Stochastic Processes 2
Exercises 5 (21th April 2008)
– Solve 4 problems. –

Theorem 0.1 (Backwards martingales) *Let* $(\mathcal{F}_n)_{n=1}^{\infty}$ *be a* **decreasing** *sequence of* σ*-algebras,* $\mathcal{F}_\infty := \bigcap_n \mathcal{F}_n$*, and let* $Z \in L_1$*. Then one has that*

$$\lim_n \mathbb{E}(Z|\mathcal{F}_n) = \mathbb{E}(Z|\mathcal{F}_\infty) \text{ in } L_1 \text{ and a.s.}$$

Let $\varepsilon_1, \varepsilon_2, ... : \Omega \to \mathbb{R}$ iid Bernoulli random variables, i.e. $\mathbb{P}(\varepsilon_k = \pm 1) = 1/2$, and let $S_n := \varepsilon_1 + \cdots + \varepsilon_n$.

(1) Prove that

$$\mathbb{E}(\varepsilon_1|\sigma(S_n, S_{n+1}, S_{n+2}, ...)) = \frac{S_n}{n} \quad a.s.$$

Hint: See also Exercise 8 of Series 4.

(2) Prove that

$$\mathcal{F}_\infty := \bigcap_n \sigma(S_n, S_{n+1}, S_{n+2}, ...)$$

contains only sets of measure one or zero.

Hint: Prove that $A \in \mathcal{F}_\infty$ does not depend on ε_k for any fixed $k = 1, 2, ...$ and use the ?-? law of ?.

(3) Using Theorem 0.1 prove that

$$\lim_n \frac{S_n}{n} = m \text{ in } L_1 \text{ and a.s.}$$

where $m = \mathbb{E}\varepsilon_1 = 0$. Did we have something like that already in the course?

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(4) Let $\mathbb{P}_k$ and μ_k be measures on $\Omega_k := \{-1, 1\}$ with $\mathcal{F}_k := 2^{\Omega_k}$ such that $\mathbb{P}_k(\{-1\}) = \mathbb{P}_k(\{1\}) = 1/2$, and $\mu_k(\{-1\}) = p_k$ and $\mu_k(\{1\}) = q_k$ with $p_k + q_k = 1$, where

$$p_k := \frac{1}{2} + \sqrt{\frac{1}{k} - \frac{1}{k^2}} \quad \text{for} \quad k \geq 4$$

and $p_k, q_k \in (0, 1)$ for $k = 1, 2, 3, 4$. Decide whether $\times_{n=1}^{\infty} \mu_k$ is absolutely continuous with respect to $\times_{k=1}^{\infty} \mathbb{P}_k$.

Hint: Check that $((1 - \frac{1}{k})^2 - \frac{1}{2})^2 = p_k q_k$ for $k \geq 4$ and use a discrete version of Proposition 3.9.7.

Let $\Pi = \{(x, y) : x^2 + y^2 = 1\}$ be the unit circle in $\mathbb{R}^2$ and $A_n : \Pi \to \Pi$ a rotation by $\frac{2\pi}{2^n}$ degree, $n = 0, 1, 2, ...$. Let $\mathcal{F}_n$ be the set of all Borel sets $B \subseteq \Pi$ such that $A_n(B) = B$.

(5) Show that $\mathcal{F}_1 \supseteq \mathcal{F}_2 \supseteq \mathcal{F}_3 \supseteq \cdots$.

(6) Given $f \in L_1(\Pi, \lambda)$, compute $\mathbb{E}(f|\mathcal{F}_n)$, where λ is the normalized Lebesgue measure on Π.