Stochastic Processes 1

-3- (08.02.2010)

(1) Assume a random variable $f : \Omega \to [0, \infty)$.

(a) For f given by

$$f(\omega) = \sum_{k=1}^{n} a_k \mathbb{1}_{A_k}(\omega), \quad \omega \in \Omega$$

such that $0 \le \alpha_1 \le \cdots \le \alpha_n$ and $A_1, ..., A_n \in \mathcal{F}$ with $\Omega = \bigcup_{i=1}^{n} A_i$ and $A_i \cap A_j = \emptyset$ for $i \neq j$ one shows that

$$\int_\Omega f(\omega) d\mathbb{P}(\omega) = \int_0^\infty \mathbb{P}(f > \lambda) d\lambda.$$

(b) Using the relation

$$\int_\Omega f(\omega) d\mathbb{P}(\omega) = \int_0^\infty \mathbb{P}(f > \lambda) d\lambda,$$

which can be assumed to be known (both sides may be infinite), one shows that

$$\int_\Omega f(\omega) d\mathbb{P}(\omega) < \infty \Longleftrightarrow \sum_{n=1}^{\infty} \mathbb{P}(f > n) < \infty.$$

(2) Let $(\xi_n)_{n=1}^\infty$ be a sequence of independent random variables with $0 \le |\xi_n(\omega)| \le 1$ for all $\omega \in \Omega$ such that $\sum_{n=1}^\infty \mathbb{E}|\xi_n| < \infty$.

(a) Using the two-series-theorem one shows that

$$\mathbb{P}\left(\sum_{n=1}^{\infty} \xi_n \text{ converges} \right) = 1.$$

(b) Prove directly by monotone convergence that

$$\mathbb{P}\left(\sum_{n=1}^{\infty} |\xi_n| < \infty \right) = 1.$$

(3) Let $(\xi_n)_{n=1}^\infty$ be a sequence of independent random variables with $\xi_n(\omega) \ge 0$ for all $\omega \in \Omega$. One shows that

$$\sum_{n=1}^{\infty} \mathbb{E} \frac{\xi_n}{1 + \xi_n} < \infty \Longrightarrow \mathbb{P}\left(\sum_{n=1}^{\infty} \xi_n \text{ converges} \right) = 1.$$

(4) From

$$\lim_{x\to\infty} \frac{\int_x^\infty e^{-\frac{y^2}{2}}\,dy}{\frac{1}{x}e^{-\frac{x^2}{2}}} = 1$$

deduce the following: If $(g_n)_{n=1}^\infty$ are independent random variables such that $g_n \sim N(0,1)$, then

$$\sum_{n=1}^\infty \mathbb{P}\big(g_n > (1-\varepsilon)\sqrt{2\log n}\big) = \infty \quad \text{for all} \quad 0 \le \varepsilon < 1$$

and

$$\sum_{n=1}^\infty \mathbb{P}\big(g_n > (1+\varepsilon)\sqrt{2\log n}\big) < \infty \quad \text{for all} \quad \varepsilon > 0.$$