

Stochastic Processes in continuous time

Stefan Geiss

April 28, 2009

Contents

1	Introduction	5
2	Stochastic processes in continuous time	7
2.1	Some definitions	7
2.2	Two basic examples of stochastic processes	12
2.3	Gaussian processes	14
2.4	Brownian motion	25
2.5	Stopping and optional times	29
2.6	A short excursion to Markov processes	33
3	Stochastic integration	35
3.1	Definition of the stochastic integral	36
3.2	Itô's formula	53
3.3	Proof of ITÔ's formula in a simple case	63
4	Stochastic differential equations	67
4.1	What is a stochastic differential equation?	67
4.2	Strong Uniqueness of SDE's	70
4.3	Existence of strong solutions of SDE's	73
4.4	Solutions of SDE's by a transformation of drift	75
4.5	Weak solutions	81

Chapter 1

Introduction

One goal of the lecture is to study stochastic differential equations (SDE's). So let us start with a (hopefully) motivating example: Assume that X_t is the share price of a company at time $t \geq 0$ where we assume without loss of generality that $X_0 := 1$. To get an idea of the dynamics of X let us consider the relative increments (these are the increments which are relevant in financial markets)

$$\frac{X_{t+\Delta} - X_t}{X_t} \sim b\Delta + \sigma Y_{t,\Delta}$$

with $b \in \mathbb{R}$, $\sigma > 0$, and $\Delta > 0$ being small. Here $b\Delta$ describes a general trend and $\sigma Y_{t,\Delta}$ some random events (perturbations). Asking several people about this approach we probably get answers like that:

- Statisticians: the random variables $Y_{t,\Delta}$ should be centered GAUSSIAN random variables.
- Mathematicians: the perturbations should not have a memory, otherwise the problem gets too difficult. Hence $Y_{t,\Delta}$ is independent from X_t .
- Then, in addition, probably both of them agree to assume that the perturbations behave additively, that means

$$Y_{t,\Delta} = Y_{t,\frac{\Delta}{2}} + Y_{t+\frac{\Delta}{2},\frac{\Delta}{2}}$$

so that $\text{var}(Y_{t,\Delta}) = \Delta$ is a good choice.

An approach like this yields to the famous BLACK-SCHOLES option pricing model. Is it possible to make out of this a correct mathematical theory? Yes it is, if we proceed for example in the following way:

Step 1: The random variables $Y_{t,\Delta}$ will be replaced by a continuous time stochastic process $W = (W_t)_{t \geq 0}$, called Brownian motion, such that

$$Y_{t,\Delta} = W_{t+\Delta} - W_t.$$

Consequently, we have to introduce and study the Brownian motion.

Step 2: Our formal equation reads now as

$$X_{t+\Delta} - X_t \sim (bX_t)\Delta + (\sigma X_t)(W_{t+\Delta} - W_t)$$

with $X_0 = 1$ which looks nicer. Letting $\Delta \downarrow 0$ we hope to get a stochastic differential equation

$$dX_t = bX_t dt + \sigma X_t dW_t$$

we are able to explain. To this end we introduce stochastic integrals to be in a position to write the differential equation as an integral equation

$$X_t = X_0 + \int_0^t bX_u du + \int_0^t \sigma X_u dW_u.$$

Step 3: We have to solve this equation. In particular we have to study whether the solutions are unique and to find possible ways to obtain them.

Chapter 2

Stochastic processes in continuous time

2.1 Some definitions

In this section we introduce some basic concepts concerning continuous time stochastic processes used freely later on. Let us fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and recall that a map $Z : \Omega \rightarrow \mathbb{R}$ is called a random variable if Z is measurable as a map from (Ω, \mathcal{F}) into $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ where $\mathcal{B}(\mathbb{R})$ is the Borel σ -algebra on \mathbb{R} .

What is a continuous time stochastic process? For us it is simply a family of random variables:

Definition 2.1.1 *A family of random variables $X = (X_t)_{t \geq 0}$ with $X_t : \Omega \rightarrow \mathbb{R}$ is called stochastic process with index set $I = [0, \infty)$.*

Throughout this lecture we only consider processes with values in \mathbb{R} and over the time interval $[0, \infty)$. The definition of a stochastic process can be given more generally. There are two different views on the stochastic process X :

- (1) The family $X = (X_t)_{t \geq 0}$ describes random functions by

$$\omega \rightarrow f(\omega) = (X_t(\omega))_{t \geq 0}.$$

The function $f(\omega) = (t \rightarrow X_t(\omega))$ is called *path* or *trajectory* of X .

- (2) The family $X = (X_t)_{t \geq 0}$ describes a *process*, which is, with respect to the time variable t , an ordered family of random variables $t \rightarrow X_t$.

The two approaches differ by the roles of ω and t . Next we ask, when do two stochastic processes X and Y coincide? It turns out to be useful to have several 'degrees of coincidence' as described now.

Definition 2.1.2 Let $X = (X_t)_{t \geq 0}$ and $Y = (Y_t)_{t \geq 0}$ be stochastic processes on $(\Omega, \mathcal{F}, \mathbb{P})$. The processes X and Y are *indistinguishable* if and only if

$$\mathbb{P}(X_t = Y_t, t \geq 0) = 1.$$

It should be noted that the definition automatically requires that the set $\{\omega \in \Omega : X_t(\omega) = Y_t(\omega), t \geq 0\}$ is measurable which is not the case in general as shown by the

Example 2.1.3 Let $\Omega = [0, 2)$, $Y_t = 0$,

$$X_t(\omega) := \begin{cases} 0 & : \omega \in [0, 1] \\ 0 & : \omega \in (1, 2), t \neq \omega \\ 1 & : \omega \in (1, 2), t = \omega \end{cases},$$

and $\mathcal{F} := \sigma(X_t : t \geq 0) = \sigma(\{t\} : t \in (1, 2))$. But

$$\{\omega \in \Omega : X_t(\omega) = Y_t(\omega), t \geq 0\} = [0, 1] \notin \mathcal{F}.$$

Another form of coincidence is the following:

Definition 2.1.4 Let $X = (X_t)_{t \geq 0}$ and $Y = (Y_t)_{t \geq 0}$ be stochastic processes on $(\Omega, \mathcal{F}, \mathbb{P})$. The processes X and Y are *modifications* of each other provided that

$$\mathbb{P}(X_t = Y_t) = 1 \quad \text{for all } t \geq 0.$$

Up to now we have to have that the processes are defined on the same probability space. This can be relaxed as follows:

Definition 2.1.5 Let $X = (X_t)_{t \geq 0}$ and $Y = (Y_t)_{t \geq 0}$ be stochastic processes on $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\Omega', \mathcal{F}', \mathbb{P}')$, respectively. Then X and Y have the same *finite-dimensional distributions* if

$$\mathbb{P}((X_{t_1}, \dots, X_{t_n}) \in B) = \mathbb{P}'((Y_{t_1}, \dots, Y_{t_n}) \in B)$$

for all $0 \leq t_1 < \dots < t_n < \infty$, where $n = 1, 2, \dots$ and $B \in \mathcal{B}(\mathbb{R}^n)$.

Proposition 2.1.6 (i) *If X and Y are indistinguishable, then they are modifications of each other. The converse implication is not true in general.*

(ii) *If X and Y are modifications from each other, then they have the same finite-dimensional distributions. There are examples of stochastic processes defined on the same probability space having the same finite-dimensional distributions but which are not modifications of each other.*

Proof. (i) Fixing $t \geq 0$ we immediately get that

$$\mathbb{P}(X_t = Y_t) \geq \mathbb{P}(X_s = Y_s, s \geq 0) = 1.$$

To construct a counterexample for the converse implication let $T : \Omega \rightarrow \mathbb{R}$ be a random variable such that $T \geq 0$ and $\mathbb{P}(T = t) = 0$ for all $t \geq 0$. Define $X_t := 0$ and

$$Y_t(\omega) := \begin{cases} 0 & : t \neq T(\omega) \\ 1 & : t = T(\omega) \end{cases} = \chi_{\{T(\omega)=t\}}.$$

The map Y_t is a random variable and

$$\mathbb{P}(X_t = Y_t) = \mathbb{P}(0 = Y_t) = \mathbb{P}(T \neq t) = 1,$$

but

$$\{\omega \in \Omega : X_t(\omega) = Y_t(\omega), t \geq 0\} = \{\omega \in \Omega : 0 = Y_t(\omega), t \geq 0\} = \emptyset.$$

(ii) Let $N_t := \{X_t \neq Y_t\}$ so that $\mathbb{P}(N_t) = 0$. Then, for $B \in \mathcal{B}(\mathbb{R}^d)$,

$$\begin{aligned} \mathbb{P}(X_{t_1}, \dots, X_{t_n} \in B) &= \mathbb{P}(\{X_{t_1}, \dots, X_{t_n} \in B\} \setminus (N_{t_1} \cup \dots \cup N_{t_n})) \\ &= \mathbb{P}(\{Y_{t_1}, \dots, Y_{t_n} \in B\} \setminus (N_{t_1} \cup \dots \cup N_{t_n})) \\ &= \mathbb{P}(Y_{t_1}, \dots, Y_{t_n} \in B). \end{aligned}$$

For the counterexample we let $\Omega = [0, 1]$ and \mathbb{P} be the Lebesgue measure on $[0, 1]$. We choose $X_t = X_0$, $Y_t = Y_0$, $X_0(t) := t$, and $Y_0(t) := 1 - t$. \square

There are situations in which two processes are indistinguishable when they are modifications of each other.

Proposition 2.1.7 *Assume that X and Y are modifications of each other and that all trajectories of X and Y are left-continuous (or right-continuous). Then the processes X and Y are indistinguishable.*

Proof. In both cases we have that

$$A := \{X_t = Y_t, t \geq 0\} = \{X_t = Y_t, t \in Q \cap [0, \infty)\}$$

so that $A \in \mathcal{F}$ and

$$\begin{aligned} \mathbb{P}(A) &= \mathbb{P}(X_t = Y_t, t \in Q \cap [0, \infty)) \\ &= 1 - \mathbb{P}(X_t \neq Y_t \text{ for some } t \in Q \cap [0, \infty)) \\ &\geq 1 - \sum_{t \in Q \cap [0, \infty)} \mathbb{P}(X_t \neq Y_t) \\ &= 1. \end{aligned}$$

\square

We also need different types of measurability for our stochastic processes. First let us recall the notion of a filtration and a stochastic basis.

Definition 2.1.8 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A family of σ -algebras $(\mathcal{F}_t)_{t \geq 0}$ is called filtration if $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for all $0 \leq s \leq t < \infty$. The quadruple $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$ is called stochastic basis.

The different types of measurability are given by

Definition 2.1.9 Let $X = (X_t)_{t \geq 0}$, $X_t : \Omega \rightarrow \mathbb{R}$ be a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$ and let $(\mathcal{F}_t)_{t \geq 0}$ be a filtration.

- (i) The process X is called *measurable* provided that the function $(\omega, t) \rightarrow X_t(\omega)$ considered as map between $\Omega \times [0, \infty)$ and \mathbb{R} is measurable with respect to $\mathcal{F} \times \mathcal{B}([0, \infty))$ and $\mathcal{B}(\mathbb{R})$.
- (ii) The process X is called *progressively measurable* with respect to a filtration $(\mathcal{F}_t)_{t \geq 0}$ provided that for all $T \geq 0$ the function $(\omega, t) \rightarrow X_t(\omega)$ considered as map between $\Omega \times [0, T]$ and \mathbb{R} is measurable with respect to $\mathcal{F}_T \times \mathcal{B}([0, T])$ and $\mathcal{B}(\mathbb{R})$.
- (iii) The process X is called *adapted* with respect to a filtration $(\mathcal{F}_t)_{t \geq 0}$ provided that for all $t \geq 0$ one has that X_t is \mathcal{F}_t -measurable.

Proposition 2.1.10 *A process which is progressively measurable is measurable and adapted. All other implications between progressively measurable, measurable, and adapted do not hold true in general.*

Proof. (a) Assume that X is progressively measurable. We show that X is measurable as well. Given $n = 1, 2, \dots$ we have that $X^n : \Omega \times [0, n] \rightarrow \mathbb{R}$ given by $X^n(\omega, t) := X_t(\omega)$ is measurable with respect to $\mathcal{F}_n \times \mathcal{B}([0, n])$ by assumption. Hence the extension $\tilde{X}^n : \Omega \times [0, \infty) \rightarrow \mathbb{R}$ given by

$$\tilde{X}^n(\omega, t) := X_{t \wedge n}(\omega)$$

is measurable with respect to $\mathcal{F}_n \times \mathcal{B}([0, \infty))$ and henceforth with respect to $\mathcal{F} \times \mathcal{B}([0, \infty))$. This can be checked considering $\tilde{X}^n = X^n \circ J_n$ with $J_n : \Omega \times [0, \infty) \rightarrow \Omega \times [0, n]$ given by

$$J_n(\omega, t) := (\omega, t \wedge n).$$

Finally we observe that

$$X_t(\omega) = \lim_{n \rightarrow \infty} X_{t \wedge n}(\omega)$$

and get that X is $\mathcal{F} \times \mathcal{B}([0, \infty))$ -measurable as a limit of measurable mappings.

(b) Fix $t \geq 0$. Then $X^t : \Omega \times [0, t] \rightarrow \mathbb{R}$ is measurable with respect to $\mathcal{F}_t \times \mathcal{B}([0, t])$ by assumption. Then FUBINI's theorem gives that X_t is \mathcal{F}_t -measurable.

(c) Let $B \subseteq [0, \infty)$ be a non-measurable set and define $X_t := 1$ if $t \in B$ and $X_t := 0$ if $t \notin B$. Then X_t is constant, but

$$\{(\omega, t) \in \Omega \times [0, \infty) : X_t(\omega) = 1\} = \Omega \times B$$

is not measurable. Hence the process is adapted but not measurable or even progressively measurable.

(d) Examples of measurable, but not progressively measurable, processes are trivial. \square

Proposition 2.1.11 *An adapted process such that all trajectories are left-continuous (or right-continuous) is progressively measurable.*

Proof. We only consider the case of right-continuous paths. For $T = 0$ we easily have that the map $(\omega, 0) \mapsto X_0(\omega)$ is measurable when considered between $(\Omega \times \{0\}, \mathcal{F}_0 \times \mathcal{B}(\{0\}))$ and $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Consider now $T > 0$. We have to show that $(\omega, t) \mapsto X_t(\omega)$ is measurable when considered between $(\Omega \times [0, T], \mathcal{F}_T \times \mathcal{B}([0, T]))$ and $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. For $n \in \{1, 2, \dots\}$ we define $F_n : \Omega \times [0, T] \rightarrow \mathbb{R}$ by $(\omega, t) \mapsto X_t^{(n)}(\omega)$ with

$$X_t^{(n)}(\omega) := X_{\frac{k}{2^n}T}(\omega)$$

for $\frac{k-1}{2^n}T < t \leq \frac{k}{2^n}T$ and $k = 1, \dots, 2^n$, and $X_0^{(n)}(\omega) := X_0(\omega)$. It is clear that F_n is measurable when considered between $(\Omega \times [0, T], \mathcal{F}_T \times \mathcal{B}([0, T]))$ and $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Hence our claim follows since

$$X_t(\omega) = \lim_{n \rightarrow \infty} F_n(\omega, t)$$

by the path-wise right-hand side continuity of X . \square

Finally let us recall the notion of a martingale.

Definition 2.1.12 Let $(X_t)_{t \geq 0}$ be $(\mathcal{F}_t)_{t \geq 0}$ -adapted and such that $\mathbb{E}|X_t| < \infty$ for all $t \geq 0$.

(i) X is called *martingale* provided that for all $0 \leq s \leq t < \infty$ one has

$$\mathbb{E}(X_t | \mathcal{F}_s) = X_s \text{ a.s.}$$

(ii) X is called *sub-martingale* provided that for all $0 \leq s \leq t < \infty$ one has

$$\mathbb{E}(X_t | \mathcal{F}_s) \geq X_s \text{ a.s.}$$

- (iii) X is called *super-martingale* provided that for all $0 \leq s \leq t < \infty$ one has that

$$\mathbb{E}(X_t | \mathcal{F}_s) \leq X_s \text{ a.s.}$$

Finally we need some properties of the trajectories:

Definition 2.1.13 Let $X = (X_t)_{t \geq 0}$ be a stochastic process.

- (i) The process X is *continuous* provided that $t \rightarrow X_t(\omega)$ is continuous for all $\omega \in \Omega$.
- (ii) The process X is *càdlàg* (*continue à droite, limites à gauche*) provided that $t \rightarrow X_t(\omega)$ is right-continuous and has left limits for all $\omega \in \Omega$.
- (iii) The process X is *càglàd* (*continue à gauche, limites à droite*) provided that $t \rightarrow X_t(\omega)$ is left-continuous and has right limits for all $\omega \in \Omega$.

2.2 Two basic examples of stochastic processes

Brownian motion. The two-dimensional Brownian motion was observed in 1828 by ROBERT BROWN as diffusion of pollen in water. Later the one-dimensional Brownian motion was used by LOUIS BACHELIER around 1900 in modeling of financial markets and in 1905 by ALBERT EINSTEIN. A first rigorous proof of its (mathematical) existence was given by NORBERT WIENER in 1921. Later on, various different proofs of its existence were given.

Proposition 2.2.1 *There exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a filtration $(\mathcal{F}_t)_{t \geq 0}$, and a process $W = (W_t)_{t \geq 0}$ with $W_0 \equiv 0$ such that*

- (i) $(W_t)_{t \geq 0}$ is $(\mathcal{F}_t)_{t \geq 0}$ -adapted,
- (ii) $(W_t)_{t \geq 0}$ is continuous,
- (iii) for all $0 \leq s < t < \infty$ the random variable $W_t - W_s$ is independent of \mathcal{F}_s (*independent increments*),
- (iv) for all $0 \leq s < t < \infty$ one has $W_t - W_s \sim \mathcal{N}(0, t - s)$ (*stationary increments*).

Definition 2.2.2 A process satisfying the properties of Proposition 2.2.1 is called *standard $(\mathcal{F}_t)_{t \geq 0}$ -Brownian motion*.

Proposition 2.2.3 $W = (W_t)_{t \geq 0}$ is a continuous martingale.

Proof. From $W_t = W_t - W_0 \sim \mathcal{N}(0, t)$ we get immediately that $\mathbb{E}|W_t| < \infty$. Moreover, for $0 \leq s < t < \infty$ we have that, a.s.,

$$\begin{aligned} \mathbb{E}(W_t | \mathcal{F}_s) &= \mathbb{E}((W_t - W_s) | \mathcal{F}_s) + W_s \\ &= \mathbb{E}(W_t - W_s) + W_s \\ &= W_s. \end{aligned}$$

□

Poisson process. Let us assume that we have a lamp . Statistics says that the probability that a bulb breaks down is at any time the same. Hence it does not make sense to change a bulb before it is broken down.

