Stochastic Processes 1

-4- (15.02.2010)

(1) Show using (4) of -3- and the Lemma of Borel-Cantelli that

$$\mathbb{P}\Big(\limsup_{n\to\infty (n\geq 2)} \frac{g_n}{\sqrt{2\log n}} = 1\Big) = 1.$$

(2) Let $\xi_1, \xi_2, \ldots$ be a sequence of independent random variables. Show

$$\sum_{n=1}^{\infty} \xi_n^2 < \infty \quad a.s. \Longleftrightarrow \sum_{n=1}^{\infty} \mathbb{E}\frac{\xi_n^2}{1+\xi_n^2} < \infty.$$

(3) Let $(\xi_n)_{n=1}^{\infty}$, $\xi_n : \Omega \to [0,1]$ be a sequence of independent random variables. Use the Two Series Theorem to show that

$$\mathbb{P}\left(\sum_{n=1}^{\infty} \xi_n \text{ converges } \right) = 1 \iff \sum_{n=1}^{\infty} \mathbb{E}\xi_n < \infty.$$

(4) Let $(f_n)_{n=1}^{\infty}$ be a sequence of independent and identically distributed random variables $f_n : \Omega \to \mathbb{R}$ and let $\varepsilon > 0$. Using the equivalence proved in exercise (1) -3- one observes that the following assertions are equivalent:

(a) $\mathbb{E}|f_1| < \infty$.

(b) $\sum_{n=1}^{\infty} \mathbb{P}\big(\frac{|f_n|}{n} > \varepsilon\big) < \infty$.

Using the Lemma of Borel-Cantelli one shows that (b) implies

$$\frac{f_n(\omega)}{n} \to 0 \qquad \text{for almost all} \qquad \omega \in \Omega.$$