Stochastic Modeling, 07.09.09

Note Title 7.9.2009

Introduction

mathematical modeling:

needed information

- natural laws
  - laws from economy
  - social laws
- scientific data

1. Weather forecast

1 day = 8am - 8am next day

day wet = if ≥ 0.01 inches precipitation

day dry = if < 0.01 inches precipitation

1948 - 1983

January

$$(X_{ij})_{i=1,\, j=1948}^{31,\, 1983} \qquad X_{ij} = \begin{cases} 1 & \text{wet} \\ 0 & \text{dry} \end{cases}$$

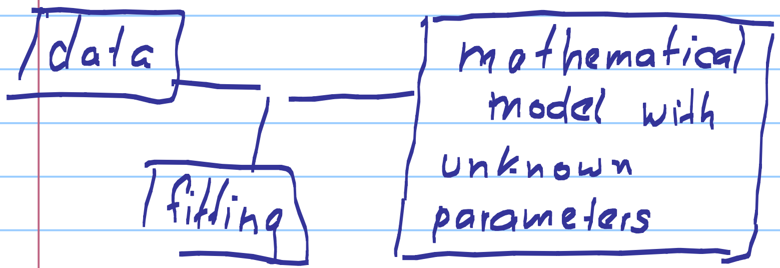
2. Share prices

NOKIA share, 1 year:

- price in Euro
- price taken 12:00 each trading

} rules

3 Modeling



model with parameters

describing the considered process → future events

2. Concepts of Stochastics

Probability space $(\Omega, \mathcal{F}, \mathbb{P})$

Ω is a set

$\Omega = \{\omega_1, \dots, \omega_n\}$

$\Omega = \{x, \; x \in [0,1]\}$

$\Omega \neq \emptyset$

$\mathcal{F}$ is a σ-algebra (σ-field)

Definition: σ algebra

Let $\Omega \neq \emptyset$. The the collection $\underline{\mathcal{F}}$ of subsets $A \subseteq \Omega$ is called a $\underline{\sigma\text{-algebra}}$

(1) $\emptyset, \Omega \in \mathcal{F}$

(2) $A \in \mathcal{F} \Rightarrow A^c = \Omega \setminus A \in \mathcal{F}$

(3) $A_1, A_2, \dots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

Remark: $A_1, A_2, A_3, \ldots \in \mathcal{F}$

$$\Longrightarrow \bigcap_{i=1}^{\infty} A_i \in \mathcal{F}.$$

Example Ω is given

1. Largest σ-algebra:

$\mathcal{F} = 2^{\Omega}$ = all possible subsets $A \subseteq \Omega$

2. The smallest σ-algebra:

$\mathcal{F} = \{\emptyset, \Omega\}$

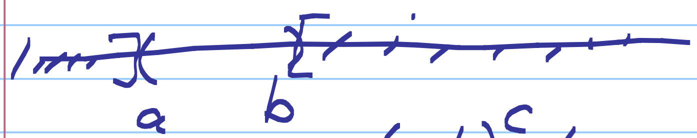
$\emptyset^c = \Omega, \quad \Omega^c = \emptyset$

3. $\mathcal{F} = \{\emptyset, \Omega, A, A^c\}$

$\emptyset \neq A \neq \Omega$

4. $\mathcal{F} = \mathcal{B}(\mathbb{R})$ = Borel sets of the real line

= smallest σ-algebra, which contains all open intervalls (a,b)