How to model this? First we assume that we have discrete time. The probability that a bulb breaks down at time 1 is θ , at time 2 is $(1 - \theta)\theta$, at time 3 is $(1 - \theta)^2\theta$, and finally at time k is $(1 - \theta)^{k-1}\theta$. This yields a geometric distribution. This gives us the right guess for the continuous time case: We take a random variable $T : \Omega \rightarrow \mathbb{R}$ with $T \geq 0$ and

$$\mathbb{P}(T \in B) = \int_B \lambda e^{-\lambda t} dt$$

for $B \in \mathcal{B}(\mathbb{R})$ where $\lambda > 0$ is a parameter to model the time for breaking down. The random variable T has an exponential distribution with parameter $\lambda > 0$. Letting T_1, T_2, \dots be independent copies of T the time

$$S_n := T_1 + \dots + T_n$$

with $S_0 = 0$ gives the time that the n th bulb broke down. The inverse function

$$N_t := \max\{n \geq 0 : S_n \leq t\}$$

describes the number of bulbs which were broken down until time t .

Definition 2.2.4 $(N_t)_{t \geq 0}$ is called *Poisson process with intensity $\lambda > 0$* . Moreover, $\mathcal{F}_t^N := \sigma(N_s : s \in [0, t])$.

Proposition 2.2.5 (i) $(N_t)_{t \geq 0}$ is a càdlàg process adapted to $(\mathcal{F}_t^N)_{t \geq 0}$.

(ii) $N_t - N_s$ is independent from \mathcal{F}_s^N for $0 \leq s < t$.

(iii) $N_t - N_s$ has a Poisson distribution with parameter $\lambda(t - s)$, that means

$$\mathbb{P}(N_t - N_s = k) = \frac{\mu^k}{k!} e^{-\mu} \text{ for } \mu = \lambda(t - s).$$

There is a connection to martingales. To see this let us first compute the expected value

$$\mathbb{E}N_t = \sum_{k=0}^{\infty} k \frac{(\lambda t)^k}{k!} e^{-\lambda t} = \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{(k-1)!} e^{-\lambda t} = \lambda t \sum_{k=0}^{\infty} \frac{(\lambda t)^{k-1}}{(k-1)!} e^{-\lambda t} = \lambda t$$

to define the compensated Poisson process:

Definition 2.2.6 The process $(M_t)_{t \geq 0}$ with $M_t := N_t - \lambda t$ is called *compensated Poisson process with intensity $\lambda > 0$* .

Proposition 2.2.7 *The compensated Poisson process is a càdlàg martingale with respect to $(\mathcal{F}_t^N)_{t \geq 0}$.*

Proof. We have, a.s.,

$$\begin{aligned} \mathbb{E}(M_t - M_s \mid \mathcal{F}_s^N) &= \mathbb{E}(N_t - N_s \mid \mathcal{F}_s^N) - \lambda(t - s) \\ &= \mathbb{E}(N_t - N_s) - \lambda(t - s) \\ &= \mathbb{E}(N_{t-s}) - \lambda(t - s) \\ &= 0. \end{aligned}$$

□

2.3 Gaussian processes

Gaussian processes form a class of stochastic processes used in several branches in pure mathematics and in applied mathematics. Some typical examples are the following:

- The modeling of telecommunication traffic, where the fractional Brownian motion is used.
- In real analysis the Laplace operator is directly connected to the Brownian motion.
- In the theory of stochastic processes many processes can be represented and investigated as transformations of the Brownian motion.

We introduce Gaussian processes in two steps. First we recall Gaussian random variables with values in \mathbb{R}^n , then we turn to the processes.

Definition 2.3.1 (i) A random variable $f : \Omega \rightarrow \mathbb{R}$ is called *Gaussian* provided that $\mathbb{P}(f = m) = 1$ for some $m \in \mathbb{R}$ or there are $m \in \mathbb{R}$ and $\sigma > 0$ such that

$$\mathbb{P}(f \in B) = \int_B e^{-\frac{(x-m)^2}{2\sigma^2}} \frac{dx}{\sqrt{2\pi}\sigma}$$

for all $B \in \mathcal{B}(\mathbb{R})$. The parameters m and σ^2 are called *expected value* and *variance*, respectively.

- (ii) A random vector $f = (f_1, \dots, f_n) : \Omega \rightarrow \mathbb{R}^n$ is called *Gaussian* provided that for all $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ one has that

$$\langle f(\omega), a \rangle := \sum_{i=1}^n a_i f_i(\omega)$$

is Gaussian. The parameters $m = (m_1, \dots, m_n)$ with $m_i := \mathbb{E}f_i$ and $\sigma = (\sigma_{ij})_{i,j=1}^n$ with

$$\sigma_{ij} := \mathbb{E}(f_i - m_i)(f_j - m_j)$$

are called *mean (vector)* and *covariance (matrix)*, respectively.

For a Gaussian random variable we can compute the expected value and the variance by

$$m = \mathbb{E}f \quad \text{and} \quad \sigma^2 = \mathbb{E}(f - m)^2.$$

Proposition 2.3.2 *Assume Gaussian random variables $f, g : \Omega \rightarrow \mathbb{R}^n$ with the same parameters (m, σ) . Then f and g have the same laws.*

Proof. If $\widehat{f}(a) := \mathbb{E}e^{i\langle f, a \rangle}$ and $\widehat{g}(a) := \mathbb{E}e^{i\langle g, a \rangle}$ are the characteristic functions, then by the uniqueness theorem we need to show that $\widehat{f}(a) = \widehat{g}(a)$ for all $a \in \mathbb{R}^d$. For this it is sufficient to have that the distributions of $\langle f, a \rangle$ and $\langle g, a \rangle$ are the same. By assumption both random variables are Gaussian random variables. Hence we only have to check the expected values and the variances. We get

$$\mathbb{E}\langle f, a \rangle = \sum_{i=1}^n a_i \mathbb{E}f_i = \sum_{i=1}^n a_i m_i = \sum_{i=1}^n a_i \mathbb{E}g_i = \mathbb{E}\langle g, a \rangle$$

and

$$\begin{aligned} \mathbb{E}(\langle f, a \rangle - \mathbb{E}\langle f, a \rangle)^2 &= \sum_{i,j=1}^n a_i a_j \mathbb{E}(f_i - m_i)(f_j - m_j) \\ &= \langle \sigma a, a \rangle \\ &= \sum_{i,j=1}^n a_i a_j \mathbb{E}(g_i - m_i)(g_j - m_j) \\ &= \mathbb{E}(\langle g, a \rangle - \mathbb{E}\langle g, a \rangle)^2 \end{aligned}$$

and are done. □

Now we introduce Gaussian processes.

Definition 2.3.3 A stochastic process $X = (X_t)_{t \geq 0}$, $X_t : \Omega \rightarrow \mathbb{R}$, is called *Gaussian* provided that for all $n = 1, 2, \dots$ and all $0 \leq t_1 < t_2 < \dots < t_n < \infty$ one has that

$$(X_{t_1}, \dots, X_{t_n}) : \Omega \rightarrow \mathbb{R}^n$$

is a Gaussian random vector. Moreover, we let

$$m_t := \mathbb{E}X_t \quad \text{and} \quad \Gamma(s, t) := \mathbb{E}(X_s - m_s)(X_t - m_t).$$

The process $m = (m_t)_{t \geq 0}$ is called *mean (process)* and the process $(\Gamma(s, t))_{s, t \geq 0}$ *covariance (process)*.

Up to now we only defined Gaussian processes, however we do not know yet whether they exist. We will prove the existence by analyzing the finite dimensional distributions of a stochastic process $X = (X_t)_{t \geq 0}$. What are the properties we can expect? From now on we use the index set

$$\Delta := \{(t_1, \dots, t_n) : n \geq 1, t_1, \dots, t_n \text{ are distinct}\}.$$

Then the family $(\mu_{t_1, \dots, t_n})_{(t_1, \dots, t_n) \in \Delta}$ with

$$\mu_{t_1, \dots, t_n}(B) := \mathbb{P}((X_{t_1}, \dots, X_{t_n}) \in B)$$

defines a family of measures such that

$$\begin{aligned} \mu_{t_1, \dots, t_n}(B_1 \times \dots \times B_n) &= \mu_{(t_{\pi(1)}, \dots, t_{\pi(n)})}(B_{\pi(1)} \times \dots \times B_{\pi(n)}), \\ \mu_{t_1, \dots, t_n}(B_1 \times \dots \times B_{n-1} \times \mathbb{R}) &= \mu_{t_1, \dots, t_{n-1}}(B_1 \times \dots \times B_{n-1}) \end{aligned}$$

for all $B_1, \dots, B_n \in \mathcal{B}(\mathbb{R})$ and all permutations $\pi : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$. This is our starting point:

Definition 2.3.4 A family of probability measures $(\mu_{t_1, \dots, t_n})_{(t_1, \dots, t_n) \in \Delta}$, where μ_{t_1, \dots, t_n} is a measure on $\mathcal{B}(\mathbb{R}^n)$ is called *consistent* provided that

- (i) $\mu_{t_1, \dots, t_n}(B_1 \times \dots \times B_n) = \mu_{(t_{\pi(1)}, \dots, t_{\pi(n)})}(B_{\pi(1)} \times \dots \times B_{\pi(n)})$ for all $n = 1, 2, \dots$, $B_1, \dots, B_n \in \mathcal{B}(\mathbb{R})$, and all permutations $\pi : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$,
- (ii) $\mu_{t_1, \dots, t_n}(B_1 \times \dots \times B_{n-1} \times \mathbb{R}) = \mu_{t_1, \dots, t_{n-1}}(B_1 \times \dots \times B_{n-1})$ for all $n \geq 2$ and $B_1, \dots, B_{n-1} \in \mathcal{B}(\mathbb{R})$.

We show that a consistent family of measures can be derived from one measure. The measure will be defined on the following σ -algebra:

Definition 2.3.5 We let $\sigma(\mathbb{R}^{[0, \infty)})$ be the smallest σ -algebra which contains all cylinder sets

$$B := \{(\xi_t)_{t \geq 0} : (\xi_{t_1}, \dots, \xi_{t_n}) \in A\}$$

for $(t_1, \dots, t_n) \in \Delta$ and $A \in \mathcal{B}(\mathbb{R}^n)$.

Proposition 2.3.6 (Daniell 1918, Kolmogorov 1933) *Assume a consistent family $(\mu_{t_1, \dots, t_n})_{(t_1, \dots, t_n) \in \Delta}$ of probability measures. Then there exists a probability measure μ on $\mathcal{B}(\mathbb{R}^{[0, \infty)})$ such that*

$$\mu((\xi_t)_{t \geq 0} : (\xi_{t_1}, \dots, \xi_{t_n}) \in A) = \mu_{t_1, \dots, t_n}(A)$$

for all $(t_1, \dots, t_n) \in \Delta$ and $A \in \mathcal{B}(\mathbb{R}^n)$.

Proof. We only give the idea of the proof. Let \mathcal{A} be the algebra of cylinder sets

$$B := \{(\xi_t)_{t \geq 0} : (\xi_{t_1}, \dots, \xi_{t_n}) \in A\}$$

with $(t_1, \dots, t_n) \in \Delta$ and $A \in \mathcal{B}(\mathbb{R}^n)$, that means we have that

- $\mathbb{R}^{[0, \infty)} \in \mathcal{A}$,
- $B_1, \dots, B_n \in \mathcal{B}$ implies that $B_1 \cup \dots \cup B_n \in \mathcal{B}$,
- $B \in \mathcal{A}$ implies that $B^c \in \mathcal{A}$.

Now we define $\nu : \mathcal{A} \rightarrow [0, 1]$ by

$$\nu((\xi_t)_{t \geq 0} : (\xi_{t_1}, \dots, \xi_{t_n}) \in A) := \mu_{t_1, \dots, t_n}(A).$$

In fact, the definition is correct. Assume that

$$\{(\xi_t)_{t \geq 0} : (\xi_{s_1}, \dots, \xi_{s_m}) \in A\} = \{(\xi_t)_{t \geq 0} : (\xi_{t_1}, \dots, \xi_{t_n}) \in B\}.$$

Let $(r_1, \dots, r_N) \in \Delta$ such that $\{r_1, \dots, r_N\} = \{s_1, \dots, s_m, t_1, \dots, t_n\}$. By adding coordinates we find an $C \in \mathcal{B}(\mathbb{R}^N)$ such that the sets above are equal to

$$\{(\xi_t)_{t \geq 0} : (\xi_{r_1}, \dots, \xi_{r_N}) \in C\}.$$

By the consistency we have that

$$\mu_{s_1, \dots, s_m}(A) = \mu_{r_1, \dots, r_N}(C) = \mu_{t_1, \dots, t_n}(B).$$

More difficult would be to check that ν is σ -additive on \mathcal{A} , which means that

$$\nu \left(\bigcup_{n=1}^{\infty} B_n \right) = \sum_{n=1}^{\infty} \nu(B_n)$$

for $B_1, B_2, \dots \in \mathcal{A}$, $B_i \cap B_j = \emptyset$ for $i \neq j$, and $\bigcup_{n=1}^{\infty} B_n \in \mathcal{A}$. This we leave out for the moment. Having this we might finish with CARATHÉODORY's extension theorem. \square

As an application we get the following

Proposition 2.3.7 *Let $(\Gamma(s, t))_{s, t \geq 0}$ be positive semi-definite and symmetric, that means*

$$\sum_{i, j=1}^n \Gamma(t_i, t_j) a_i a_j \geq 0 \quad \text{and} \quad \Gamma(s, t) = \Gamma(t, s)$$

for all $s, t, t_1, \dots, t_n \geq 0$ and $a_1, \dots, a_n \in \mathbb{R}$. Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a Gaussian process $X = (X_t)_{t \geq 0}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with

- (i) $\mathbb{E}X_t = 0$,
- (ii) $\mathbb{E}X_s X_t = \Gamma(s, t)$.

Remark 2.3.8 Given any stochastic process $X = (X_t)_{t \geq 0} \subseteq L_2$ with $\mathbb{E}X_t = 0$ and $\Gamma(s, t) := \mathbb{E}X_s X_t$ we always have that Γ is positive semi-definite and symmetric.

Proof of Proposition 2.3.7. We will construct a consistent family of probability measures. Given $(t_1, \dots, t_n) \in \Delta$, we let μ_{t_1, \dots, t_n} be the Gaussian measure on \mathbb{R}^n with mean zero and covariance

$$\int_{\mathbb{R}^n} \xi_i \xi_j d\mu_{t_1, \dots, t_n}(\xi_1, \dots, \xi_n) = \Gamma(t_i, t_j).$$

If the measure exists, then it is unique. To obtain the measure we let $C := (\Gamma(t_i, t_j))_{i, j=1}^n$, so that C is symmetric and positive semi-definite. We know from algebra that there is a matrix A such that $C = AA^T$. Let γ_n be the standard Gaussian measure on \mathbb{R}^n and μ be the image with respect to $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$. Then

$$\int_{\mathbb{R}^d} \langle x, e_i \rangle d\mu(x) = \int_{\mathbb{R}^d} \langle Ax, e_i \rangle d\gamma_n(x) = 0$$

and

$$\begin{aligned} \int_{\mathbb{R}^n} \langle x, e_i \rangle \langle x, e_j \rangle d\mu(x) &= \int_{\mathbb{R}^n} \langle Ax, e_i \rangle \langle Ax, e_j \rangle d\gamma_n(x) \\ &= \int_{\mathbb{R}^n} \langle x, A^T e_i \rangle \langle x, A^T e_j \rangle d\gamma_n(x) \\ &= \langle A^T e_i, A^T e_j \rangle \\ &= \langle e_i, AA^T e_j \rangle \\ &= \langle e_i, C e_j \rangle. \end{aligned}$$

The defined family of measures is easily seen to be consistent: given a permutation $\pi : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ we have that the covariance of $\mu_{t_{\pi(1)}, \dots, t_{\pi(n)}}$

is $\Gamma(t_{\pi(i)}, t_{\pi(j)})$ which proves property (i). Hence $\mu_{t_{\pi(1)}, \dots, t_{\pi(n)}}$ can be obtained from μ by permutation of the coordinates. To prove that

$$\mu_{t_1, \dots, t_{n-1}, t_n}(B_1 \times \dots \times B_{n-1} \times \mathbb{R}) = \mu_{t_1, \dots, t_{n-1}}(B_1 \times \dots \times B_{n-1})$$

we consider the linear map $A : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ with $A(\xi_1, \dots, \xi_n) := (\xi_1, \dots, \xi_{n-1})$ so that

$$A^{-1}(B_1 \times \dots \times B_{n-1}) = B_1 \times \dots \times B_{n-1} \times \mathbb{R}$$

and

$$\mu_{t_1, \dots, t_{n-1}, t_n}(B_1 \times \dots \times B_{n-1} \times \mathbb{R}) = \mu_{t_1, \dots, t_{n-1}, t_n}(A^{-1}(B_1 \times \dots \times B_{n-1}))$$

and we need to show that

$$\nu := \mu_{t_1, \dots, t_{n-1}, t_n}(A^{-1}(\cdot)) = \mu_{t_1, \dots, t_{n-1}}.$$

The measure ν is the image measure of $\mu_{t_1, \dots, t_{n-1}, t_n}$ with respect to A so that it is a Gaussian measure. Moreover,

$$\begin{aligned} & \int_{\mathbb{R}^{n-1}} \eta_i \eta_j d\nu(\eta_1, \dots, \eta_{n-1}) \\ &= \int_{\mathbb{R}^n} \langle A\xi, e_i \rangle \langle A\xi, e_j \rangle d\mu_{t_1, \dots, t_n}(\xi) \\ &= \int_{\mathbb{R}^n} \langle \xi, A^T e_i \rangle \langle \xi, A^T e_j \rangle d\mu_{t_1, \dots, t_n}(\xi) \\ &= \sum_{k, l=1}^n \langle e_k, A^T e_i \rangle \langle e_l, A^T e_j \rangle \int_{\mathbb{R}^n} \langle \xi, e_k \rangle \langle \xi, e_l \rangle d\mu_{t_1, \dots, t_n}(\xi) \\ &= \sum_{k, l=1}^n \langle e_k, A^T e_i \rangle \langle e_l, A^T e_j \rangle \sigma_{kl} \\ &= \sum_{k, l=1}^n \langle Ae_k, e_i \rangle \langle Ae_l, e_j \rangle \sigma_{kl} \\ &= \sum_{k, l=1}^{n-1} \langle e_k, e_i \rangle \langle e_l, e_j \rangle \sigma_{kl} \\ &= \sigma_{ij}. \end{aligned}$$

Now the process $X = (X_t)_{t \geq 0}$ is obtained by $X_t : \mathbb{R}^{[0, \infty)} \rightarrow \mathbb{R}$ with $X_t((\xi_s)_{s \geq 0}) := \xi_t$. \square

Let us consider our first examples:

Example 2.3.9 (Brownian motion) We let

$$\Gamma(s, t) := \min \{s, t\} = \int_0^\infty \chi_{[0, s]}(\xi) \chi_{[0, t]}(\xi) d\xi$$

so that

$$\begin{aligned} \sum_{i,j=1}^n \Gamma(t_i, t_j) a_i a_j &= \int_0^\infty \sum_{i,j=1}^n a_i \chi_{[0,t_i]}(\xi) a_j \chi_{[0,t_j]}(\xi) d\xi \\ &= \int_0^\infty \left(\sum_{i=1}^n a_i \chi_{[0,t_i]}(\xi) \right)^2 d\xi \\ &\geq 0. \end{aligned}$$

Example 2.3.10 (Brownian bridge) Here we take for a moment the time-interval $[0, 1]$ instead of $[0, \infty)$ and believe that all things from before can be done in the same way. We let

$$\Gamma(s, t) := \begin{cases} s(1-t) & : 0 \leq s \leq t \leq 1 \\ t(1-s) & : 0 \leq t \leq s \leq 1 \end{cases}$$

and want to get a *Gaussian* process returning to zero at time $T = 1$. The easiest way to show that Γ is positive semi-definite is to find one realization of this process: we take the Brownian motion $W = (W_t)_{t \geq 0}$ like in Example 2.3.9, let

$$X_t := W_t - tW_1$$

and get that

$$\begin{aligned} \mathbb{E}X_s X_t &= \mathbb{E}(W_s - sW_1)(W_t - tW_1) \\ &= \mathbb{E}W_s W_t - t\mathbb{E}W_s W_1 - s\mathbb{E}W_1 W_t + st\mathbb{E}W_1^2 \\ &= s - ts - st + st \\ &= s(1-t) \end{aligned}$$

for $0 \leq s \leq t \leq 1$.

