Stochastic Processes 1

-2- (01.02.2010)

(1) Let $(\xi_n)_{n=1}^\infty$, $\xi_n : \Omega \to \mathbb{R}$, be random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Show that $\bigcup_{n=1}^\infty \sigma(\xi_1, ..., \xi_n)$ is an algebra on Ω.

(2) Let $\xi_1, \xi_2, ... : \Omega \to [0,1]$ be a sequence of independent random variables and $c \in [0,1]$.

(a) Using Kolmogorow's 0-1-law, one shows that

$$p_c = \mathbb{P}\left(\limsup_n \xi_n = c\right) \in \{0,1\}.$$

(b) Can one choose $c \in [0,1]$ such that $p_c = 1$?

(3) Let $-\infty < A < B < \infty$ and $f_n := \varepsilon_1 + \cdots + \varepsilon_n$, where $\varepsilon_1, \varepsilon_2, ... : \Omega \to \mathbb{R}$ are independent such that $\mathbb{P}(\varepsilon_k = -1) = \mathbb{P}(\varepsilon_k = 1) = 1/2$. We say that a trajectory $(f_n(\omega))_{n=1}^\infty$ has infinitely many up-crossings provided that there exists a sequence $n_1(\omega) < N_1(\omega) < n_2(\omega) < N_2(\omega) < \cdots$ such that

$$f_{n_k(\omega)}(\omega) < A < B < f_{N_k(\omega)}(\omega)$$

for $k = 1, 2, ...$ Let C be the event that $(f_n)_{n=1}^\infty$ has infinitely many up-crossings.

(a) Prove by Hewitt-Savage that $\mathbb{P}(C) \in \{0,1\}$.

(b) Use another statement from the course to prove that $\mathbb{P}(C) = 1$.

(4) Assume independent random variables $\xi_1, \xi_2, ... : \Omega \to \mathbb{R}$ such that $\mathbb{P}(\xi_k = -1) = 1/4$, $\mathbb{P}(\xi_k = 0) = 1/2$, and $\mathbb{P}(\xi_k = 1) = 1/4$. Let $f_n := \xi_1 + \cdots + \xi_n$. Does the random walk leave the strip $[-10, 10]$ with probability one?