

Stochastic Processes 1

Exercises 1 (21th January 2008)

(1) Proposition 1.1.1.

(a) Prove that

$$r_x = \alpha + \beta(p/q)^x$$

is the general form of a solution for $p \neq q$.

(b) Prove Proposition 1.1.1(i) for $p = q = \frac{1}{2}$ directly.

(c) Deduce Proposition 1.1.1(i) for $p = q = \frac{1}{2}$ from Proposition 1.1.1(i) for $p \neq q$ by L'Hospital's rule.

(d) Prove Proposition 1.1.1(ii).

(2) Prove Lemma 2.1.7 from the course using the following monotone class theorem:

Let (Ω, \mathcal{F}) be a measurable space. If a system of subsets $\mathcal{G} \subseteq \mathcal{F}$ is a monotone class, that means $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$ with $A_k \in \mathcal{G}$ implies that $\bigcup_k A_k \in \mathcal{G}$, and if \mathcal{G} is an algebra such that $\sigma(\mathcal{G}) = \mathcal{F}$, then $\mathcal{G} = \mathcal{F}$.

(3) Let $\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots : \Omega \rightarrow \mathbb{R}$ independent random variables such that $\mathbb{P}(\varepsilon_k = -1) = p$ and $\mathbb{P}(\varepsilon_k = 1) = q = 1-p$ with $p \in (0, 1)$. Let $f_n := \varepsilon_1 + \dots + \varepsilon_n$. Given an integer $k \geq 0$, compute

$$\mathbb{P}(f_n = k).$$

(4) Given independent random variables $f_1, f_2, f_3, \dots : \Omega \rightarrow \mathbb{R}$ and $S_n := f_1 + \dots + f_n$. What is the probability of

$$\left\{ \omega \in \Omega : \limsup_{n \rightarrow \infty} \frac{S_n(\omega)}{n^2} \geq 1 \right\}?$$