Example 2.3.11 (Fractional Brownian motion) The Fractional Brownian motion was considered in 1941 by KOLMOGOROV in connection with turbulence and in 1968 by MANDELROT and VAN NESS as fractional Gaussian noise. Let $H \in (0, 1)$ the HURST index of the fractional Brownian motion (HURST was an English hydrologist) and define the covariance function Γ as

$$\Gamma(s, t) := \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}).$$

For $H = 1/2$ we get $\Gamma(s, t) = \min\{s, t\}$, that means the Brownian motion from Example 2.3.9. The main problem consists in showing that Γ is positive semi-definite. To give the idea for this proof let $t_0 := 0$ and $a_0 := -\sum_{i=1}^n a_i$ so that $\sum_{i=0}^n a_i = 0$ and

$$\sum_{i,j=1}^n \Gamma(t_i, t_j) a_i a_j = -\frac{1}{2} \sum_{i,j=0}^n |t_i - t_j|^{2H} a_i a_j.$$

Take $\varepsilon > 0$ so that

$$\begin{aligned} \sum_{i,j=0}^n e^{-\varepsilon|t_i-t_j|^{2H}} a_i a_j &= \sum_{i,j=0}^n \left(e^{-\varepsilon|t_i-t_j|^{2H}} - 1 \right) a_i a_j \\ &= -\varepsilon \sum_{i,j=0}^n |t_i - t_j|^{2H} a_i a_j + o(\varepsilon) \\ &= 2\varepsilon \sum_{i,j=1}^n \Gamma(t_i, t_j) a_i a_j + o(\varepsilon). \end{aligned}$$

Hence it is sufficient to show that

$$\sum_{i,j=0}^n e^{-\varepsilon|t_i-t_j|^{2H}} a_i a_j \geq 0.$$

The following fact shows that there exists a random variable Z such that

$$\mathbb{E}e^{itZ} = e^{-\varepsilon|t|^{2H}}.$$

Fact 2.3.12 (Polya) *Let $\varphi : \mathbb{R} \rightarrow [0, \infty)$ be*

- (i) *continuous,*
- (ii) *even (i.e. $\varphi(x) = \varphi(-x)$),*
- (iii) *convex on $[0, \infty)$,*
- (iv) *and assume that $\varphi(0) = 1$ and $\lim_{x \rightarrow \infty} \varphi(x) = 0$.*

Then there exists a probability measure μ on \mathbb{R} such that $\widehat{\mu}(x) = \varphi(x)$.

The random variable we obtain in our situation is $2H$ -stable (a random variable Z is p -stable for some $0 < p \leq 2$ if $\alpha Z + \beta Z'$ and $(|\alpha|^p + |\beta|^p)^{1/p} Z$ have the same distribution if $\alpha, \beta \in \mathbb{R}$ and Z' is an independent copy of Z). Since characteristic functions are positive semi-definite, we are done.

It should be remarked that, for $0 < p < 2$, one gets p -stable random variables also as

$$Z = \sum_{j=1}^{\infty} \Gamma_j^{-\frac{1}{p}} \eta_j$$

where $(\Gamma_j)_{j=1}^{\infty}$ are the jumps of a standard Poisson process and η_1, η_2, \dots are iid copies of a symmetric p -integrable random variable.

Up to now we have constructed stochastic processes with certain finite-dimensional distributions. In the case of Gaussian processes this can be done through the covariance structure. Now we go the next step and provide the path-properties we would like to have. Here we use the fundamental

Proposition 2.3.13 (KOLMOGOROV) *Let $X = (X_t)_{t \in [0,1]}$, $X_t : \Omega \rightarrow \mathbb{R}$, be a family of random variables such that there are constants $c, \varepsilon > 0$ and $p \in [1, \infty)$ with*

$$\mathbb{E}|X_t - X_s|^p \leq c|t - s|^{1+\varepsilon}.$$

Then there is a modification Y of the process X such that

$$\mathbb{E} \sup_{s \neq t} \left(\frac{|Y_t - Y_s|}{|t - s|^\alpha} \right)^p < \infty$$

for all $0 < \alpha < \frac{\varepsilon}{p}$ and that all trajectories are continuous.

Remark 2.3.14 In particular we get from the proof of the proposition two things:

(i) The function $f : \Omega \rightarrow [0, \infty]$ given by

$$f(\omega) := \sup_{s \neq t} \frac{|Y_t(\omega) - Y_s(\omega)|}{|t - s|^\alpha}$$

is a measurable function.

(ii) The function f is almost surely finite (otherwise $\mathbb{E}|f|^p$ would be infinite), so that there is a set Ω_0 of measure one such that for all $\omega \in \Omega_0$ there is a $c(\omega) > 0$ such that

$$|Y_t(\omega) - Y_s(\omega)| \leq c(\omega)|t - s|^\alpha$$

for all $s, t \in [0, 1]$ and $\omega \in \Omega_0$. In particular, the trajectories $t \rightarrow Y_t(\omega)$ are continuous for $\omega \in \Omega_0$.

Before we prove KOLMOGOROV's theorem we consider our fundamental example, the Brownian motion.

Proposition 2.3.15 *Let $W = (W_t)_{t \geq 0}$ be a Gaussian process with mean $m(t) = 0$ and covariance $\Gamma(s, t) = \mathbb{E}W_s W_t = \min\{s, t\}$. Then there is a modification $B = (B_t)_{t \geq 0}$ of $W = (W_t)_{t \geq 0}$ such that all trajectories are continuous and*

$$\mathbb{E} \left(\sup_{0 \leq s < t \leq T} \frac{|B_t - B_s|}{|t - s|^\alpha} \right)^p < \infty$$

for all $0 < \alpha < \frac{1}{2}$, $0 < p < \infty$, and $T > 0$.

Proof. First we fix $T > 0$ and define

$$X_t := W_{tT}$$

for $t \in [0, 1]$. Then, for $p \in (0, \infty)$,

$$\begin{aligned}
\mathbb{E}|X_t - X_s|^p &= \mathbb{E}|W_{tT} - W_{sT}|^p \\
&= \mathbb{E}|W_{(t-s)T}|^p \\
&= \frac{1}{\sqrt{2\pi(t-s)T}} \int_{\mathbb{R}} |\xi|^p e^{-\frac{\xi^2}{2(t-s)T}} d\xi \\
&= ((t-s)T)^{\frac{p}{2}} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |\xi|^p e^{-\frac{\xi^2}{2}} d\xi \\
&= \gamma_p (t-s)^{\frac{p}{2}} T^{\frac{p}{2}}
\end{aligned}$$

where we used that the covariance structure implies $W_b - W_a \sim N(0, b-a)$ for $0 \leq a < b < \infty$. Now fix $\alpha \in (0, 1/2)$ and $p \in (2, \infty)$ such that

$$\frac{1}{\frac{1}{2} - \alpha} < p < \infty \quad \text{and} \quad 0 < \alpha < \frac{\varepsilon}{p} = \frac{1}{2} - \frac{1}{p}$$

and

$$\mathbb{E}|X_t - X_s|^p \leq \gamma_p T^{\frac{p}{2}} (t-s)^{1+\varepsilon}.$$

Proposition 2.3.13 implies the existence of a path-wise continuous modification $Y = Y(\alpha, p)$ of X such that

$$\mathbb{E} \sup_{0 \leq s < t \leq 1} \left(\frac{|Y_t(\alpha, p) - Y_s(\alpha, p)|}{|t-s|^\alpha} \right)^p < \infty. \quad (2.1)$$

Replacing p by $q \in (0, p)$ the same inequality remains true since $\|\cdot\|_{L_q} \leq \|\cdot\|_{L_p}$ for $0 < q < p < \infty$. Hence for all $0 < \alpha < 1/2$ and $0 < p < \infty$ we find a modification $Y(\alpha, p)$ such that (2.1) is satisfied. However, since $Y(\alpha_1, p_1)$ and $Y(\alpha_2, p_2)$ are continuous and modifications of each other, they are indistinguishable. Hence we can pick one process $Y = Y(p_0, \alpha_0)$ which satisfies (2.1) for all $0 < \alpha < 1/2$ and all $0 < p < \infty$. Coming back to our original time-scale we have found a continuous modification $(B_t^T)_{t \in [0, T]}$ of $(W_t)_{t \in [0, T]}$ such that

$$\mathbb{E} \sup_{0 \leq s < t \leq T} \left(\frac{|B_s^T - B_t^T|}{|t-s|^\alpha} \right)^p < \infty$$

for all $0 < \alpha < 1/2$ and $0 < p < \infty$. We are close to the end, we only have to remove the remaining parameter T . For this purpose we let

$$\Omega_T := \{\omega \in \Omega : B_t^T(\omega) = W_t(\omega), t \in Q \cap [0, T]\}$$

and $\tilde{\Omega} := \bigcap_{N=1}^{\infty} \Omega_N$ so that $\mathbb{P}(\tilde{\Omega}) = 1$ and

$$B_t^{N_1}(\omega) = B_t^{N_2}(\omega) \quad \text{for } t \in Q \cap [0, \min\{N_1, N_2\}]$$

and $\omega \in \tilde{\Omega}$. Since $(B_t^{N_i})_{t \in [0, N_i]}$ are continuous processes we derive that

$$B_t^{N_1}(\omega) = B_t^{N_2}(\omega) \quad \text{for } t \in [0, \min\{N_1, N_2\}]$$

whenever $\omega \in \tilde{\Omega}$. Hence we have found one process $(B_t)_{t \geq 0}$ on $\tilde{\Omega}$ and may set the process B zero on $\Omega \setminus \tilde{\Omega}$. \square

Proof of Proposition 2.3.13. (a) For $m = 1, 2, \dots$ we let

$$D_m := \left\{ 0, \frac{1}{2^m}, \dots, \frac{2^m}{2^m} \right\} \quad \text{and} \quad D := \bigcup_{m=1}^{\infty} D_m.$$

Moreover, we set

$$\Delta_m := \left\{ (s, t) \in D_m \times D_m : |s - t| = 2^{-m} \right\} \quad \text{and} \quad K_m := \sup_{(s,t) \in \Delta_m} |X_t - X_s|.$$

Then $\text{card}(\Delta_m) \leq 2 \cdot 2^m$ and

$$\begin{aligned} \mathbb{E}K_m^p &= \mathbb{E} \sup_{(s,t) \in \Delta_m} |X_t - X_s|^p \\ &\leq \sum_{(s,t) \in \Delta_m} \mathbb{E}|X_t - X_s|^p \\ &\leq \text{card}(\Delta_m) c \left(\frac{1}{2^m} \right)^{1+\varepsilon} \\ &\leq 2 \cdot 2^m c 2^{-m} 2^{-m\varepsilon} \\ &= 2c 2^{-m\varepsilon}. \end{aligned}$$

(b) Let $s, t \in D$ and

$$\begin{aligned} S_k &:= \max \{ s_k \in D_k : s_k \leq s \} \in D_k \\ T_k &:= \max \{ t_k \in D_k : t_k \leq t \} \in D_k \end{aligned}$$

so that $S_k \uparrow s$, $T_k \uparrow t$, and $S_k = s$ and $T_k = t$ for $k \geq k_0$. For $|t - s| \leq 2^{-m}$ we get that

$$X_s - X_t = \sum_{i=m}^{\infty} (X_{S_{i+1}} - X_{S_i}) + X_{S_m} + \sum_{i=m}^{\infty} (X_{T_i} - X_{T_{i+1}}) - X_{T_m}$$

where we note that the sums are finite sums, that $|T_m - S_m| \in \{0, 2^{-m}\}$, $S_{i+1} - S_i \in \{0, 2^{-(i+1)}\}$, and that $T_{i+1} - T_i \in \{0, 2^{-(i+1)}\}$. Hence

$$|X_t - X_s| \leq K_m + 2 \sum_{i=1}^m K_{i+1} \leq 2 \sum_{i=m}^{\infty} K_i.$$

(c) Let

$$M_\alpha := \sup \left\{ \frac{|X_t - X_s|}{|t - s|^\alpha} : s, t \in D, s \neq t \right\}.$$

Now we estimate M_α from above by

$$M_\alpha = \sup_{m=0,1,\dots} \sup \left\{ \frac{|X_t - X_s|}{|t - s|^\alpha} : s, t \in D, s \neq t, 2^{-m-1} \leq |t - s| \leq 2^{-m} \right\}$$

$$\begin{aligned}
&\leq \sup_{m=0,1,\dots} 2^{(m+1)\alpha} \sup \{ |X_t - X_s| : s, t \in D, s \neq t, |t - s| \leq 2^{-m} \} \\
&\leq 2 \sup_{m=0,1,\dots} 2^{(m+1)\alpha} \sum_{i=m}^{\infty} K_i \\
&\leq 2^{1+\alpha} \sum_{i=0}^{\infty} 2^{\alpha i} K_i,
\end{aligned}$$

where we used step (b), and

$$\begin{aligned}
\|M_\alpha\|_{L_p} &\leq 2^{1+\alpha} \sum_{i=0}^{\infty} 2^{\alpha i} \|K_i\|_{L_p} \\
&\leq 2^{1+\alpha} \sum_{i=0}^{\infty} 2^{\alpha i} (2c)^{\frac{1}{p}} 2^{-\frac{i\varepsilon}{p}} \\
&= 2^{1+\alpha} (2c)^{\frac{1}{p}} \sum_{i=0}^{\infty} 2^{i(\alpha - \frac{\varepsilon}{p})} \\
&< \infty
\end{aligned}$$

where we used step (a).

(d) Hence there is a set $\Omega_0 \subseteq \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that $t \rightarrow X_t(\omega)$ is uniformly continuous on D for $\omega \in \Omega_0$. We define

$$Y_t(\omega) := \begin{cases} X_t(\omega) & : \omega \in \Omega_0, t \in D \\ \lim_{s \uparrow t, s \in D} X_s(\omega) & : \omega \in \Omega_0, t \notin D \\ 0 & : \omega \notin \Omega_0 \end{cases}.$$

It remains to show that $\mathbb{P}(X_t = Y_t) = 1$. Because of our assumption we have that

$$\|X_{t_n} - X_t\|_{L_p} \rightarrow 0 \quad \text{as } t_n \uparrow t.$$

Take $t_n \in D$ and find a subsequence $(n_k)_{k=1}^{\infty}$ such that

$$\mathbb{P}(\lim_k X_{t_{n_k}} = X_t) = 1.$$

Since $\mathbb{P}(\lim_k X_{t_{n_k}} = Y_t) = 1$ by construction, we are done. \square

2.4 Brownian motion

In this section we prove the existence of the Brownian motion introduced already earlier. To this end we start with a setting which does not use a filtration. This leads, in our first step, to a Brownian motion formally a bit different than presented in Definition 2.2.2.

Definition 2.4.1 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $B = (B_t)_{t \geq 0}$ be a stochastic process. The process B is called *standard Brownian motion* provided that

- (i) $B_0 \equiv 0$,
- (ii) for all $0 \leq s < t < \infty$ the random variable $B_t - B_s$ is independent from $(B_u)_{u \in [0, s]}$, which means that for all $0 \leq s_1 \leq \dots \leq s_n \leq s$ and $A, A_1, \dots, A_n \in \mathcal{B}(\mathbb{R})$ one has

$$\begin{aligned} \mathbb{P}(B_{s_1} \in A_1, \dots, B_{s_n} \in A_n, B_t - B_s \in A) = \\ \mathbb{P}(B_{s_1} \in A_1, \dots, B_{s_n} \in A_n) \mathbb{P}(B_t - B_s \in A), \end{aligned}$$

- (iii) for all $0 \leq s < t < \infty$ and for all $A \in \mathcal{B}(\mathbb{R})$ one has

$$\mathbb{P}(B_t - B_s \in A) = \frac{1}{\sqrt{2\pi(t-s)}} \int_A e^{-\frac{x^2}{2(t-s)}} dx,$$

- (iv) all paths $t \rightarrow B_t(\omega)$ are continuous.

Proposition 2.4.2 *The standard Brownian motion exists.*

Proof. We take the process $B = (B_t)_{t \geq 0}$ from Proposition 2.3.15.

(i) Since $\mathbb{E}B_0B_0 = 0$ so that $B_0 = 0$ a.s. we can set the whole process B on the null-set $\{B_0 \neq 0\}$ to zero and the conclusion of Proposition 2.3.15 is still satisfied.

(iv) follows directly from Proposition 2.3.15.

(iii) follows from $\mathbb{E}(B_t - B_s) = 0$,

$$\mathbb{E}(B_t - B_s)^2 = t - 2 \min\{t, s\} + s = t - s,$$

and the fact that $B_t - B_s$ is a Gaussian random variable.

