Stochastic Modeling 9, 05.10.09

Note Title 5.10.2009

Proposition $(f_i)_{i=0}^{\infty}$ homogeneous Markov chain

$$X = X_T \cup X_p, \quad X_T \cap X_p = \emptyset$$

$$\boxed{X_p = \bigcup_m C_m}$$

disjoint, closed, irreducible.

Proof

$k \in X_p$

$$C(k) = \{ \ell \in X, \ k \to \ell \}$$

k is persistent: so $k \to \ell \Rightarrow \exists\, i \geq 1$

$$\mathbb{P}(f_i = k \mid f_0 = \ell) > 0$$

$k \to k \Rightarrow \boxed{k \in C(k)}$

$\boxed{C(k) \text{ is closed}}$ $m \in C(k)$, $\ell \in X \setminus C(k)$

assume $\mathbb{P}(f_n = \ell \mid f_0 = m) > 0$ for some $n \geq 1$.

so $m \to \ell$, but because of $k \to m$ we have $k \to \ell$.

This implies that $\ell \in C(k)$ by df of $C(k)$.

$\boxed{C(k) \text{ is irreducible}}$ $\ell, m \in C(k)$

$$\left.\begin{array}{l} k \to \ell \text{ by df of } C(k) \\ \ell \to k \text{ by persistence of } k \\ \quad \text{and } k \to \ell \end{array}\right\} \underline{k \leftrightarrow \ell}$$

In the same way $\underline{k \leftrightarrow m}$

$\leftrightarrow$ equivalence relation, especially

$$m \leftrightarrow e$$

$$\boxed{\text{Either } C(k) \cap C(e) = \emptyset \text{ or } C(k) = C(e)}$$

$m \in C(k) \cap C(e)$. Then $k \to m$

$m \leftrightarrow \bar{e} \quad \forall \bar{e} \in C(e)$

$$\Rightarrow k \to \bar{e} \quad \forall \bar{e} \in C(e)$$

By df of $C(k)$ $\quad \bar{e} \in C(k) \quad \forall \bar{e} \in C(e)$

or $\quad C(e) \subseteq C(k)$

<u>By symmetry</u>

$$C(k) \subseteq C(e)$$

so $\quad C(k) = C(e)$

$\square$

Summary of the classification

arithmetic properties of T^n

or $p^{(n)}(k,l) =: p^{(n)}_{k,l}$

$\boxed{k \text{ is absorbing:}}$ $p_{kk} = 1$

$\boxed{k \to l}$ $\exists\, n \geq 1,\ p^{(n)}_{kl} > 0$

$\boxed{k \leftrightarrow l}$ $k \to l$ and $l \to k$

$\boxed{k \text{ has the period } d(k)}$

$d(k)$ is the largest m such that

$p^{(n)}_{kk} = 0$ for all n which are not multiples of m.

$\boxed{\text{decomposition of } X}$

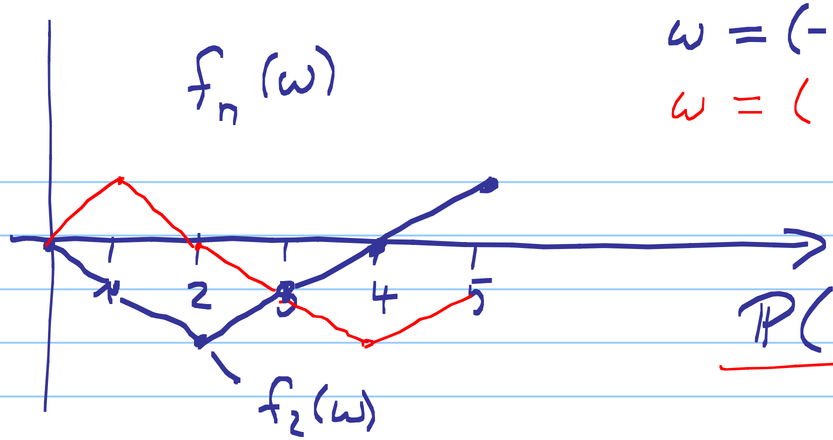
$$X = \bigcup_m X_m$$

disjoint

$$k \leftrightarrow l \iff k, l \in X_m \text{ for some } m$$

<u>Classification according to asymptotic properties of $p_{kl}^{(n)}$</u>

Recall $\quad \Omega_n = \{ \omega = (\varepsilon_1, .., \varepsilon_n) \quad \varepsilon_i = \{1,-1\} \}$



$\omega = (-1, -1, 1, 1, 1)$

$\omega = (1, -1, -1, -1, 1)$

$n = 5$

Assume

$\mathbb{P}(f_0 = k) = 1$ (*)

