Stochastic Processes 1

-5- (22.02.2010)

(1) Let $\Omega = (0,1]$, $\mathcal{F} := \mathcal{B}((0,1])$, and $\mathbb{P}$ be the Lebesgue measure. Define a sub-σ-algebra by $\mathcal{G} := \sigma\big(\big(\frac{1}{n+1}, \frac{1}{n}\big], n = 1,2,...\big)$ and $f : \Omega \to \mathbb{R}$ by $f(x) := \log x$. Compute $\mathbb{E}(f|\mathcal{G})$.

(2) Let $\varepsilon_1, \varepsilon_2, \varepsilon_3 : \Omega \to \mathbb{R}$ be independent random variables such that $\mathbb{P}(\varepsilon_n = 1) = \mathbb{P}(\varepsilon_n = -1) = 1/2$. Define $f := \varepsilon_1(\varepsilon_2 + \varepsilon_3)(\varepsilon_1 + \varepsilon_3)$ and $\mathcal{G} := \sigma\big(\varepsilon_1, \varepsilon_2\big)$. Compute

$$\mathbb{E}\big(f|\mathcal{G}\big) : \Omega \to \mathbb{R}.$$

(3) Show Proposition 3.1.8 (ix).

(4) Let $\Omega := [0,1]$, $\mathcal{F} := \mathcal{B}([0,1])$ and let λ be the Lebesgue measure on $[0,1]$. Define $f(x) := x^2$ and

$$\mathcal{G} := \sigma\left([0, \frac{1}{2}], A \subseteq (\frac{1}{2}, 1], A \in \mathcal{F} \right).$$

Compute

$$\mathbb{E}\big(f|\mathcal{G}\big) : [0,1] \to \mathbb{R}.$$

(5) For $f \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ prove that

$$\big(\mathbb{E}\big(|f||\mathcal{G}\big)\big)^2 \le \mathbb{E}\big(|f|^2|\mathcal{G}\big) \; a.s.$$

Hint: Use Proposition 3.1.8 to show that

$$\mathbb{E}\big(|f|^2|\mathcal{G}\big) = \mathbb{E}\big(\big[|f| - \mathbb{E}\big(|f| \; |\mathcal{G}\big)\big]^2 \,|\mathcal{G}\big) + \big[\mathbb{E}\big(|f| \; |\mathcal{G}\big)\big]^2.$$