(ii) The random variables $B_t - B_s, B_s - B_{s_n}, \dots, B_{s_2} - B_{s_1}, B_{s_1}$ are independent since they are Gaussian random variables and any two of them are uncorrelated. Consequently

$$\begin{aligned} \mathbb{P}(B_{s_1} \in A_1, \dots, B_{s_n} \in A_n, B_t - B_s \in A) \\ = \mathbb{P}((B_{s_1}, \dots, B_{s_n}) \in A_1 \times \dots \times A_n, B_t - B_s \in A) \\ = \mathbb{P}((B_{s_1}, B_{s_2} - B_{s_1}, \dots, B_{s_n} - B_{s_{n-1}}) \in C, B_t - B_s \in A) \\ = \mathbb{P}((B_{s_1}, B_{s_2} - B_{s_1}, \dots, B_{s_n} - B_{s_{n-1}}) \in C) \mathbb{P}(B_t - B_s \in A) \\ = \mathbb{P}(B_{s_1} \in A_1, \dots, B_{s_n} \in A_n) \mathbb{P}(B_t - B_s \in A) \end{aligned}$$

where

$$C := \{(y_1, \dots, y_n) \in \mathbb{R}^n : y_1 \in A_1, y_1 + y_2 \in A_2, \dots, y_1 + \dots + y_n \in A_n\}.$$

□

Proposition 2.4.3 *Let $X = (X_t)_{t \geq 0}$ be a stochastic process such that all trajectories are continuous and such that $X_0 \equiv 0$. Then the following assertions are equivalent:*

- (i) *The process X is a standard Brownian motion.*
- (ii) *The process X is a Gaussian process with mean $m(t) \equiv 0$ and covariance $\Gamma(s, t) = \min\{s, t\}$.*

Proof. (ii) \Rightarrow (i) From Proposition 2.3.15 (see Proposition 2.4.2) we get that there is a modification \tilde{B} of B which is a standard Brownian motion. Since both processes are continuous they are indistinguishable. Hence B is a standard Brownian motion as well.

(i) \Rightarrow (ii) The mean is zero by definition and the covariance is obtained by, for $0 \leq s < t < \infty$,

$$\mathbb{E}B_s B_t = \mathbb{E}B_s(B_t - B_s) + \mathbb{E}B_s^2 = \mathbb{E}B_s \mathbb{E}(B_t - B_s) + s = s.$$

It remains to show that B is a Gaussian process. To this end we have to check whether for all $n = 1, 2, \dots$ and $0 \leq t_1 < t_2 < \dots < t_n < \infty$

$$(B_{t_1}, \dots, B_{t_n}) : \Omega \rightarrow \mathbb{R}^n$$

is a Gaussian random vector. We know that $B_{t_n} - B_{t_{n-1}}, \dots, B_{t_2} - B_{t_1}, B_{t_1}$ are independent Gaussian random variables so that $(B_{t_n} - B_{t_{n-1}}, \dots, B_{t_2} - B_{t_1}, B_{t_1})$ is a Gaussian random vector. Using the linear transformation $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by $A(\xi_1, \dots, \xi_n) := (\xi_1 + \dots + \xi_n, \xi_1 + \dots + \xi_{n-1}, \dots, \xi_1)$ we obtain

$$A((B_{t_n} - B_{t_{n-1}}, \dots, B_{t_2} - B_{t_1}, B_{t_1})) = (B_{t_n}, B_{t_{n-1}}, \dots, B_{t_1})$$

and are done since the linear image of a Gaussian random vector is a Gaussian random vector. \square

Proposition 2.4.4 *The trajectories of the standard Brownian motion are Hölder continuous with exponent $\alpha \in (0, 1/2)$, i.e. the set*

$$A_{\alpha, T} := \left\{ \omega \in \Omega : \sup_{0 \leq s < t \leq T} \frac{|B_t(\omega) - B_s(\omega)|}{|t - s|^\alpha} < \infty \right\}$$

is measurable and of measure one for all $\alpha \in (0, 1/2)$ and $T > 0$.

Proof. Since the Brownian motion is continuous we get that $A_{\alpha, T} \in \mathcal{F}$. Moreover, $\mathbb{E}B_s B_t = \min\{s, t\}$ implies by Proposition 2.3.15 that there is a continuous modification $\tilde{B} = (\tilde{B}_t)_{t \geq 0}$ of B such that the corresponding set $\tilde{A}_{\alpha, T}$ has measure one. However, B and \tilde{B} are indistinguishable, so that $\mathbb{P}(A_{\alpha, T}) = 1$ as well. \square

For our later purpose we need a slight modification of the definition of the Brownian motion. We include a filtration as follows:

Definition 2.4.5 Let $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_t)_{t \geq 0})$ be a stochastic basis. An *adapted* stochastic process $B = (B_t)_{t \geq 0}$, $B_t : \Omega \rightarrow \mathbb{R}$, is called (standard) $(\mathcal{F}_t)_{t \geq 0}$ -Brownian motion provided that

- (i) $B_0 \equiv 0$,
- (ii) for all $0 \leq s < t < \infty$ the random variable $B_t - B_s$ is independent from \mathcal{F}_s that means that

$$\mathbb{P}(C \cap \{B_t - B_s \in A\}) = \mathbb{P}(C)\mathbb{P}(B_t - B_s \in A)$$

for $C \in \mathcal{F}_s$ and $A \in \mathcal{B}(\mathbb{R})$,

- (iii) for all $0 \leq s < t < \infty$ one has $B_t - B_s \sim N(0, t - s)$,
- (iv) for all $\omega \in \Omega$ the trajectories $t \rightarrow B_t(\omega)$ are continuous.

Now we link this form of definition to the previous one.

Proposition 2.4.6 Let $B = (B_t)_{t \geq 0}$ be a standard Brownian motion in the sense of Definition 2.4.1 and let $(\mathcal{F}_t^B)_{t \geq 0}$ be its natural filtration, i.e. $\mathcal{F}_t^B := \sigma(B_s : s \in [0, t])$. Then B is an $(\mathcal{F}_t^B)_{t \geq 0}$ -Brownian motion.

Proof. Comparing Definitions 2.4.1 and 2.4.5 we only need to check that Definition 2.4.1(ii) implies Definition 2.4.5(ii) which is left as an exercise. \square

For technical reason we have to go one step further: we have to augment the natural filtration. First we recall the completion of a probability space.

Lemma 2.4.7 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and

$$\mathcal{N} := \{A \subseteq \Omega : \text{there exists a } B \in \mathcal{F} \text{ with } A \subseteq B \text{ and } \mathbb{P}(B) = 0\} \cup \{\emptyset\}.$$

- (i) Let \mathcal{G} be a sub- σ -algebra of \mathcal{F} . Then $B \in \mathcal{G} \vee \mathcal{N}$ if and only if there is a $A \in \mathcal{G}$ such that $A \Delta B \in \mathcal{N}$.
- (ii) The measure \mathbb{P} can be extended to a measure $\tilde{\mathbb{P}}$ on $\tilde{\mathcal{F}} := \mathcal{F} \vee \mathcal{N}$ by $\tilde{\mathbb{P}}(B) := \mathbb{P}(A)$ for $A \in \mathcal{F}$ such that $A \Delta B \in \mathcal{N}$.

Definition 2.4.8 The probability space $(\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ is called *completion* of $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 2.4.9 Let $X = (X_t)_{t \geq 0}$, $X_t : \Omega \rightarrow \mathbb{R}$, be a stochastic process,

$$\mathcal{F}_\infty^X := \sigma(X_s : s \geq 0), \quad \text{and} \quad \mathcal{F}_t^X := \sigma(X_s : s \in [0, t])$$

for $t \geq 0$. Define

$$\mathcal{N} := \{A \subseteq \Omega : \text{there exists a } B \in \mathcal{F}_\infty^X \text{ with } A \subseteq B \text{ and } \mathbb{P}(B) = 0\} \cup \{\emptyset\}.$$

Then $(\mathcal{F}_t)_{t \geq 0}$ with $\mathcal{F}_t := \mathcal{F}_t^X \vee \mathcal{N}$ is called *augmentation* of $(\mathcal{F}_t^X)_{t \geq 0}$.

Proposition 2.4.10 *Let $B = (B_t)_{t \geq 0}$ be a standard Brownian motion, $(\mathcal{F}_t^B)_{t \geq 0}$ be its natural filtration. and $(\mathcal{F}_t)_{t \geq 0}$ be the augmentation of $(\mathcal{F}_t^B)_{t \geq 0}$. Then*

- (i) *the process B is an $(\mathcal{F}_t)_{t \geq 0}$ -Brownian motion,*
- (ii) *the filtration $(\mathcal{F}_t)_{t \geq 0}$ is right-continuous that means that*

$$\mathcal{F}_t = \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}.$$

Proof. (i) We only have to check that $B_t - B_s$ is independent of \mathcal{F}_s for $0 \leq s < t < \infty$. Assume $C \in \mathcal{F}_s$ and find an $\tilde{C} \in \mathcal{F}_s^B$ such that $\mathbb{P}(C \Delta \tilde{C}) = 0$ where we denote the extension $\tilde{\mathbb{P}}$ of \mathbb{P} again by \mathbb{P} . Taking $A \in \mathcal{B}(\mathbb{R})$ we get that

$$\begin{aligned} \mathbb{P}(\{B_t - B_s \in A\} \cap C) &= \mathbb{P}(\{B_t - B_s \in A\} \cap \tilde{C}) \\ &= \mathbb{P}(B_t - B_s \in A) \mathbb{P}(\tilde{C}) = \mathbb{P}(B_t - B_s \in A) \mathbb{P}(C). \end{aligned}$$

- (ii) The right-hand side continuity of the filtration $(\mathcal{F}_t)_{t \geq 0}$ is not proved here. \square

Definition 2.4.11 The stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_t)_{t \geq 0})$ satisfies the *usual conditions* provided that

- (i) $(\Omega, \mathcal{F}, \mathbb{P})$ is complete,
- (ii) $A \in \mathcal{F}_t$ for all $A \in \mathcal{F}$ with $\mathbb{P}(A) = 0$ and $t \geq 0$,
- (iii) the filtration $(\mathcal{F}_t)_{t \geq 0}$ is right-continuous that means that

$$\mathcal{F}_t = \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}.$$

2.5 Stopping and optional times

Stopping times and optional times are random times that form an important tool in the theory of stochastic processes. To demonstrate their usefulness let us briefly consider the *Reflection Principle* for a standard Brownian motion $B = (B_t)_{t \geq 0}$. Given $b > 0$, we are interested in the distribution of

$$\tau_b := \inf\{t \geq 0 : B_t = b\}.$$

First we write

$$\mathbb{P}(\tau_b < t) = \mathbb{P}(\tau_b < t, B_t > b) + \mathbb{P}(\tau_b < t, B_t < b).$$

Then our heuristic *Reflection Principle* says that

$$\mathbb{P}(\tau_b < t, B_t < b) = \mathbb{P}(\tau_b < t, B_t > b). \quad (2.2)$$

On the other hand

$$\mathbb{P}(\tau_b < t, B_t > b) = \mathbb{P}(B_t > b)$$

which implies

$$\begin{aligned} \mathbb{P}(\tau_b < t) &= \mathbb{P}(\tau_b < t, B_t > b) + \mathbb{P}(\tau_b < t, B_t < b) \\ &= 2\mathbb{P}(\tau_b < t, B_t > b) \\ &= 2\mathbb{P}(B_t > b). \end{aligned}$$

From this one can deduce at least two things:

- The Brownian motion reaches with probability one any level because

$$\begin{aligned} \mathbb{P}(\tau_b < \infty) &= \lim_{t \rightarrow \infty} \mathbb{P}(\tau_b < t) = 2 \lim_{t \rightarrow \infty} \mathbb{P}(B_t > b) \\ &= 2 \lim_{t \rightarrow \infty} \mathbb{P}\left(B_1 > \frac{b}{\sqrt{t}}\right) = 1. \end{aligned}$$

- One can deduce the distribution of the running maximum of the Brownian motion $M_t(\omega) := \sup_{s \in [0, t]} B_s(\omega)$ because

$$\{M_t \geq b\} = \{\tau_b \leq t\} \quad \text{so that} \quad \mathbb{P}(M_t \geq b) = 2\mathbb{P}(B_t > b).$$

To justify (2.2) would require a considerable amount of work. Here we only introduce the concepts around the random time $\tau_b : \Omega \rightarrow [0, \infty]$ like stopping times and optional times, and their relations to each other.

Definition 2.5.1 Assume a measurable space (Ω, \mathcal{F}) equipped with a filtration $(\mathcal{F}_t)_{t \geq 0}$.

- (i) The map $\tau : \Omega \rightarrow [0, \infty]$ is called *stopping time* with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$ provided that

$$\{\tau \leq t\} \in \mathcal{F}_t$$

for all $t \geq 0$. Moreover,

$$\mathcal{F}_\tau := \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}.$$

- (ii) The map $\tau : \Omega \rightarrow [0, \infty]$ is called *optional time* with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$ provided that

$$\{\tau < t\} \in \mathcal{F}_t$$

for all $t \geq 0$. Moreover,

$$\mathcal{F}_{\tau+} := \{A \in \mathcal{F} : A \cap \{\tau < t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}.$$

One might say that \mathcal{F}_τ contains those events that can be decided until time τ and that $\mathcal{F}_{\tau+}$ contains those events that can be decided right after the random time τ occurs. Let us give a simple example of an optional time that is not a stopping time:

Example 2.5.2 Let $\Omega := \{1, 2, 3\}$, $\mathcal{F} = 2^\Omega$, $\mathbb{P}(\{k\}) = 1/3$, and $X_t(\omega) := (t - \omega)^+$ for $t \geq 0$. Then the natural filtration computes as

$$\mathcal{F}_t^X = \begin{cases} \{\emptyset, \Omega\} & : t \in [0, 1] \\ \{\emptyset, \Omega, \{1\}, \{2, 3\}\} & : t \in (1, 2] \\ 2^\Omega & : t \geq 2 \end{cases} .$$

Let $\tau(\omega) := \inf \{t > 0 : X_t(\omega) > 0\}$ so that $\tau(\omega) = \omega$. The time τ is not a stopping time since

$$\{\tau \leq 1\} = \{1\} \notin \mathcal{F}_1.$$

But it is an optional time since $\{\tau < t\} \in \mathcal{F}_t$ for all $t \geq 0$.

Proposition 2.5.3 *Let $\tau : \Omega \rightarrow [0, \infty]$ be a stopping time. Then*

- (i) *the system of sets \mathcal{F}_τ is a σ -algebra,*
- (ii) *one has $\{\tau \leq s\} \in \mathcal{F}_\tau$ for $s \geq 0$ so that τ is an extended \mathcal{F}_τ -measurable random variable.*

Proof. (i) Since $\emptyset \cap \{\tau \leq t\} = \emptyset \in \mathcal{F}_t$ we have $\emptyset \in \mathcal{F}_\tau$. Assume that $B_1, B_2, \dots \in \mathcal{F}_\tau$. Then

$$\left(\bigcup_{n=1}^{\infty} B_n \right) \cap \{\tau \leq t\} = \bigcup_{n=1}^{\infty} (B_n \cap \{\tau \leq t\}) \in \mathcal{F}_t.$$

Finally, for $B \in \mathcal{F}_\tau$ we get that

$$B^c \cap \{\tau \leq t\} = \{\tau \leq t\} \setminus (B \cap \{\tau \leq t\}) \in \mathcal{F}_t.$$

(ii) For $s, t \in [0, \infty)$ we get that

$$\{\tau \leq s\} \cap \{\tau \leq t\} = \{\tau \leq \min\{s, t\}\} \in \mathcal{F}_{\min\{s, t\}} \subseteq \mathcal{F}_t.$$

We conclude the proof by remarking that the system $\{x \in \mathbb{R} : x \leq t\}, t \in \mathbb{R}$, generates the Borel σ -algebra and that, trivially, $\{\tau \leq t\} = \emptyset \in \mathcal{F}_\tau$ for $t < 0$. \square

The pendant for the optional times reads as

Proposition 2.5.4 *Let $\tau : \Omega \rightarrow [0, \infty]$ be an optional time. Then*

- (i) *the system of sets $\mathcal{F}_{\tau+}$ is a σ -algebra,*

- (ii) one has $\{\tau < s\} \in \mathcal{F}_{\tau+}$ for all $s \geq 0$ so that τ is an extended $\mathcal{F}_{\tau+}$ -measurable random variable.

The proof is an exercise. Optional times are stopping times with respect to the filtration $(\mathcal{F}_t^+)_{t \geq 0}$:

Proposition 2.5.5 *Let (Ω, \mathcal{F}) be a measurable space equipped with a filtration $(\mathcal{F}_t)_{t \geq 0}$ and $\tau : \Omega \rightarrow [0, \infty]$.*

- (i) *Then τ is an optional time if and only if, for all $t \geq 0$,*

$$\{\tau \leq t\} \in \mathcal{F}_{t+} := \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}.$$

- (ii) *If τ is an optional time, then $A \in \mathcal{F}_{\tau+}$ if and only if $A \cap \{\tau \leq t\} \in \mathcal{F}_{t+}$ for $t \geq 0$.*

The proof is an exercise. The general relation between stopping and optional times is as follows:

Proposition 2.5.6 (i) *Every stopping time τ is an optional time. In this case, $\mathcal{F}_{\tau} \subseteq \mathcal{F}_{\tau+}$.*

- (ii) *If the filtration is right-continuous, then any optional time τ is a stopping time. In this case $\mathcal{F}_{\tau} = \mathcal{F}_{\tau+}$.*

Proof. (i) This follows from

$$\{\tau < t\} = \bigcup_{n=1}^{\infty} \left\{ \tau \leq t - \frac{1}{n} \right\}$$

and

$$\left\{ \tau \leq t - \frac{1}{n} \right\} \in \mathcal{F}_{\max\{0, t - \frac{1}{n}\}} \subseteq \mathcal{F}_t.$$

Hence τ is an optional time. Let $A \in \mathcal{F}_{\tau}$ so that by definition $A \cap \{\tau \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$. Hence

$$A \cap \{\tau < t\} = \bigcup_{n=1}^{\infty} A \cap \left\{ \tau \leq t - \frac{1}{n} \right\} \in \mathcal{F}_t$$

because

$$A \cap \left\{ \tau \leq t - \frac{1}{n} \right\} \in \mathcal{F}_{\max\{0, t - \frac{1}{n}\}} \subseteq \mathcal{F}_t.$$

- (ii) The second assertion follows from Proposition 2.5.5 since $\mathcal{F}_t = \mathcal{F}_{t+}$. \square

Now we consider two basic examples.

