with $\mathcal{F}_0 := \{\emptyset, \Omega\}$ and $\mathcal{F}_k := \sigma(\varepsilon_1, ..., \varepsilon_k)$. Assume that $-a + b > 0$. Why there cannot exist random variables $\varepsilon_1, ..., \varepsilon_n : \Omega \to \{-1, 1\}$ such that $(X_k)_{k=0}^n$ is a martingale?