Stokastiset differentiaaliyhtälöt

Harjoitus 11 (28. Huhtikuuta 2009)

Assume that B is a standard Brownian motion defined on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ satisfying the usual assumptions.

(1) Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$|\sigma(x) - \sigma(y)| \leq K|x - y| \quad \text{and} \quad |\sigma(x)| \leq K[1 + |x|].$$

Assume that $X = (X_t)_{t\geq 0}$ is the solution of the SDE

$$dX_t = \sigma(X_t)dB_t \quad \text{with} \quad X_0 = 1.$$

(a) Show that there are constants $A, B > 0$ such that

$$\mathbb{E}X_t^2 \leq Ae^{Bt} \quad \text{for all} \quad t \geq 0.$$

Hint: Gronwall-lemma.

(b) Is the exponential order in t of e^{Bt} optimal?

Hint: Compute $\mathbb{E}\left(e^{B_t - \frac{t}{2}}\right)^2$.

(2) Prove Proposition 4.4.3 for an process L with $L_t := 1$ for $t \in [0, 1]$ and $L_t := 2$ for $t > 1$.

Hint: Proposition 3.1.15.

(3) Consider the SDE

$$dS_t = S_t dB_t + (\mu S_t - tS_t)dt \quad \text{where} \quad S_0 = 1$$

and let $T > 0$.

(a) Solve the SDE by the help of Proposition 4.4.9 (S_t is a function, which depends on B_t, μ, and t only).

(b) Check the solution by Itô's formula.

(4) Consider the differential equation

$$dX_t = 2X_t dB_t + 5X_t dt \quad \text{with} \quad X_0 = 1.$$

(a) Given $T > 0$, find a measure Q_T equivalent to P such that the solution X is a martingale on $[0, T]$ with respect to Q.

(b) Given an arbitrary solution X. Why does X have almost surely positive trajectories.

(5*) Let $W_t = (B_{t,1}, ..., B_{t,3})$ be a 3-dimensional standard Brownian motion starting in zero and $R_t := |x_0 + W_t|$ be a 3-dimensional Bessel process starting at x_0 with $|x_0| = r > c > 0$. Assume that You know that

$$\mathbb{P}(\inf R_t \leq c) = \frac{c}{r}.$$

Deduce intuitively that $\mathbb{P}(\lim_{t\to\infty} |R_t| = \infty) = 1$, i.e. the 3-dimensional Brownian motion is transient.