$(a,b)^c \neq$ open interval

$(\Omega, \mathcal{F}, \mathbb{P})$

$\uparrow \quad \uparrow \quad \uparrow$

Definition: probability measure

Let $\Omega \neq \emptyset$, $\mathcal{F}$ is a σ-algebra on Ω,
then the map

$$\mathbb{P} : \mathcal{F} \longrightarrow [0,1]$$

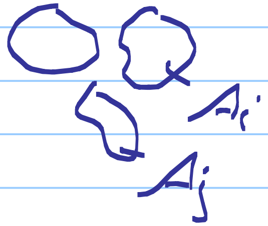
is called a probability measure if

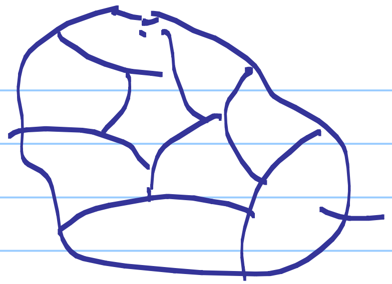
(1) $\mathbb{P}(\Omega) = 1$,

(2) $A_1, A_2, \ldots \in \mathcal{F}$ such that

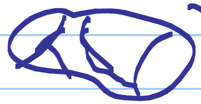
$$A_i \cap A_j = \emptyset, \quad i \neq j$$

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$





A_j

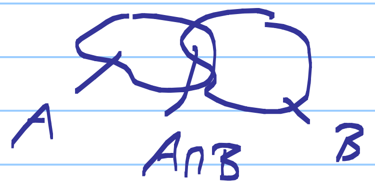
$\mathbb{P}\Big($  $\Big) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$

$\mathbb{P}(\Omega) = 1$

Properties of $\mathbb{P}$

1. $\mathbb{P}(\emptyset) = 0$,
2. $A, B \in \mathcal{F}$

   $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$



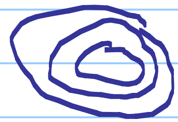
3. $A \in \mathcal{F}$

   $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$

4. Continuity of $\mathbb{P}$ from below

   $A_1, A_2, \ldots \in \mathcal{F}$

   $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$



$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i \to \infty} \mathbb{P}(A_i).$$

Example: Binomial distribution

$(\Omega, \mathcal{F}, \mathbb{P})$

$\Omega = \{0, 1, \dots, n\}$

$\mathcal{F}$ = set of all subsets of $\Omega = 2^{\Omega}$

$\mathbb{P}(\{k\}) = \binom{n}{k} p^k (1-p)^{n-k}$

$0 < p < 1$

$k = 0, 1, \dots, n$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$\boxed{\mathbb{P}(A) = \sum_{k \in A} \mathbb{P}(\{k\})}$$

$A \in \mathcal{F}$, $A \subseteq \Omega$ $\qquad A = \{0, 2\}$

$\mathbb{P}(\{0, 2\}) = \mathbb{P}(\{0\}) + \mathbb{P}(\{2\})$.

3. Independence and conditional probability

Example: rolling a die

$$\mathbb{P}(1.\text{ time } 3,\ 2.\text{ time } 6)$$
$$= \underbrace{\mathbb{P}(1 \text{ times } 3)}_{=\frac{1}{6}}\underbrace{\mathbb{P}(1 \text{ times } 6)}_{\frac{1}{6}}$$
$$= \frac{1}{36}.$$

<u>Definition</u> (Independence $A, B \in \mathcal{F}$)

A and B ($\in \mathcal{F}$) are <u>independent</u> if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

Definition (Conditional Probability)

Let $A \in \mathcal{F}$, $\mathbb{P}(A) > 0$. Then

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}$$

is called the conditional probability of B given A.

Notice if A and B are independent:

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A)\mathbb{P}(B)}{\mathbb{P}(A)}$$

$$= \mathbb{P}(B).$$

Definition: Independence of $A_1, \dots, A_n \in \mathcal{F}$

$A_1, \dots, A_n$ are independent,

if for $k = 2, 3, \dots, n$, $1 \le i_1 < i_2 < \dots < i_k \le n$

$$\mathbb{P}(A_{i_1} \cap \dots \cap A_{i_k}) = \mathbb{P}(A_{i_1}) \cdots \mathbb{P}(A_{i_k})$$

Assume : $\mathbb{P}(A_1 \cap A_2) \neq \mathbb{P}(A_1)\,\mathbb{P}(A_2)$

choose $A_3 = \emptyset$

$$A_1 \cap A_2 \cap A_3 = \emptyset$$

$$\mathbb{P}(A_1 \cap A_2 \cap A_3) = 0 = \mathbb{P}(A_1) \times \mathbb{P}(A_2) \cdot \underset{\mathbb{P}(A_3)}{\underset{\parallel}{0}}$$