Stochastic Processes 1

- 1 - (26.01.2010)

(1) Proposition 1.1.1.

(a) Prove that
$$r_x = \alpha + \beta(p/q)^x$$
is the general form of a solution for $p \neq q$.

(b) Prove Proposition 1.1.1(i) for $p = q = \frac{1}{2}$ directly.

(c) Deduce Proposition 1.1.1(i) for $p = q = \frac{1}{2}$ from Proposition 1.1.1(i) for $p \neq q$ by L'Hospital's rule.

(d) Prove Proposition 1.1.1(ii).

(2) Prove Lemma 2.1.7 from the course using the following monotone class theorem:

Let $(\Omega, \mathcal{F})$ be a measurable space. If a system of subsets $\mathcal{G} \subseteq \mathcal{F}$ is a monotone class, that means $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ with $A_k \in \mathcal{G}$ implies that $\bigcup_k A_k \in \mathcal{G}$, and if $\mathcal{G}$ is an algebra such that $\sigma(\mathcal{G}) = \mathcal{F}$, then $\mathcal{G} = \mathcal{F}$.

(3) Let $\varepsilon_1, \varepsilon_2, \varepsilon_3, ... : \Omega \to \mathbb{R}$ independent random variables such that $\mathbb{P}(\varepsilon_k = -1) = p$ and $\mathbb{P}(\varepsilon_k = 1) = q = 1-p$ with $p \in (0,1)$. Let $f_n := \varepsilon_1 + \cdots + \varepsilon_n$. Given an integer $k \geq 0$, compute
$$\mathbb{P}(f_n = k).$$

(4) Given independent random variables $f_1, f_2, f_3, ... : \Omega \to \mathbb{R}$ and $S_n := f_1 + \cdots + f_n$. Show that
$$\mathbb{P}\left(\omega \in \Omega : \limsup_{n\to\infty} \frac{S_n(\omega)}{n^2} \geq 1\right) \in \{0, 1\}.$$