Example 2.5.7 Let $X = (X_t)_{t \geq 0}$ be continuous and adapted, $\Gamma \subseteq \mathbb{R}$ be non-empty, and define the hitting time

$$\tau_\Gamma := \inf\{t \geq 0 : X_t \in \Gamma\}$$

with the convention that $\inf \emptyset := \infty$.

- (i) If Γ is open, then τ_Γ is an optional time.
- (ii) If Γ is closed, then τ_Γ is a stopping time.

Proof. We only show (ii), part (i) is an exercise. Given $t \geq 0$ we have to show that $\{\tau_\Gamma \leq t\} \in \mathcal{F}_t$. The condition $\tau_\Gamma(\omega) \leq t$ implies the existence of a sequence $t_1 \geq t_2 \geq t_3 \geq \dots \geq 0$ such that $X_{t_n}(\omega) \in \Gamma$ and $s := \lim_n t_n \leq t$. By the continuity of X we have $X_s(\omega) = \lim_n X_{t_n}(\omega)$ and by the closeness of Γ that $X_s(\omega) \in \Gamma$. Hence

$$\begin{aligned} \{\tau_\Gamma \leq t\} &= \{\omega \in \Omega : \text{there exists an } s \in [0, t] \text{ such that } X_s(\omega) \in \Gamma\} \\ &= \{\omega \in \Omega : \inf_{s \in \mathbb{Q}, s \leq t} d(X_s(\omega), \Gamma) = 0\} \in \mathcal{F}_t \end{aligned}$$

where $d(x, \Gamma) := \inf\{|x - y| : y \in \Gamma\}$ is a continuous function since

$$|d(x, \Gamma) - d(y, \Gamma)| \leq |x - y|.$$

□

2.6 A short excursion to Markov processes

To give a rigorous justification of the reflection principle one would need to introduce the strong Markov property. In this course we restrict ourselves to the introduction of the basic concept. Because of the continuity properties of the filtration (right-hand side continuous or not) we first work with the optional times and not with the, probably more intuitive, stopping times.

Definition 2.6.1 Let $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_t)_{t \geq 0})$ be a stochastic basis and $X = (X_t)_{t \geq 0}$, $X_t : \Omega \rightarrow \mathbb{R}$, be a stochastic process.

- (i) The process X is a *Markov process* provided that X is adapted and for all $s, t \geq 0$ and $B \in \mathcal{B}(\mathbb{R})$ one has that

$$\mathbb{P}(X_{s+t} \in B | \mathcal{F}_s) = \mathbb{P}(X_{s+t} \in B | \sigma(X_s)) \text{ a.s.}$$

- (ii) The process X is a *strong Markov process* provided that X is progressively measurable and for all $t \geq 0$, optional times $\tau : \Omega \rightarrow [0, \infty]$, and $B \in \mathcal{B}(\mathbb{R})$ one has that, a.s.,

$$\mathbb{P}(\{X_{\tau+t} \in B\} \cap \{\tau < \infty\} | \mathcal{F}_{\tau+}) = \mathbb{P}(\{X_{\tau+t} \in B\} \cap \{\tau < \infty\} | \sigma(X_\tau)),$$

where $\sigma(X_\tau) := \sigma(\tau^{-1}(\infty), \{X_\tau^{-1}(B) \cap \{\tau < \infty\} : B \in \mathcal{B}(\mathbb{R})\})$.

To make the definition rigorous we need

Lemma 2.6.2 *Let X be progressively measurable and $\tau : \Omega \rightarrow [0, \infty]$ be an optional time, then*

$$X_\tau^{-1}(B) \cap \{\tau < \infty\} \in \mathcal{F}_{\tau+}$$

for all $B \in \mathcal{B}(\mathbb{R})$.

We give an easy example for a Markov process which is not a strong Markov process.

Example 2.6.3 We consider $\Omega := \{-1, 1\}$, $\mathcal{F} := 2^\Omega$, and $\mathbb{P}(\{-1\}) = \mathbb{P}(\{1\}) = 1/2$. As stochastic process we take $X_t(1) := \max\{t-1, 0\}$ and $X_t(-1) := \min\{-t+1, 0\}$. The filtration is the natural filtration $\mathcal{F}_t := \sigma(X_s : s \in [0, t])$. Taking the optional time $\tau \equiv 1$, one checks that $\sigma(X_\tau) = \{\emptyset, \Omega\}$ and $\mathcal{F}_{\tau+} = 2^\Omega$. But this gives that

$$\mathbb{P}(X_2 \in B | \mathcal{F}_{\tau+}) = \mathbb{P}(X_2 \in B | \sigma(X_\tau)) \text{ a.s.}$$

cannot be true for all $B \in \mathcal{B}(\mathbb{R})$.

For us, the main (positive) example is

Proposition 2.6.4 *Assume a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_t)_{t \geq 0})$ and an $(\mathcal{F}_t)_{t \geq 0}$ -Brownian motion $B = (B_t)_{t \geq 0}$ like in Definition 2.4.5. Then B is a strong Markov process.*

Finally, without proof we state

Proposition 2.6.5 *Assume a process $X = (X_t)_{t \geq 0}$ which is a strong Markov process with respect to its natural filtration $(\mathcal{F}_t^X)_{t \geq 0}$ with $\mathcal{F}_t^X := \sigma(X_u : u \in [0, t])$. Then the augmented filtration is right continuous.*

Chapter 3

Stochastic integration

Given a Brownian motion $B = (B_t)_{t \geq 0}$, we would like to define

$$\int_0^T L_t dB_t$$

for a large class of stochastic processes $L = (L_t)_{t \geq 0}$. A first approach would be to write

$$\int_0^T L_t dB_t = \int_0^T L_t \frac{dB_t}{dt} dt.$$

However this is not possible (at least in this naive form) because of the following

Proposition 3.0.1 (PALEY, WIENER-ZYGMUND) *Given a standard Brownian motion in the sense of Definition 2.4.1, then the set*

$$\{\omega \in \Omega : t \rightarrow B_t(\omega) \text{ is nowhere differentiable}\}$$

contains a set of measure one.

So we have to proceed differently. We will first define the stochastic integral for simple processes and extend then the definition to an appropriate class of processes. Throughout the whole section we assume that

- the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is complete,
- the filtration $(\mathcal{F}_t)_{t \geq 0}$ is right-continuous that means $\bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon} = \mathcal{F}_t$ for all $t \geq 0$,
- all null-sets of \mathcal{F} are contained in \mathcal{F}_0 ,
- the process $B = (B_t)_{t \geq 0}$ is an $(\mathcal{F}_t)_{t \geq 0}$ -Brownian motion.

3.1 Definition of the stochastic integral

First we need simple processes we are able to integrate:

Definition 3.1.1 A stochastic process $L = (L_t)_{t \geq 0}$ is called *simple* provided that there exists

- (i) a sequence $0 = t_0 < t_1 < t_2 < \dots$ with $\lim_n t_n = \infty$,
- (ii) \mathcal{F}_{t_i} -measurable random variables $v_i : \Omega \rightarrow \mathbb{R}$, $i = 0, 1, \dots$, with $\sup_{i, \omega} |v_i(\omega)| < \infty$,

such that

$$L_t(\omega) = \sum_{i=1}^{\infty} \chi_{(t_{i-1}, t_i]}(t) v_{i-1}(\omega).$$

The class of these processes is denoted by \mathcal{L}_0 .

Now we can define our first stochastic integral:

Definition 3.1.2 (Stochastic integrals for \mathcal{L}_0 -integrands) For $L \in \mathcal{L}_0$ and $t \geq 0$ we let

$$I_t(L)(\omega) := \sum_{i=1}^{\infty} v_{i-1}(\omega) (B_{t_i \wedge t}(\omega) - B_{t_{i-1} \wedge t}(\omega)).$$

For the investigation of the stochastic integral and its extension we need the following class of martingales:

Definition 3.1.3 A martingale $M = (M_t)_{t \geq 0}$ belongs to \mathcal{M}_2^c provided that

- (i) $\mathbb{E}|M_t|^2 < \infty$ for all $t \geq 0$,
- (ii) the trajectories $t \rightarrow M_t(\omega)$ are continuous for all $\omega \in \Omega$.

In case that $M_0 \equiv 0$ we write $M \in \mathcal{M}_2^{c,0}$.

Example 3.1.4 For the Brownian motion one has $B = (B_t)_{t \geq 0} \in \mathcal{M}_2^{c,0}$.

Example 3.1.5 For the geometric Brownian motion $S = (S_t)_{t \geq 0}$ defined by

$$S_t := e^{B_t - \frac{t}{2}},$$

one has $S = (S_t)_{t \geq 0} \in \mathcal{M}_2^c$.

Proof. For all $0 < p < \infty$ one has that $\mathbb{E}S_t^p < \infty$ since, for $t > 0$,

$$\begin{aligned}\mathbb{E}S_t^p &= \int_{\mathbb{R}} e^{px} e^{-\frac{x^2}{2t}} \frac{dx}{\sqrt{2\pi t}} e^{-p\frac{t}{2}} \\ &= \int_{\mathbb{R}} e^{-\frac{(x-tp)^2}{2t}} \frac{dx}{\sqrt{2\pi t}} e^{\frac{tp^2}{2} - p\frac{t}{2}} \\ &= e^{\frac{tp^2}{2} - p\frac{t}{2}}.\end{aligned}$$

If B_t is \mathcal{F}_t -measurable, then $e^{B_t - \frac{t}{2}}$ is as a deterministic functional of B_t measurable with respect to \mathcal{F}_t as well. Finally letting $0 \leq s < t < \infty$, we conclude by (a.s.)

$$\begin{aligned}\mathbb{E}(S|\mathcal{F}_t) &= \mathbb{E}(e^{B_t - \frac{t}{2}}|\mathcal{F}_s) \\ &= \mathbb{E}(e^{(B_t - B_s) - \frac{t-s}{2}} e^{B_s - \frac{s}{2}}|\mathcal{F}_s) \\ &= S_s \mathbb{E}(e^{(B_t - B_s) - \frac{t-s}{2}}|\mathcal{F}_s) \\ &= S_s \mathbb{E}e^{(B_t - B_s) - \frac{t-s}{2}} \\ &= S_s.\end{aligned}$$

□

Proposition 3.1.6 For $L \in \mathcal{L}_0$ one has that $(I_t(L))_{t \geq 0} \in \mathcal{M}_2^{c,0}$.

Proof. By definition we have that $I_0(L) \equiv 0$ and that the process $t \rightarrow I_t(L)(\omega)$ is continuous for all $\omega \in \Omega$. Since

$$v_{i-1}(B_{t_i \wedge t} - B_{t_{i-1} \wedge t}) = \begin{cases} 0 & : t \leq t_{i-1} \\ v_{i-1}(B_{t_i \wedge t} - B_{t_{i-1}}) & : t > t_{i-1} \end{cases}$$

we get that $I_t(L)$ is \mathcal{F}_t -measurable. Now we observe that

$$\mathbb{E}|v_{i-1}(B_b - B_{t_{i-1}})|^2 \leq c^2 \mathbb{E}|B_b - B_{t_{i-1}}|^2 = c^2(b - t_{i-1})$$

for $t_{i-1} \leq b < \infty$ so that

$$\begin{aligned}(\mathbb{E}|I_t(L)|^2)^{\frac{1}{2}} &\leq \left(\mathbb{E} \left| \sum_{i=1}^{n_0} v_{i-1}(B_{t_i \wedge t} - B_{t_{i-1} \wedge t}) \right|^2 \right)^{\frac{1}{2}} \\ &\leq \sum_{i=1}^{n_0} (\mathbb{E}|v_{i-1}(B_{t_i \wedge t} - B_{t_{i-1} \wedge t})|^2)^{\frac{1}{2}} < \infty\end{aligned}$$

whenever $t_{n_0-1} < t \leq t_{n_0}$. It remains to show the martingale property

$$\mathbb{E}(I_t(L)|\mathcal{F}_s) = I_s(L) \text{ a.s.}$$

for $0 \leq s \leq t < \infty$. We only check $0 < s \leq t < \infty$ and find n_0 and m_0 such that $t_{n_0-1} < t \leq t_{n_0}$ and $t_{m_0-1} < s \leq t_{m_0}$. Then, a.s.,

$$\begin{aligned} \mathbb{E}(I_t(L)|\mathcal{F}_s) &= \mathbb{E}\left(\sum_{i=1}^{n_0} v_{i-1}(B_{t_i \wedge t} - B_{t_{i-1} \wedge t})|\mathcal{F}_s\right) \\ &= \sum_{i=1}^{n_0} \mathbb{E}(v_{i-1}(B_{t_i \wedge t} - B_{t_{i-1} \wedge t})|\mathcal{F}_s). \end{aligned}$$

For $1 \leq i \leq m_0 - 1$ we get

$$v_{i-1}(B_{t_i \wedge t} - B_{t_{i-1} \wedge t}) = v_{i-1}(B_{t_i} - B_{t_{i-1}}) = v_{i-1}(B_{t_i \wedge s} - B_{t_{i-1} \wedge s})$$

which is \mathcal{F}_s -measurable. In the case $n_0 \geq i \geq m_0 + 1$ we may deduce, a.s., that

$$\begin{aligned} \mathbb{E}(v_{i-1}(B_{t_i \wedge t} - B_{t_{i-1} \wedge t})|\mathcal{F}_s) &= \mathbb{E}(v_{i-1}(B_{t_i \wedge t} - B_{t_{i-1}})|\mathcal{F}_s) \\ &= \mathbb{E}(\mathbb{E}(v_{i-1}(B_{t_i \wedge t} - B_{t_{i-1}})|\mathcal{F}_{t_{i-1}})|\mathcal{F}_s) \\ &= \mathbb{E}(v_{i-1}\mathbb{E}(B_{t_i \wedge t} - B_{t_{i-1}}|\mathcal{F}_{t_{i-1}})|\mathcal{F}_s) \\ &= \mathbb{E}(v_{i-1}(B_{t_i \wedge t \wedge t_{i-1}} - B_{t_{i-1}})|\mathcal{F}_s) \\ &= 0. \end{aligned}$$

Finally, for $i = m_0$ one obtains, a.s., that

$$\begin{aligned} \mathbb{E}(v_{i-1}(B_{t_i \wedge t} - B_{t_{i-1} \wedge t})|\mathcal{F}_s) &= v_{i-1}\mathbb{E}(B_{t_i \wedge t} - B_{t_{i-1} \wedge t}|\mathcal{F}_s) \\ &= v_{i-1}(B_{t_i \wedge t \wedge s} - B_{t_{i-1}}) \\ &= v_{i-1}(B_{t_i \wedge s} - B_{t_{i-1} \wedge s}). \end{aligned}$$

□

Now we turn to the Itô-isometry for stochastic integrals over simple integrands. Before we do this, we need the following

Lemma 3.1.7 *Let $M = (M_k)_{k=0}^N$ be a martingale with respect to $(\mathcal{G}_k)_{k=0}^N$ such that $\mathbb{E}M_k^2 < \infty$ for $k = 0, \dots, N$. Then, a.s.,*

$$\mathbb{E}((M_N - M_n)^2|\mathcal{G}_n) = \mathbb{E}\left(\sum_{l=n+1}^N (dM_l)^2|\mathcal{G}_n\right)$$

for $0 \leq n < N$ where $dM_l := M_l - M_{l-1}$.

Proof. We have that

$$\mathbb{E}(M_N - M_n)^2|\mathcal{G}_n = \sum_{i,j=n+1}^N \mathbb{E}(dM_i dM_j|\mathcal{G}_n).$$

We are done if we can show that

$$\mathbb{E}(dM_i dM_j | \mathcal{F}_n) = 0 \text{ a.s.}$$

for $k < i < j \leq N$. But this follows from, a.s.,

$$\mathbb{E}(dM_i dM_j | \mathcal{G}_n) = \mathbb{E}(\mathbb{E}(dM_i dM_j | \mathcal{G}_i) | \mathcal{G}_n) = \mathbb{E}(dM_i \mathbb{E}(dM_j | \mathcal{G}_i) | \mathcal{G}_n) = 0.$$

□

Now we can prove our first Itô-isometry:

Proposition 3.1.8 *For $L \in \mathcal{L}_0$ and $0 \leq s \leq t < \infty$ one has that*

$$\mathbb{E}([I_t(L) - I_s(L)]^2 | \mathcal{F}_s) = \mathbb{E}\left(\int_s^t L_u^2 du | \mathcal{F}_s\right) \text{ a.s.}$$

Proof. By introducing new time knots we can assume without loss of generality that $s = t_n$ and $t = t_N$. Let

$$M_k := I_{t_k}(L) \quad \text{and} \quad \mathcal{G}_k := \mathcal{F}_{t_k}$$

so that $(M_k)_{k=0}^N$ is a martingale with respect to $(\mathcal{G}_k)_{k=0}^N$. Hence, a.s.,

$$\begin{aligned} \mathbb{E}([I_t(L) - I_s(L)]^2 | \mathcal{F}_s) &= \mathbb{E}([M_N - M_n]^2 | \mathcal{G}_n) \\ &= \mathbb{E}\left(\sum_{l=n+1}^N (dM_l)^2 | \mathcal{G}_n\right) \\ &= \mathbb{E}\left(\sum_{l=n+1}^N v_{l-1}^2 (B_{t_l} - B_{t_{l-1}})^2 | \mathcal{F}_{t_n}\right) \\ &= \sum_{l=n+1}^N \mathbb{E}(\mathbb{E}(v_{l-1}^2 (B_{t_l} - B_{t_{l-1}})^2 | \mathcal{F}_{t_{l-1}}) | \mathcal{F}_{t_n}) \\ &= \sum_{l=n+1}^N \mathbb{E}(v_{l-1}^2 \mathbb{E}((B_{t_l} - B_{t_{l-1}})^2 | \mathcal{F}_{t_{l-1}}) | \mathcal{F}_{t_n}) \\ &= \sum_{l=n+1}^N \mathbb{E}(v_{l-1}^2 \mathbb{E}(B_{t_l} - B_{t_{l-1}})^2 | \mathcal{F}_{t_n}) \\ &= \sum_{l=n+1}^N \mathbb{E}(v_{l-1}^2 (t_l - t_{l-1}) | \mathcal{F}_{t_n}) \\ &= \mathbb{E}\left(\int_{t_n}^{t_N} L_u^2 du | \mathcal{F}_{t_n}\right) \\ &= \mathbb{E}\left(\int_s^t L_u^2 du | \mathcal{F}_s\right). \end{aligned}$$

□

Proposition 3.1.9 For $K, L \in \mathcal{L}_0$ and $\alpha, \beta \in \mathbb{R}$ one has that

$$I_t(\alpha L + \beta K) = \alpha I_t(L) + \beta I_t(K).$$

Now we want to extend this to a larger class of integrands. So far we have a map

$$I : \mathcal{L}_0 \longrightarrow \mathcal{M}_2^{c,0}$$

which is defined ω -wise.

