Stochastic Processes 1

-6- (01.03.2010)

(1) Let $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ be independent random variables such that $\mathbb{P}(\varepsilon_i = 1) = 1/4$ and $\mathbb{P}(\varepsilon_i = -1) = 3/4$. Let $\mathcal{G} := \sigma\big(\varepsilon_3, \varepsilon_4\big)$. Compute

$$\mathbb{E}\Big(\varepsilon_1 + \varepsilon_1\varepsilon_2 + \varepsilon_3 + (\varepsilon_1\varepsilon_2 + \varepsilon_3)\varepsilon_4 \Big| \mathcal{G}\Big).$$

(2) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Assume that the joint distribution $\mathbb{P}_{f,g}$ of the random variables f, g has the density $h_{f,g}(x,y)$ i.e.

$$\mathbb{P}((f,g) \in B) = \int_B h_{f,g}(x,y)dxdy \quad \forall B \in \mathcal{B}(\mathbb{R}^2).$$

Let h_g be the density of $\mathbb{P}_g$. Define

$$h_{f|g}(x,y) = \begin{cases} h_{f,g}(x,y)/h_g(y) & \text{if} \quad h_g(y) \neq 0 \\ 0 & \text{if} \quad h_g(y) = 0 \end{cases}.$$

Show that

$$\mathbb{P}(f \in A | g = y) = \int_A h_{f|g}(x,y)dx$$

by checking that

$$\mathbb{P}(f \in A, g \in B) = \int_B \int_A h_{f|g}(x,y)dxd\mathbb{P}_g(y) \quad \forall B \in \mathcal{B}(\mathbb{R})$$

where you can use that $\mathbb{P}(f \in A, g \in B) = \int_B \mathbb{P}(f \in A | g = y)d\mathbb{P}_g(y)$.

(3) Let $\Omega := [0,1)$ and let $(h_n)_{n=0}^{\infty}$, be the sequence of Haar-functions defined by $h_0 \equiv 1$ and

$$h_{2^{n-1}+k}(t) := \begin{cases} -1 & : \quad t \in \left[\frac{2k}{2^n}, \frac{2k+1}{2^n}\right) \\ 1 & : \quad t \in \left[\frac{2k+1}{2^n}, \frac{2k+2}{2^n}\right) \\ 0 & : \quad \text{else} \end{cases}$$

for $k = 0, ..., 2^{n-1} - 1$ (draw pictures of the first functions) if $n \geq 1$. Define $\mathcal{F}_n := \sigma(h_0, ..., h_n)$ for $n = 0, 1, ...$

(a) Describe $\mathcal{F}_n$ as easy as possible.

(b) Is $(M_n)_{n=0}^{\infty}$ with $M_n := h_0 + \cdots + h_n$ a martingale?

(4) Prove Proposition 3.1.8 (x) by using monotone convergence.