$T_k(\omega) = \inf\{ n \geq 1 ; f_n(\omega) = k \}$

$T_0(\omega) = 2 \qquad T_0(\omega) = 4.$

T_k is random (depends on ω)
is the recurrence time

k is persistent (recurrent) $\Leftrightarrow f_{kk} = 1.$

$$f_{kk} \overset{=}{\text{df}} \sum_{n=1}^{\infty} f_{kk}^{(n)}$$

$$f_{kk}^{(n)} = \mathbb{P}(f_n = k, f_{n-1} \neq k, \dots, f_1 \neq k \mid f_0 = k)$$

$$= \mathbb{P}(f_n = k, f_{n-1} \neq k, \dots, f_1 \neq k)$$

(*)

$$= \mathbb{P}(T_k = n)$$

So if k is persistent $\boxed{\mathbb{P}_{T_k} = \sum_{n=1}^{\infty} f_{kk}^{(n)} \delta_n}$

discrete measure

$$\delta_n(B) = \begin{cases} 1 & n \in B \\ 0 & n \notin B \end{cases}$$

$$\mathbb{P}_{T_k}(\{e\}) = \mathbb{P}(T_k = e) = f_{kk}^{(e)}$$

$$\sum_{n=1}^{\infty} f_{kk}^{(n)} \delta_n(\{e\}) = 0 + 0 + f_{kk}^{(e)} + 0 + 0 \cdots$$

$$\mu_k = \mathbb{E}\, T_k = \sum_{n=1}^{\infty} n \underbrace{f_{kk}^{(n)}}_{0 \le \cdot \le 1} \ge 0$$

mean recurrence time.

$\mu_k = \infty$ is possible

$\Rightarrow \mu_k^{-1} := 0$

X

$X_P = \{ e \in X,\ \sum_{n=1}^{\infty} p_{ee}^{(n)} = \infty \}$

$X_T = \{ e \in X,\ \sum_{n=1}^{\infty} p_{ee}^{(n)} < \infty \}$

positive persistent

$\mu_k^{-1} > 0$

null persistent

$\mu_k^{-1} = 0$

Ergodic Theorem (first version)

$\# X < \infty$

$$T = \begin{pmatrix} 0.602 & 0.398 \\ 0.166 & 0.695 \end{pmatrix}$$

$$T^5 = \begin{pmatrix} \boxed{0.305 \quad 0.695} \\ \boxed{0.296 \quad 0.710} \end{pmatrix} \qquad \square \approx \square$$

Definition Let $[(f_i)_{i=0}^{\infty}, P, T]$ be a Markov chain with $X = \{0, 1, \dots, K\}$

$p(k) > 0, \quad k \in X,$

$$T = \begin{pmatrix} p_{00} & \cdots & p_{0K} \\ \vdots & & \vdots \\ p_{K0} & \cdots & p_{KK} \end{pmatrix}$$

$$T^n = \underbrace{T \circ \cdots \circ T}_{n} = \left(p_{kl}^{(n)} \right)_{k,l=0}^{K}$$

1. Markov chain is ergodic $\overset{}{\underset{df}{\Longleftrightarrow}}$ $\exists\, s_l > 0, \; l \in X$

(a) $s_0 + s_1 + \cdots + s_K = 1$

(b) $\lim\limits_{n \to \infty} p_{kl}^{(n)} = s_l \quad \forall k, l \in X$

2. A distribution $s = (s_0, \ldots, s_K)$ is called stationary $\underset{df}{\Longleftrightarrow}$

(a) $s_0 + s_1 + \cdots + s_K = 1, \quad s_l \in [0, 1],$

(b) $\quad s \circ T = s$ i.e. $\sum_{m=0}^{K} s_m \, p_{me} = s_e$.

Proposition:

An ergodic Markov chain has a unique stationary distribution.

Proof:

$$T^n \longrightarrow \begin{pmatrix} s_0 & s_1 & \cdots & s_K \\ s_0 & s_1 & \cdots & s_K \\ \vdots & & & \\ s_0 & s_1 & \cdots & s_K \end{pmatrix} = A$$

becaus $p_{kc}^{(n)} \longrightarrow s_e$.

$$A = \lim_{n\to\infty} T^n = \left(\lim_{n\to\infty} T^n\right) \circ T$$

$$= A \circ T$$

$$\Rightarrow \quad (s_0, \dots, s_K) \circ T = (s_0, \dots, s_K)$$

is a stationary distribution.