Definition 3.1.10 Let \mathcal{L}_2 ¹ be the set of all progressively measurable processes $L = (L_t)_{t \geq 0}$, $L_t : \Omega \rightarrow \mathbb{R}$, such that

$$[L]_T := \left(\mathbb{E} \int_0^T L_t^2 dt \right)^{\frac{1}{2}} < \infty \quad \text{for all } T > 0.$$

Moreover, we let

$$d(K, L) := \sum_{n=1}^{\infty} 2^{-n} \min \{1, [K - L]_n\} \quad \text{for } K, L \in \mathcal{L}_2.$$

Proposition 3.1.11 One has that $\mathcal{L}_0 \subseteq \mathcal{L}_2$. Moreover the inclusion is dense, that means for all $L \in \mathcal{L}_2$ there is a sequence $(L^{(n)})_{n=1}^{\infty} \subseteq \mathcal{L}_0$ such that

$$\lim_n d(L^{(n)}, L) = 0.$$

Proof. (a) Assume that $\sup_{t, \omega} |L_t(\omega)| < \infty$ and that $t \rightarrow L_t(\omega)$ is continuous for all $\omega \in \Omega$. For $T > 0$ we define

$$L_t^{(n)}(\omega) := \sum_{k=0}^{2^n - 1} L_{\frac{kT}{2^n}}(\omega) \chi_{\left(\frac{kT}{2^n}, \frac{(k+1)T}{2^n}\right]}(t).$$

Then $\lim_{n \rightarrow \infty} L_t^{(n)}(\omega) = L_t(\omega)$ for all $t \in (0, T]$ and, by dominated convergence,

$$\lim_{n \rightarrow \infty} \mathbb{E} \int_0^T (L_t^{(n)} - L_t)^2 dt = 0.$$

(b) Let L be progressively measurable with $\sup_{t, \omega} |L_t(\omega)| < \infty$ and let $T > 0$. For $t \geq 0$ and $m = 1, 2, \dots$ define

$$\tilde{L}_t^{(m)}(\omega) := m \int_{(t - \frac{1}{m})^+}^t L_s(\omega) ds.$$

¹Working with \mathcal{L}_2 we will identify $K, L \in \mathcal{L}_2$ if $(\lambda \times \mathbb{P})((t, \omega) \in [0, \infty) \times \Omega : L_t(\omega) \neq K_t(\omega)) = 0$.

Hence $(\tilde{L}^{(m)})_{t \geq 0}$ is continuous and adapted (we can write $\tilde{L}_t^{(m)}(\omega) = m[K_t(\omega) - K_{(t-\frac{1}{m}) \vee 0}(\omega)]$ with $K_t(\omega) := \int_0^{t \wedge T} L_s(\omega) ds$). Moreover,

$$|\tilde{L}_t^{(m)}(\omega)| \leq \sup_{s \geq 0} |L_s(\omega)|$$

and we can apply step (a) and find

$$\tilde{L}^{(m,n)} = \left(\tilde{L}_t^{(m,n)} \right)_{t \geq 0} \in \mathcal{L}_0$$

such that

$$\lim_{n \rightarrow \infty} \mathbb{E} \int_0^T |\tilde{L}_t^{(m,n)} - \tilde{L}_t^{(m)}|^2 dt = 0. \quad (3.1)$$

Let

$$\mathcal{A} := \left\{ (t, \omega) \in [0, T] \times \Omega : \lim_{m \rightarrow \infty} \tilde{L}_t^{(m)}(\omega) = L_t(\omega) \right\}^c \in \mathcal{B}([0, T]) \times \mathcal{F}_T.$$

Let

$$\mathcal{A}_\omega := \{t \in [0, T] : (t, \omega) \in \mathcal{A}\}.$$

FUBINI'S theorem implies that $\mathcal{A}_\omega \in \mathcal{B}([0, T])$. Moreover, $t \in \mathcal{A}_\omega^c$ whenever

$$\tilde{L}_t^{(m)}(\omega) = m \int_{(t-\frac{1}{m}) \vee 0}^t L_s(\omega) ds \rightarrow_m L_t(\omega).$$

By the fundamental theorem of calculus we get that

$$\lambda(\mathcal{A}_\omega^c) = T \quad \text{or} \quad \lambda(\mathcal{A}_\omega) = 0$$

where λ is the LEBESGUE measure. Hence by FUBINI'S theorem,

$$(\lambda \times \mathbb{P})(\mathcal{A}) = 0$$

so that by majorized convergence,

$$\lim_m \mathbb{E} \int_0^T |\tilde{L}_t^{(m)} - L_t|^2 dt = 0. \quad (3.2)$$

Combining (3.1) and (3.2) gives the existence of $L^{(n)} = (L_t^{(n)})_{t \geq 0} \in \mathcal{L}_0$ such that

$$\lim_n \mathbb{E} \int_0^T |L_t^{(n)} - L_t|^2 dt = 0.$$

(c) From step (b) we get that there are $L^{(N)} = (L_t^{(N)})_{t \geq 0} \in \mathcal{L}_0$ such that

$$\mathbb{E} \int_0^N |L_t^{(N)} - L_t|^2 dt \leq \frac{1}{N^2}$$

for $N = 1, 2, \dots$. Hence

$$[L^{(N)} - L]_T \leq \frac{1}{N} \quad \text{for } N \geq T$$

and $d(L^{(N)}, L) \rightarrow_N 0$ as $N \rightarrow \infty$.

(d) Now we remove the condition that L is bounded. For $k = 1, 2, \dots$ we let

$$L_t^{(k)} := \begin{cases} k & : L_t \geq k \\ L_t & : L_t \in [-k, k] \\ -k & : L_t \leq -k \end{cases} .$$

By dominated convergence we have that $[L - L^{(k)}]_T \rightarrow 0$ as $k \rightarrow \infty$ for all $T > 0$. But the $L^{(k)}$ can be approximated by elements of \mathcal{L}_0 so that we are done. \square

Proposition 3.1.12 *The map $I : \mathcal{L}_0 \rightarrow \mathcal{M}_2^{c,0}$ can be extended to a map $J : \mathcal{L}_2 \rightarrow \mathcal{M}_2^{c,0}$ such that the following is true:*

(i) *Linearity:* for $\alpha, \beta \in \mathbb{R}$ and $K, L \in \mathcal{L}_2$ one has

$$J_t(\alpha K + \beta L) = \alpha J_t(K) + \beta J_t(L)$$

for $t \geq 0$ a.s.

(ii) *Extension property:* if $L \in \mathcal{L}_0$, then

$$I_t(L) = J_t(L), t \geq 0, \text{ a.s.}$$

(iii) *Isometry property:* if $L \in \mathcal{L}_2$, then, for $t \geq 0$,

$$\|J_t(L)\|_{L_2} = \left(\mathbb{E} \int_0^t L_u^2 du \right)^{\frac{1}{2}} .$$

(iv) *Continuity property:* if $A^{(n)}, L \in \mathcal{L}_2$ such that $d(A^{(n)}, L) \rightarrow 0$ as $n \rightarrow \infty$, then

$$\mathbb{E} \sup_{t \in [0, T]} |J_t(L) - J_t(A^{(n)})|^2 \rightarrow_n 0.$$

(v) *Uniqueness property:* if $J' : \mathcal{L}_2 \rightarrow \mathcal{M}_2^{c,0}$ is another mapping satisfying (i), ..., (iv), then one has for all $T > 0$ that

$$\mathbb{P}(J_t(L) = J'_t(L) : t \geq 0) = 1 \quad \text{for all } L \in \mathcal{L}_2.$$

Before we prove this proposition let us turn to a simple example, the WIENER integral.

Example 3.1.13 For a continuous function $L : [0, \infty) \rightarrow \mathbb{R}$ the process

$$X_t := J_t(L)$$

is a Gaussian process with mean zero and covariance

$$\Gamma(s, t) = \mathbb{E}X_s X_t = \int_0^{\min\{s, t\}} L_u^2 du.$$

Proof. (a) First we remark that an L_2 -limit of Gaussian random variables is again a Gaussian random variable, which is left as an exercise.

(b) Defining

$$L_t^{(n)} := \sum_{k=1}^{\infty} \chi_{(\frac{k-1}{2^n}, \frac{k}{2^n}]}(t) L\left(\frac{k-1}{2^n}\right)$$

we obtain that $d(L, L^{(n)}) \rightarrow_n 0$. Hence by Proposition 3.1.12 (iv)

$$J_t(L^{(n)}) \rightarrow_{L_2} J_t(L).$$

But $J_t(L^{(n)}) = I_t(L^{(n)})$ are Gaussian random variables so $J_t(L)$ is Gaussian by step (a) as well.

(c) To check that we have a Gaussian process we let $0 \leq t_1 \leq \dots \leq t_n \leq T$ and $\alpha_1, \dots, \alpha_n \in \mathbb{R}$. It is easy to see that

$$\sum_{k=1}^n \alpha_k J_{t_k}(L^{(n)})$$

is a Gaussian random variable which converges in L_2 to the random variable $\sum_{k=1}^n \alpha_k J_{t_k}(L)$. Hence the latter is Gaussian and we have a Gaussian process.

(d) To get the covariance structure we simply use the ITÔ-isometry since one can show that

$$J_r(L) = J_T(K^{(r)}) \text{ a.s.}$$

for $K_u^{(r)} := L_u$ for $u \in [0, r]$ and $K_u^{(r)} := 0$ for $u > r$, where $T \geq r$. \square

Next we consider our first real example.

Example 3.1.14 One has that

$$J_t(B) = \frac{1}{2}(B_t^2 - t) \quad \text{for } t \geq 0 \text{ a.s.}$$

Proof. Since the left and right-hand side are continuous processes starting in zero we only have to show that $J_T(B) = \frac{1}{2}(B_T^2 - T)$ a.s. for $T > 0$. Let

$$L_t^{(n)} := \sum_{i=1}^{\infty} \chi_{(T\frac{i-1}{2^n}, T\frac{i}{2^n}]}(t) B_{T\frac{i-1}{2^n}}.$$

By majorized convergence one can show that

$$\lim_n d(L^{(n)}, B) = 0.$$

In the next step we would need to show that

$$J_t(L^{(n)}) = \sum_{i=1}^{2^n} B_{T \frac{i-1}{2^n}} (B_{T \frac{i}{2^n}} - B_{T \frac{i-1}{2^n}})$$

(note that $L^{(n)}$ is not a simple integrand according to our definition). This can be easily done by a truncation of the coefficients $B_{T \frac{i-1}{2^n}}$ which brings us back to the simple integrands. So we are left to show that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left| \sum_{i=1}^{2^n} B_{T \frac{i-1}{2^n}} (B_{T \frac{i}{2^n}} - B_{T \frac{i-1}{2^n}}) - \frac{1}{2} (B_T^2 - T) \right|^2 = 0.$$

Let $N = 2^n$. Then

$$\begin{aligned} & \mathbb{E} \left| \sum_{i=1}^N B_{T \frac{i-1}{N}} (B_{T \frac{i}{N}} - B_{T \frac{i-1}{N}}) - \frac{1}{2} (B_T^2 - T) \right|^2 \\ &= \mathbb{E} \left| \sum_{i=1}^N \left[B_{T \frac{i-1}{N}} (B_{T \frac{i}{N}} - B_{T \frac{i-1}{N}}) - \frac{1}{2} (B_{T \frac{i}{N}})^2 + \frac{1}{2} (B_{T \frac{i-1}{N}})^2 \right] + \frac{T}{2} \right|^2 \\ &= \frac{1}{4} \mathbb{E} \left| T - \sum_{i=1}^N [B_{T \frac{i}{N}} - B_{T \frac{i-1}{N}}]^2 \right|^2 \\ &= \frac{T^2}{4} \mathbb{E} \left| 1 - \sum_{i=1}^N [B_{\frac{i}{N}} - B_{\frac{i-1}{N}}]^2 \right|^2 \\ &= \frac{T^2}{4} \mathbb{E} \left| \sum_{i=1}^N \left[[B_{\frac{i}{N}} - B_{\frac{i-1}{N}}]^2 - \frac{1}{N} \right] \right|^2 \\ &= \frac{T^2}{4} \sum_{i=1}^N \mathbb{E} \left| \left[[B_{\frac{i}{N}} - B_{\frac{i-1}{N}}]^2 - \frac{1}{N} \right] \right|^2 \\ &= \frac{T^2}{4} \sum_{i=1}^N [\mathbb{E} B_1^4 - 1] \frac{1}{N^2} \rightarrow_N 0. \end{aligned}$$

□

Looking at the above proof we also got

Proposition 3.1.15 *For the Brownian motion $B = (B_t)_{t \geq 0}$ and $T \geq 0$ one has that*

$$\lim_{N \rightarrow \infty} \sum_{i=1}^N [B_{T \frac{i}{N}} - B_{T \frac{i-1}{N}}]^2 = T$$

where the convergence in probability is taken.

To proceed we need

Proposition 3.1.16 (Doob's maximal inequalities) *Let $M = (M_t)_{t \geq 0}$ be a right-continuous martingale or a right-continuous positive sub-martingale and let $M_t^* := \sup_{s \in [0, t]} |M_s|$. Then one has, for $\lambda, t \geq 0$ and $p \in (1, \infty)$, that*

$$\lambda \mathbb{P}(M_t^* \geq \lambda) \leq \mathbb{E}|M_t|,$$

and

$$\mathbb{E}(M_t^*)^p \leq \left(\frac{p}{p-1} \right)^p \mathbb{E}|M_t|^p.$$

For the proof we use the discrete time version proved in [3]:

Lemma 3.1.17 *Let $M = (M_n)_{n=0}^N$ be martingale or positive sub-martingale and let $M_n^* := \sup_{k=0, \dots, n} |M_k|$. Then one has, for $\lambda \geq 0$, $n = 0, \dots, N$, and $p \in (1, \infty)$, that*

$$\lambda \mathbb{P}(M_n^* \geq \lambda) \leq \mathbb{E}|M_n|,$$

and

$$\mathbb{E}(M_n^*)^p \leq \left(\frac{p}{p-1} \right)^p \mathbb{E}|M_n|^p.$$

Proof of Proposition 3.1.16. First we remark that $M_t^* : \Omega \rightarrow \mathbb{R}$ is measurable, since

$$M_t^* = \sup_{s \in [0, t] \cap \mathbb{Q}} (|M_s| \vee |M_t|)$$

which is a countable supremum of measurable functions. From Lemma 3.1.17 it follows that, for

$$M_t^{n,*} := \sup_{\substack{s = \frac{k}{2^n} t \\ k=0, \dots, 2^n}} |M_s|,$$

one has that

$$\lambda \mathbb{P}(M_t^{n,*} \geq \lambda) \leq \mathbb{E}|M_t| \quad \text{and} \quad \mathbb{E}(M_t^{n,*})^p \leq \left(\frac{p}{p-1} \right)^p \mathbb{E}|M_t|^p.$$

Since $M_t^{n,*} \uparrow M_t^*$ a.s. as $n \rightarrow \infty$ we are done. \square

Proof Proposition 3.1.12. (a) Before we really start, we show that (ii) and (iv) imply (v): we find $L^{(n)} \in \mathcal{L}_0$ with $d(L, L^{(n)}) \rightarrow_n 0$ where we use Proposition 3.1.11. Hence

$$J_t(L) =_{L_2} \lim_n J_t(L^{(n)}) =_{L_2} \lim_n I_t(L^{(n)}) =_{L_2} \lim_n J'_t(L^{(n)}) =_{L_2} J'_t(L).$$

Since $(J_t(L))_{t \geq 0}$ and $(J'_t(L))_{t \geq 0}$ are continuous, the processes are indistinguishable.

(b) Conditions (ii) and (iv) imply (i): By Proposition 3.1.11 we find $K^{(n)}, L^{(n)} \in \mathcal{L}_0$ such that $d(K, K^{(n)}) \rightarrow_n 0$ and $d(L, L^{(n)}) \rightarrow_n 0$. Consequently,

$$\begin{aligned}
J_t(\alpha K + \beta L) &=_{L_2} \lim_n J_t(\alpha K^{(n)} + \beta L^{(n)}) \\
&=_{L_2} \lim_n I_t(\alpha K^{(n)} + \beta L^{(n)}) \\
&=_{L_2} \lim_n [\alpha I_t(K^{(n)}) + \beta I_t(L^{(n)})] \\
&=_{L_2} \alpha \lim_n I_t(K^{(n)}) + \beta \lim_n I_t(L^{(n)}) \\
&=_{L_2} \alpha \lim_n J_t(K^{(n)}) + \beta \lim_n J_t(L^{(n)}) \\
&=_{L_2} \alpha J_t(K) + \beta J_t(L).
\end{aligned}$$

Since the processes on both sides are continuous both sides are indistinguishable and we are done.

(c) Hence the main part of the proof is to prove the existence of an extension J of I such that assertions (ii), (iii), and (iv) are satisfied. We shall fix the sequence $L^{(n)} \in \mathcal{L}_0$ from step (a) where we assume w.l.o.g. that $L^{(1)} := 0$ (which is convenient later for us).

1. Step: for all $\varepsilon > 0$ and $t \geq 0$ there is an $n(\varepsilon, t)$ such that

$$\mathbb{E}|I_t(L^{(n)}) - I_t(L^{(m)})|^2 = \int_0^t \mathbb{E}|L_u^{(n)} - L_u^{(m)}|^2 du \leq \varepsilon^2$$

for $m, n \geq n(\varepsilon, t)$ where we used ITÔ's isometry for simple processes. Hence there exists some $Y_t \in L_2$ such that

$$\lim_n I_t(L^{(n)}) =_{L_2} Y_t.$$

Assuming a different sequence $K^{(n)} \in \mathcal{L}_0$ we get that

$$\lim_n I_t(K^{(n)}) =_{L_2} X_t.$$

However,

$$\begin{aligned}
\mathbb{E}|Y_t - X_t|^2 &= \lim_n \mathbb{E}|I_t(L^{(n)}) - I_t(K^{(n)})|^2 \\
&= \lim_n \int_0^t \mathbb{E}|L_u^{(n)} - K_u^{(n)}|^2 du \\
&= 0
\end{aligned}$$

so that the process $(X_t)_{t \geq 0}$ is unique as a process in L_2 .

2. Step: We show that there is a path-wise continuous process $(M_t)_{t \geq 0}$ with $M_0 \equiv 0$ such that $X_t = M_t$ a.s. for $t \geq 0$. It is clear that M is unique

up to indistinguishability if it exists. By DOOB's maximal inequality we have

$$\begin{aligned} \mathbb{E} \sup_{t \in [0, N]} |I_t(L^{(n)}) - I_t(L^{(m)})|^2 &\leq 4\mathbb{E}|I_N(L^{(n)}) - I_N(L^{(m)})|^2 \\ &= 4\mathbb{E} \int_0^N |L_t^{(n)} - L_t^{(m)}|^2 dt \\ &< 4\varepsilon^2 \end{aligned}$$

for $m, n \geq n(\varepsilon, N)$. Hence we find $1 = n_0 < n_1 < n_2 < \dots$ (depending on N) such that

$$\mathbb{E} \sup_{t \in [0, N]} |I_t(L^{(n_{k+1})}) - I_t(L^{(n_k)})|^2 \leq \frac{1}{2^{k+1}}$$

for $k = 1, 2, \dots$ Letting

$$J_k^N := \sup_{t \in [0, N]} |I_t(L^{(n_{k+1})}) - I_t(L^{(n_k)})|$$

implies

$$\sum_{k=0}^{\infty} (\mathbb{E}|J_k^N|^2)^{\frac{1}{2}} < \infty \quad \text{and} \quad \mathbb{P} \left(\sum_{k=0}^{\infty} J_k^N < \infty \right) = 1.$$

Setting $\Omega_N := \{\omega \in \Omega : \sum_{k=0}^{\infty} J_k^N(\omega) < \infty\}$, we define

$$M_t^{(N)}(\omega) := \begin{cases} \sum_{k=0}^{\infty} [I_t(L^{(n_{k+1})}) - I_t(L^{(n_k)})] & : \omega \in \Omega_N \\ 0 & : \omega \notin \Omega_N \end{cases}$$

for $t \geq 0$. Now we see that the process $(M_t^{(N)})_{t \in [0, N]}$ is path-wise continuous and that

$$X_t =_{L_2} \sum_{k=0}^{\infty} [I_t(L^{(n_{k+1})}) - I_t(L^{(n_k)})] = M_t^{(N)}$$

for $t \in [0, N]$. With a standard trick we find the process we are looking for: define, for $0 < N \leq M < \infty$,

$$\Omega_{N, M} := \left\{ \omega \in \Omega : M_t^{(N)}(\omega) = M_t^{(M)}(\omega) \text{ for all } t \in [0, N] \right\}$$

so that $\mathbb{P}(\Omega_{N, M}) = 1$ since $(M_t^{(N)})_{t \in [0, N]}$ and $(M_t^{(M)})_{t \in [0, N]}$ are continuous modifications of $(X_t)_{t \in [0, N]}$. Letting

$$\tilde{\Omega} := \bigcap_{0 < N \leq M < \infty} \Omega_{N, M}$$

we obtain that $\mathbb{P}(\tilde{\Omega}) = 1$ and set

$$M_t(\omega) := \begin{cases} M_t^{(N)}(\omega) & : \omega \in \tilde{\Omega} \text{ and } t \in [0, N] \\ 0 & : \omega \notin \tilde{\Omega} \end{cases}.$$

By changing the process on a null-set (we have the usual conditions) we may assume that $M_0 \equiv 0$. Since M is the L_2 -limit of martingales, we finally end up with

$$M \in \mathcal{M}_2^{c,0}.$$

3. Step, Property (ii): this follows by construction since we can take $L_n = L$ in this case.

4. Step, Property (iii): since

$$\left| \|M_t\|_{L_2} - \|I_t(L^{(n)})\|_{L_2} \right| \leq \|M_t - I_t(L^{(n)})\|_{L_2} \rightarrow_n 0$$

and

$$\left| \left(\mathbb{E} \int_0^t L_u^2 du \right)^{\frac{1}{2}} - \left(\mathbb{E} \int_0^t (L_u^{(n)})^2 du \right)^{\frac{1}{2}} \right| \leq \left| \left(\mathbb{E} \int_0^t (L_u - L_u^{(n)})^2 du \right)^{\frac{1}{2}} \right| \rightarrow 0$$

and the Itô isometry for the simple processes we get (iii).

5. Step, Property (iv): Denote now M_t by $J_t(L)$. By DOOB's inequality

$$\mathbb{E} \sup_{t \in [0, T]} |J_t(L) - J_t(A^{(n)})|^2 \leq 4\mathbb{E} |J_T(L) - J_T(A^{(n)})|^2$$

so that it remains to show that

$$\mathbb{E} |J_T(L) - J_T(A^{(n)})|^2 \rightarrow_n 0.$$

Find $A^{(n,m)}, L^{(m)} \in \mathcal{L}_0$ such that $d(A^{(n,m)}, A^{(n)}) \rightarrow_m 0$ and $d(L^{(n)}, L) \rightarrow_n 0$. Then we find a sub-sequence m_n such that

$$d(A^{(n,m_n)}, A^{(n)}) \rightarrow_n 0 \quad \text{and} \quad d(A^{(n,m_n)}, L^{(n)}) \rightarrow_n 0.$$

We conclude with

$$\begin{aligned} & \|J_T(L) - J_T(A^{(n)})\|_{L_2} \\ & \leq \|J_T(L) - J_T(L^{(n)})\|_{L_2} + \|J_T(L^{(n)}) - J_T(A^{(n,m_n)})\|_{L_2} \\ & \quad + \|J_T(A^{(n,m_n)}) - J_T(A^{(n)})\|_{L_2} \end{aligned}$$

where the first and last term go to zero by construction of J_T and the middle term by Itô's isometry for simple integrands. \square

We are close to the stochastic integral we need to have. In the last step we carry out a localizing procedure to extend the integral from \mathcal{L}_2 to $\mathcal{L}_2^{\text{loc}}$. For this purpose we first need

Lemma 3.1.18 *For any stopping time $\tau : \Omega \rightarrow [0, \infty)$ and progressively measurable process $L = (L_t)_{t \geq 0}$ one has that $L^\tau := (L_t \chi_{\{t \leq \tau\}})_{t \geq 0}$ is progressively measurable.*

Definition 3.1.19 (i) Let $\mathcal{L}_2^{\text{loc}}$ be the set of all progressively measurable processes $L = (L_t)_{t \geq 0}$ such that

$$\mathbb{P} \left(\omega \in \Omega : \int_0^t L_u^2(\omega) du < \infty \right) = 1 \quad \text{for all } t \geq 0.$$

(ii) A sequence $(\tau_n)_{n=0}^\infty$ of stopping times is called *localizing* for $L = (L_t)_{t \geq 0} \in \mathcal{L}_2^{\text{loc}}$ provided that

(a) $0 \leq \tau_0(\omega) \leq \tau_1(\omega) \leq \tau_2(\omega) \leq \dots \leq \infty$ and $\lim_n \tau_n(\omega) = \infty$ for all $\omega \in \Omega$,

(b) $L^{\tau_n} \in \mathcal{L}_2$ for all $n = 0, 1, 2, \dots$

Remark 3.1.20 (i) One has that $\mathcal{L}_2 \subseteq \mathcal{L}_2^{\text{loc}}$ since

$$\mathbb{E} \int_0^t L_u^2 du < \infty$$

implies by FUBINI's theorem that

$$\int_0^t L_u^2 du < \infty \text{ a.s.}$$

(ii) For every $L \in \mathcal{L}_2^{\text{loc}}$ there exists a localizing sequence. Let $N = 1, 2, \dots$ and

$$\Omega_N := \left\{ \omega \in \Omega : \int_0^N L_u^2(\omega) du < \infty \right\}.$$

Then $\mathbb{P}(\bigcap_{N=1}^\infty \Omega_N) = 1$ and we may set

$$\tau_n(\omega) := \begin{cases} \inf \left\{ t \geq 0 : \int_0^t L_u^2(\omega) du \geq n \right\} & : \omega \in \bigcap_{N=1}^\infty \Omega_N \\ \infty & : \text{else} \end{cases}$$

because $t \rightarrow \int_0^t L_u^2(\omega) du$ is a continuous function defined on $[0, \infty)$ with values in $[0, \infty)$ for $\omega \in \bigcap_{N=1}^\infty \Omega_N$.

Next we need

Lemma 3.1.21 Assume $L \in \mathcal{L}_2^{\text{loc}}$. Then there is a unique (up to indistinguishability) adapted and continuous process $X = (X_t)_{t \geq 0}$ with $X_0 \equiv 0$ such that for all localizing sequences $(\tau_n)_{n=0}^\infty$ of L one has

$$\mathbb{P}(J_t(L^{\tau_n}) = X_t, t \in [0, \tau_n]) = 1 \quad \text{for } n = 0, 1, 2, \dots$$

Definition 3.1.22 Let $L \in \mathcal{L}_2^{\text{loc}}$. The process X obtained in Lemma 3.1.21 is denoted by

$$\left(\int_0^t L_u dB_u \right)_{t \geq 0}$$

and is called *stochastic integral* of (the integrand) L with respect to (the integrator) B . Moreover, for $0 \leq s \leq t < \infty$ we let

$$\int_s^t L_u dB_u = \int_{(s,t]} L_u dB_u := X_t - X_s.$$

Before we summarize some properties of stochastic integrals we introduce local martingales.

Definition 3.1.23 A continuous adapted process $M = (M_t)_{t \geq 0}$ with $M_0 \equiv 0$ is called *local martingale* (we shall write $M \in \mathcal{M}_{\text{loc}}^{c,0}$) provided that there exists an increasing sequence $(\sigma_n)_{n=0}^\infty$ of stopping times $0 \leq \sigma_0(\omega) \leq \sigma_1(\omega) \leq \dots \leq \infty$ with $\lim_n \sigma_n(\omega) = \infty$ such that $M^{\sigma_n} := (M_{t \wedge \sigma_n})_{t \geq 0}$ is a martingale for all $n = 0, 1, 2, \dots$

Surprisingly it is not so easy to find local martingales which are not martingales. We indicate a construction, but do not go into any details. The example is intended as motivation for ITÔ's formula presented in the next section.

Example 3.1.24 Given $d = 1, 2, \dots$ we let $(W_t)_{t \geq 0}$ be the d -dimensional standard Brownian motion where $W_t := (B_{t,1}, \dots, B_{t,d})$, $W_0 \equiv 0$, and $(B_{t,i})_{t \geq 0}$ are independent Brownian motions. The filtration is obtained as in the one-dimensional case as the augmentation of the natural filtration. Let $d = 3$ and

$$M_t := \frac{1}{|x + W_t|}$$

with $|x| > 0$ where $|\cdot|$ is the euclidean norm on \mathbb{R}^d . Then $M = (M_t)_{t \geq 0}$ is a local martingale, but not a martingale. The process $|x + W_t|$ is a 3-dimensional BESSEL process starting in $|x|$.

Proof. To justify the construction one would need the following:

(a) For a d -dimensional standard Brownian motion W with $d \geq 2$ the sets $\{y\}$ with $y \neq 0$ are *polar sets*, that means

$$\mathbb{P}(\tau_y < \infty) = 0 \quad \text{with} \quad \tau_y := \inf \{t \geq 0 : W_t = y\}.$$

(b) For $d \geq 3$ one has that $\mathbb{P}(\lim_{t \rightarrow \infty} |W_t| = \infty) = 1$.

(c) Assuming that M is a martingale we would get

$$\mathbb{E}M_t = \mathbb{E}M_0 = \frac{1}{|x|}.$$

But a direct computation yields to

$$\mathbb{E} \frac{1}{|x + W_t|} = \mathbb{E} \frac{1}{|x + \sqrt{t}(g_1, g_2, g_3)|} \xrightarrow{t \rightarrow \infty} 0$$

where $g_1, g_2, g_3 \sim N(0, 1)$ are independent.

(d) How to show that M is a *local* martingale? This gives us a first impression of Itô-formula which will read for $f(\xi_1, \xi_2, \xi_3) := \frac{1}{\sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}}$ and $X_t := x + (B_{t,1}, B_{t,2}, B_{t,3})$ as

$$\begin{aligned} f(X_t) &= f(x) + \sum_{i=1}^3 \int_0^t \frac{\partial f}{\partial x_i}(X_u) dB_{u,i} + \frac{1}{2} \int_0^t (\Delta f)(X_u) du \text{ a.s.} \\ &= f(x) + \sum_{i=1}^3 \int_0^t \frac{\partial f}{\partial x_i}(X_u) dB_{u,i} \end{aligned}$$

where the latter term turns out to be a local martingale. \square

Now we summarize some of the properties of our stochastic integral:

Proposition 3.1.25 (i) For $L \in \mathcal{L}_2^{\text{loc}}$ one has $\left(\int_0^t L_u dB_u \right)_{t \geq 0} \in \mathcal{M}_{\text{loc}}^{c,0}$.

(ii) For $K, L \in \mathcal{L}_2^{\text{loc}}$ and $\alpha, \beta \in \mathbb{R}$ one has

$$\int_0^t (\alpha K_u + \beta L_u) dB_u = \alpha \int_0^t K_u dB_u + \beta \int_0^t L_u dB_u, \quad t \geq 0, \text{ a.s.}$$

(iii) *Itô-Isometry*: for $K, L \in \mathcal{L}_2$ and $0 \leq s < t < \infty$ one has

$$\mathbb{E} \left(\int_s^t K_u dB_u \int_s^t L_u dB_u \middle| \mathcal{F}_s \right) = \mathbb{E} \left(\int_s^t K_u L_u du \middle| \mathcal{F}_s \right) \text{ a.s.}$$

(iv) Given $K \in \mathcal{L}_2$, the process

$$\left(\left(\int_0^t K_u dB_u \right)^2 - \int_0^t K_u^2 du \right)_{t \geq 0}$$

is a continuous martingale.

(v) For $L \in \mathcal{L}_2^{\text{loc}}$ and a stopping time $\tau : \Omega \rightarrow [0, \infty]$ one has that

$$\left(\int_0^{t \wedge \tau(\omega)} L_u dB_u \right) (\omega) = \left(\int_0^t L_u \chi_{\{u \leq \tau\}} dB_u \right) (\omega)$$

for $t \geq 0$ a.s.

(vi) For $0 \leq s \leq t < \infty$, $B \in \mathcal{F}_s$, and $K \in \mathcal{L}_2^{\text{loc}}$ one has that

$$\int_s^t [\chi_B K_u] dB_u = \chi_B \int_s^t K_u dB_u \text{ a.s.}$$

with the convention that

$$\int_s^t [\chi_B K_u] dB_u := \int_s^t \tilde{K}_u dB_u$$

with

$$\tilde{K}(u) := \begin{cases} 0 & : u \in [0, s] \\ \chi_B K_u & : u > s \end{cases}.$$

Proof. (iii) \Rightarrow (iv) Let

$$M_t := \int_0^t K_u dB_u.$$

Then, a.s.,

$$\begin{aligned} & \mathbb{E} \left(M_t^2 - \int_0^t K_u^2 du \middle| \mathcal{F}_s \right) \\ &= \mathbb{E} \left((M_t - M_s + M_s)^2 - \int_0^t K_u^2 du \middle| \mathcal{F}_s \right) \\ &= \mathbb{E} \left((M_t - M_s)^2 + 2(M_t - M_s)M_s + M_s^2 - \int_0^t K_u^2 du \middle| \mathcal{F}_s \right) \\ &= \mathbb{E} \left((M_t - M_s)^2 - \int_s^t K_u^2 du \middle| \mathcal{F}_s \right) + \mathbb{E} (2(M_t - M_s)M_s \middle| \mathcal{F}_s) \\ &\quad + M_s^2 - \int_0^s K_u^2 du \\ &= M_s^2 - \int_0^s K_u^2 du \end{aligned}$$

where the first term is zero because of (i) and the second one because of

$$\mathbb{E} (2(M_t - M_s)M_s \middle| \mathcal{F}_s) = 2M_s \mathbb{E} (M_t - M_s \middle| \mathcal{F}_s) = 0 \text{ a.s.}$$

(iii) Using the polarization formula $ab = \frac{1}{4}((a+b)^2 - (a-b)^2)$ it is enough to show the assertion for $K = L$. We take $L^{(n)} \in \mathcal{L}_0$ such that $d(L^{(n)}, L) \rightarrow_n 0$ and get that

$$\mathbb{E} \int_0^t |L_u^{(n)} - L_u|^2 \rightarrow_n 0.$$

By the construction of the stochastic integral we have

$$\mathbb{E} \int_0^t |L_u^{(n)} - L_u|^2 = \mathbb{E} \left| \int_0^t L_u^{(n)} dB_u - \int_0^t L_u dB_u \right|^2 \rightarrow_n 0.$$

Fact 3.1.26 *Let (M, Σ, μ) be a probability space and $\mathcal{G} \subseteq \Sigma$ be a sub- σ -algebra. Assume random variables $f, f_n \in L_2$ such that $\mathbb{E}|f_n - f|^2 \rightarrow_n 0$. Then*

$$\mathbb{E} \left| \mathbb{E}(f_n^2 | \mathcal{G}) - \mathbb{E}(f^2 | \mathcal{G}) \right| \rightarrow_n 0.$$

Let now $f_n := \int_s^t L_u^{(n)} dB_u$ and $f := \int_s^t L_u dB_u$. Then

$$\|f_n - f\|_{L_2} = \left(\mathbb{E} \int_s^t (L_u^{(n)} - L_u)^2 du \right)^{\frac{1}{2}} \leq \left(\mathbb{E} \int_0^t (L_u^{(n)} - L_u)^2 du \right)^{\frac{1}{2}} \rightarrow_n 0.$$

Hence, by our fact,

$$\mathbb{E} \left(\left(\int_s^t L_u^{(n)} dB_u \right)^2 \middle| \mathcal{F}_s \right) \rightarrow_{L_1} \mathbb{E} \left(\left(\int_s^t L_u dB_u \right)^2 \middle| \mathcal{F}_s \right).$$

Considering the product space $\Omega \times [s, t]$ with $\mu := \mathbb{P} \times \frac{\lambda}{t-s}$ and $\mathcal{G} := \mathcal{F}_s \times \mathcal{B}([s, t])$ we get in the same way that

$$\mathbb{E} \left(\int_s^t (L_u^{(n)})^2 du \middle| \mathcal{F}_s \right) \rightarrow_{L_1} \mathbb{E} \left(\int_s^t (L_u)^2 du \middle| \mathcal{F}_s \right).$$

Now we can finish with Proposition 3.1.8.

We do not give the details for (i), (ii), (v) and (vi). □

3.2 Itô's formula

In calculus there is the fundamental formula

$$f(y) = f(x) + \int_x^y f'(u) du$$

for, say, $f \in C^1(\mathbb{R})$ and $-\infty < x < y < \infty$. Is there a similar formula for stochastic integrals? We develop such a formula for a class of processes, called ITÔ-processes.