<u>$s = (s_0, \dots, s_K)$ is unique:</u>

Let $b = (b_0, \dots, b_K)$ be a stationary distribution $\Rightarrow$ $b \circ T = b$

$$b \circ T^2 = (b \circ T) \circ T = b \circ T = b$$

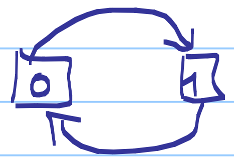
$$\lim_{n \to \infty} b \circ T^n = b \Rightarrow b \circ A = b$$

$$b_\ell = b_0 s_\ell + b_1 s_\ell + \cdots b_K s_\ell$$
$$= s_\ell (b_0 + \dots + b_K) \quad \forall \ell \in X$$

$\underbrace{\qquad\qquad\qquad}_{=1}$ b is a distri-
bution

$b = s$ or s is unique. $\square$

<u>Example</u> MC has a unique stationary distribution $\not\Rightarrow$ MC is ergodic



$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$s = (s_0, s_1)$

$s_0 + s_1 \overset{!}{=} 1$

$s_i \in [0,1]$

$s \circ T = s$

$\Leftrightarrow$ $s_1 = s_0$
$s_0 = s_1$

so we have only one choice: $s = \left(\frac{1}{2}, \frac{1}{2}\right)$

$$p_{ek}^{(n)} \xrightarrow[n\to\infty]{?} s_k \; :$$

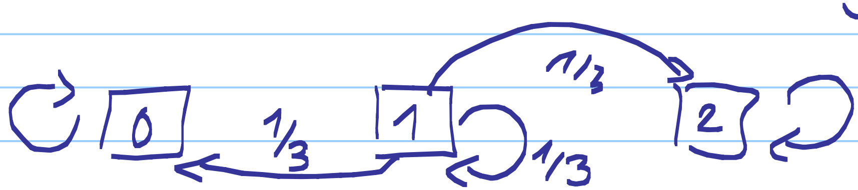
$$T^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\left.\begin{array}{l} T^3 = T^2 \circ T = T \\ T^4 = T^2 \end{array}\right\} \Rightarrow \begin{array}{l} T^{2n} = T^2 \\ T^{2n+1} = T \end{array}$$

$$\underset{n=1,2,\dots}{p_{ek}^{(n)} = 0, 1, 0, 1} \quad \text{do } \underline{\text{not}} \text{ converge}$$

$\Rightarrow$ MC is <u>not</u> ergodic!

Example: $\exists$ more than one stationary distribution



$$T = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix}$$

$$s \circ T \overset{!}{=} s \Rightarrow \left.\begin{aligned} s_0 + \tfrac{1}{3} s_1 &= s_0 \\ \tfrac{1}{3} s_1 &= s_1 \\ \tfrac{1}{3} s_1 + s_2 &= s_2 \end{aligned}\right\} s_1 = 0$$

$$s_0 + \underbrace{s_1}_{=0} + s_2 = 1 \Rightarrow s_2 = 1 - s_0$$

$$s = (s_0, 0, 1 - s_0)$$

$$s_0 \in [0, 1]$$

$\exists$ ∞ many stationary distributions

Theorem Ergodic Theorem (first version)

Let $[(f_i)_{i=0}^{\infty}, p, T]$ be a Markov chain with $X = \{0, \dots, K\}$

$T^n = T \circ \dots \circ T = \left(p_{k\ell}^{(n)}\right)_{k,\ell=0}^{K}$

Let there exists $n_0 \geq 1$ such that

$$\inf_{k,\ell} p_{k\ell}^{(n_0)} > 0. \quad \text{Then}$$

1. Markov chain is ergodic

2. The stationary distribution $s = (s_0, \dots, s_K)$ satisfies $s_\ell > 0 \ \forall \ell \in X$.

__Proof__ Assume $n_0 = 1$

$$T^n = \begin{array}{c} \begin{pmatrix} p_{00}^{(n)} & \cdots & p_{0K}^{(n)} \\ & & \\ p_{K0}^{(n)} & \cdots & p_{KK}^{(n)} \end{pmatrix} \\ \qquad\quad \ell \end{array}$$

$$m_\ell^{(n)} := \min_k p_{k\ell}^{(n)} \leq \max_k p_{k\ell}^{(n)} =: M_\ell^{(n)}$$

$\boxed{m_\ell^{(n+1)} \geq m_\ell^{(n)}}$:

$$m_\ell^{(n+1)} = \min_k p_{k\ell}^{(n+1)}$$

$$= \min_k \sum_{s=0}^{K} p_{ks} p_{s\ell}^{(n)}$$

Chapman-

Kolm.

$$\geq \min_s p_{se}^{(n)}$$

$$\geq \min_k \underbrace{\min_s p_{se}^{(n)}}_{= m_e^{(n)}} \underbrace{\sum_{s=0}^{k} p_{ks}}_{= 1}$$

$$= m_e^{(n)}$$

$\square$