Definition 3.2.1 *Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a function. Then*

$$\text{var}(f, t) := \sup_{0=t_0 \leq \dots \leq t_n=t} \sum_{k=1}^n |f(t_k) - f(t_{k-1})| \in [0, \infty].$$

Lemma 3.2.2 (i) *The function $\text{var}(f, \cdot)$ is increasing.*

(ii) *If $f : [0, \infty) \rightarrow \mathbb{R}$ is continuous, then $\text{var}(f, \cdot)$ is left-continuous.*

Definition 3.2.3 A stochastic process $A = (A_t)_{t \geq 0}$, $A_t : \Omega \rightarrow \mathbb{R}$, is called of *bounded variation* provided that

$$\text{var}(A.(\omega), t) = \sup_{0=t_0 \leq \dots \leq t_n=t} \sum_{k=1}^n |A_{t_k}(\omega) - A_{t_{k-1}}(\omega)| < \infty \text{ a.s. for all } t \geq 0.$$

Lemma 3.2.4 If $M = (M_t)_{t \geq 0} \in \mathcal{M}_2^{c,0}$ is of bounded variation, then

$$\mathbb{P}(\omega \in \Omega : M_t(\omega) = 0, t \geq 0) = 1.$$

Proof. Since M has continuous paths it is sufficient to show that

$$\mathbb{P}(M_t = 0) = 1 \quad \text{for all } t \geq 0.$$

(a) Assume that

$$\text{var}(M.(\omega), t) \leq c < \infty \text{ a.s.}$$

and let $t_i^n := \frac{it}{n}$. Then

$$\begin{aligned} \mathbb{E}M_t^2 &= \mathbb{E} \left[\sum_{i=1}^n \left[M_{t_i^n} - M_{t_{i-1}^n} \right] \right]^2 \\ &= \sum_{i=1}^n \mathbb{E} \left[M_{t_i^n} - M_{t_{i-1}^n} \right]^2 \\ &\leq \mathbb{E} \text{var}(M., t) \sup_{i=1, \dots, n} \left| M_{t_i^n} - M_{t_{i-1}^n} \right| \\ &\leq c \mathbb{E} \sup_{i=1, \dots, n} \left| M_{t_i^n} - M_{t_{i-1}^n} \right|. \end{aligned}$$

Since

$$\sup_{i=1, \dots, n} \left| M_{t_i^n}(\omega) - M_{t_{i-1}^n}(\omega) \right| \rightarrow_n 0$$

for all $\omega \in \Omega$ by the uniform continuity of the paths of M on compact intervals and

$$\sup_{i=1, \dots, n} |M_{t_i^n} - M_{t_{i-1}^n}| \leq 2 \sup_{u \in [0, t]} |M_u| \in L_2$$

by DOOB's maximal inequality, majorized convergence implies that

$$\lim_n \mathbb{E} \left| M_{t_i^n} - M_{t_{i-1}^n} \right| = 0 \quad \text{so that} \quad \mathbb{E}M_t^2 = 0.$$

(b) Now let $N \in \{1, 2, \dots\}$, $T > 0$, and

$$\tau_N(\omega) := \inf \{t \geq 0 : \text{var}(M.(\omega), t) > N\} \wedge T.$$

Because of Lemma 3.2.2 the random time τ_N is a stopping time. To check this it is sufficient to show that

$$\sigma_N(\omega) := \inf \{t \geq 0 : \text{var}(M(\omega), t) > N\}$$

is a stopping time. Indeed

$$\{t \leq \sigma_N(\omega)\} = \{\text{var}(M(\omega), t) \leq N\} \in \mathcal{F}_t$$

yields that σ_N is an optional time, so that we conclude that σ_N is a stopping time by the usual conditions. Moreover,

$$(M_{t \wedge \tau_N})_{t \geq 0} \in \mathcal{M}_2^{c,0}$$

by stopping and

$$\text{var}(M^{\tau_N}(\omega), t) \leq N.$$

By stopping we mean here that

$$\mathbb{E}(M_{t \wedge \rho} | \mathcal{F}_s) = M_{s \wedge \rho} \quad a.s. \quad (3.3)$$

for a stopping time $0 \leq \rho \leq T$. This can be proved by considering approximating stopping times $T \geq \rho_l \downarrow \rho$ where the ρ_l takes only values in $\{0, T/2^l, \dots, kT/2^l, \dots, T, s\}$. Stopping from discrete time martingales implies

$$\mathbb{E}(M_{t \wedge \rho_l} | \mathcal{F}_s) = M_{s \wedge \rho_l} \quad a.s.$$

Letting $l \rightarrow \infty$, using the pathwise continuity of M , and that (by Doob's maximal inequality) $\mathbb{E} \sup_{t \in [0, T]} M_t^2 < \infty$, we arrive at (3.3).

After all, applying (a) gives

$$\mathbb{E} M_{\tau_N \wedge T}^2 = 0.$$

Consequently,

$$\mathbb{E} M_T^2 = \mathbb{E} \lim_N M_{T \wedge \tau_N}^2 = \lim_N \mathbb{E} M_{T \wedge \tau_N}^2 = 0$$

since $\tau_N \uparrow T$ a.s. and

$$M_{T \wedge \tau_N}^2 \leq \sup_{t \in [0, T]} M_t^2 \in L_1.$$

□

Lemma 3.2.5 *If $M = (M_t)_{t \geq 0} \in \mathcal{M}_{\text{loc}}^{c,0}$ is of bounded variation, then*

$$\mathbb{P}(\omega \in \Omega : M_t(\omega) = 0, t \geq 0) = 1.$$

Proof. We assume a localizing sequence $(\sigma_n)_{n=0}^\infty$ for M . In addition, we let

$$\rho_n := \inf \{t \geq 0 : |M_t| \geq n\}$$

so that $\tau_n := \sigma_n \wedge \rho_n$ is a localizing sequence with $|M_t^{\tau_n}| \leq n$ (again, stopping is used here). The variation of M^{τ_n} is bounded by the variation of M , so that

$$\mathbb{P}(M_{t \wedge \tau_n} = 0) = 1$$

for all $t \geq 0$ and $n = 0, 1, 2, \dots$ by Lemma 3.2.4. Consequently,

$$\mathbb{P}(M_t = 0) = \mathbb{E} \left(\lim_{n \rightarrow \infty} \chi_{\{M_{t \wedge \tau_n} = 0\}} \right) = \lim_{n \rightarrow \infty} \mathbb{E} \left(\chi_{\{M_{t \wedge \tau_n} = 0\}} \right) = 1$$

where we have used dominated convergence and $\lim_n \tau_n(\omega) = \infty$ for all $\omega \in \Omega$. \square

Definition 3.2.6 A continuous and adapted process $X = (X_t)_{t \geq 0}$, $X_t : \Omega \rightarrow \mathbb{R}$, is called IT $\hat{\circ}$ -process provided there exist $L \in \mathcal{L}_2^{\text{loc}}$ and a progressively measurable process $a = (a_t)_{t \geq 0}$ with

$$\int_0^t |a_u(\omega)| du < \infty$$

for all $t \geq 0$ and $\omega \in \Omega$, and $x_0 \in \mathbb{R}$ such that

$$X_t(\omega) = x_0 + \left(\int_0^t L_u dB_u \right) (\omega) + \int_0^t a_u(\omega) du \quad \text{for } t \geq 0, \text{ a.s.}$$

Proposition 3.2.7 Assume that $X = (X_t)_{t \geq 0}$ is an IT $\hat{\circ}$ -process with representations (L, a) and (L', a') . Then

$$(\lambda \times \mathbb{P})((t, \omega) \in [0, \infty) \times \Omega : L_t(\omega) \neq L'_t(\omega) \text{ or } a_t(\omega) \neq a'_t(\omega)) = 0.$$

Proof. We only prove the part concerning the processes L and L' . From our assumption and the linearity of the stochastic integral it follows that

$$M_t := \int_0^t (L_u - L'_u) dB_u = \int_0^t (a'_u - a_u) du \quad \text{for } t \geq 0 \text{ a.s.}$$

Hence $M = (M_t)_{t \geq 0}$ is a local martingale of bounded variation because, a.s.,

$$\begin{aligned} \text{var}(M(\omega), t) &= \sup_{t_k} \left\{ \sum_{k=1}^n |M_{t_k}(\omega) - M_{t_{k-1}}(\omega)| \right\} \\ &= \sup_{t_k} \left\{ \sum_{k=1}^n \left| \int_{t_{k-1}}^{t_k} (a'_u(\omega) - a_u(\omega)) du \right| \right\} \end{aligned}$$

$$\begin{aligned}
&\leq \sup_{t_k} \left\{ \sum_{k=1}^n \int_{t_{k-1}}^{t_k} |a'_u(\omega) - a_u(\omega)| du \right\} \\
&= \int_0^t |a'_u(\omega) - a_u(\omega)| du \\
&< \infty.
\end{aligned}$$

By Lemma 3.2.5 this implies $M_t = 0$ a.s. so that by the continuity of the process M one ends up with

$$M_t = 0 \quad \text{for } t \geq 0, \text{ a.s.}$$

and

$$\int_0^t (L_u - L'_u) dB_u = 0 \quad \text{for } t \geq 0, \text{ a.s.}$$

Since $L - L' \in \mathcal{L}_2^{\text{loc}}$ we find a sequence of stopping times $0 \leq \tau_0 \leq \tau_1 \leq \dots$ converging to infinity such that $(L - L')^{\tau_n} \in \mathcal{L}_2$ and by the definition of the stochastic integral

$$\int_0^{t \wedge \tau_n} (L_u - L'_u) \chi_{\{u \leq \tau_n\}} dB_u = \int_0^{t \wedge \tau_n} (L_u - L'_u) dB_u = 0 \text{ a.s.}$$

By Proposition 3.1.25(v)

$$\int_0^t (L_u - L'_u) \chi_{\{u \leq \tau_n\}} dB_u = \int_0^{t \wedge \tau_n} (L_u - L'_u) \chi_{\{u \leq \tau_n\}} dB_u = 0 \text{ a.s.}$$

and by the ITÔ isometry,

$$\mathbb{E} \int_0^t |L_u - L'_u|^2 \chi_{\{u \leq \tau_n\}} du = 0.$$

Monotone convergence gives

$$\mathbb{E} \int_0^\infty |L_u - L'_u|^2 du = 0$$

which implies our assertion with respect to L and L' . □

To formulate ITÔ's formula we need

Definition 3.2.8 A continuous function $f : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ belongs to $C^{1,2}([0, \infty) \times \mathbb{R})$ provided that all partial derivatives $\partial f / \partial t$, $\partial f / \partial x$, and $\partial^2 f / \partial x^2$ exist on $(0, \infty) \times \mathbb{R}$, are continuous, and can be continuously extended to $[0, \infty) \times \mathbb{R}$.

Proposition 3.2.9 (ITÔ's formula) Let $X = (X_t)_{t \geq 0}$ be an ITÔ-process with representation

$$X_t = x_0 + \int_0^t L_u dB_u + \int_0^t a_u du, \quad t \geq 0, \text{ a.s.}$$

and let $f \in C^{1,2}$. Then one has that

$$\begin{aligned} f(t, X_t) = f(0, X_0) &+ \int_0^t \frac{\partial f}{\partial u}(u, X_u) du + \int_0^t \frac{\partial f}{\partial x}(u, X_u) L_u dB_u \\ &+ \int_0^t \frac{\partial f}{\partial x}(u, X_u) a_u du + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(u, X_u) L_u^2 du \end{aligned}$$

for $t \geq 0$ a.s.

Remark 3.2.10 (i) The assumptions on f and the continuity of the process X ensure that the right-hand side of ITÔ's formula is well-defined. In particular,

$$\left(\frac{\partial f}{\partial x}(u, X_u) L_u \right)_{u \geq 0} \in \mathcal{L}_2^{\text{loc}}.$$

(ii) To shorten the notation we shall use

$$\int_0^t K_u dX_u := \int_0^t K_u L_u dB_u + \int_0^t K_u a_u du$$

where we fix the decomposition of the process X in the following.

Before we discuss the proof of ITÔ's formula we consider some examples for its application.

Example 3.2.11 (Compensator) For $f(t, x) = f(x) := x^2$ we obtain

$$X_t^2 = x_0^2 + 2 \int_0^t X_u dX_u + \int_0^t L_u^2 du, \quad t \geq 0, \text{ a.s.}$$

If $a_u \equiv 0$, then we get that

$$X_t^2 - \int_0^t L_u^2 du = x_0^2 + 2 \int_0^t X_u L_u dB_u \quad t \geq 0, \text{ a.s.}$$

is a local martingale. Sometimes the term $\int_0^t L_u^2 du$ is called *compensator* (it compensates X_t^2 to get a local martingale) and denoted by $\langle X \rangle_t := \int_0^t L_u^2 du$.

Example 3.2.12 (Exponential martingale) Let $L \in C[0, \infty)$ and $X_t := \int_0^t L_u dB_u$. Then

$$\mathcal{E}(X)_t := e^{X_t - \frac{1}{2} \int_0^t L_u^2 du} = e^{X_t - \frac{1}{2} \langle X \rangle_t}$$

is a martingale and called *exponential martingale*. To check this, we let

$$f(t, x) := e^{x - \frac{1}{2} \int_0^t L(u)^2 du}.$$

Applying ITÔ's formula gives, a.s.,

$$\begin{aligned} \mathcal{E}(X)_t &= f(t, X_t) \\ &= f(0, X_0) + \int_0^t \frac{\partial f}{\partial u}(u, X_u) du + \int_0^t \frac{\partial f}{\partial x}(u, X_u) L_u dB_u \\ &\quad + \int_0^t \frac{\partial f}{\partial x}(u, X_u) a_u du + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(u, X_u) L_u^2 du \\ &= f(0, X_0) + \int_0^t f(u, X_u) \left(-\frac{1}{2}\right) L_u^2 du + \int_0^t f(u, X_u) L_u dB_u \\ &\quad + \int_0^t f(u, X_u) a_u du + \frac{1}{2} \int_0^t f(u, X_u) L_u^2 du \\ &= 1 + \int_0^t f(u, X_u) L_u dB_u \end{aligned}$$

with

$$\begin{aligned} \int_0^t \mathbb{E} |f(u, X_u) L_u|^2 du &= \int_0^t \mathbb{E} e^{2(X_u - \frac{1}{2} \int_0^u L_v^2 dv)} L_u^2 du \\ &\leq \sup_{u \in [0, t]} L_u^2 e^{-\int_0^t L_v^2 dv} \int_0^t \mathbb{E} e^{2X_u} du. \end{aligned}$$

So we have to compute $\mathbb{E} e^{2X_u}$. This is easy since X_u is a centered Gaussian random variable with variance $c(u) \leq \int_0^u L_v^2 dv$ and

$$\mathbb{E} e^{2X_u} = \mathbb{E} e^{2\sqrt{c_u} g} = e^{2c_u}.$$

This implies

$$\int_0^t \mathbb{E} |f(u, X_u) L_u|^2 du < \infty$$

and that $(\mathcal{E}(X)_t - 1)_{t \geq 0} \in \mathcal{M}_2^{c,0}$. The above integral equation can also be written as a differential equation

$$df(t, X_t) = f(t, X_t) L_t dB_t \quad \text{with} \quad f(0, X_0) = 1$$

or

$$d\mathcal{E}(X)_t = \mathcal{E}(X)_t dX_t \quad \text{with} \quad \mathcal{E}(X)_0 = 1.$$

Example 3.2.13 (Partial integration) For $\psi \in C^1[0, \infty)$ one has

$$\psi(t)X_t = \psi(0)X_0 + \int_0^t \psi(u)dX_u + \int_0^t X_u\psi'(u)du$$

which follows by using $f(t, x) := \psi(t)x$.

Example 3.2.14 Using Itô's formula we can now establish an important connection between stochastic differential equations and partial differential equations by one example. Assume the following parabolic PDE

$$\frac{\partial f}{\partial t} + \frac{x^2}{2} \frac{\partial^2 f}{\partial x^2} = 0 \quad (3.4)$$

for $(t, x) \in [0, T) \times (0, \infty)$ with the formal boundary condition $f(T, x) = \psi(x)$ for some fixed $T > 0$. The PDE is called *backwards* equation since the boundary condition is a condition about the *final* time point T . Assume now the *geometric Brownian motion*

$$S_t(\omega) := e^{B_t(\omega) - \frac{t}{2}} \quad \text{for } t \geq 0.$$

Applying Itô's formula we see that S is an Itô-process with

$$S_t = 1 + \int_0^t S_u dB_u.$$

What is the connection between the geometric Brownian motion and the PDE (3.4). Assume that $\mathbb{E}\psi(S_T)^2 < \infty$ and define

$$f(t, x) := \mathbb{E}\psi(xS_{T-t}).$$

One can show that there is some $\varepsilon > 0$ such that $f \in C^\infty((-\varepsilon, T) \times (0, \infty))$ and that f satisfies the PDE (3.4). The principal way (without details) is as follows:

Fact 3.2.15 Let $p(t, x, y) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}$ for $t > 0$ and $x, y \in \mathbb{R}$. Then

$$\frac{\partial p}{\partial t} - \frac{1}{2} \frac{\partial^2 p}{\partial x^2} = 0.$$

Moreover, letting $h : \mathbb{R} \rightarrow \mathbb{R}$ be a Borel function such that

$$\int_{\mathbb{R}} e^{-ax^2} |h(x)| dx < \infty,$$

then for

$$u(t, x) := \int_{\mathbb{R}} h(y)p(t, x, y) dy$$

one has that

$$\frac{\partial u}{\partial t} - \frac{1}{2} \frac{\partial^2 u}{\partial x^2} = 0$$

for $0 < t < \frac{1}{2a}$ and $x \in \mathbb{R}$.

Let now $\tilde{\psi}(x) := \psi(\exp(x - (T/2)))$. Applying the above fact we get an $\varepsilon > 0$ (without proof) such that for

$$\tilde{f}(t, x) := \mathbb{E}\tilde{\psi}(x + B_{T-t})$$

one has

$$\frac{\partial \tilde{f}}{\partial t} + \frac{1}{2} \frac{\partial^2 \tilde{f}}{\partial x^2} = 0$$

for $(t, x) \in (-\varepsilon, T) \times \mathbb{R}$. Using

$$f(t, x) = \mathbb{E}\psi(xS_{T-t}) = \mathbb{E}\tilde{\psi}\left(B_{T-t} + \frac{t}{2} + \log x\right) = \tilde{f}\left(t, \frac{t}{2} + \log x\right)$$

we can derive that f satisfies the PDE (3.4). Applying ITÔ's formula gives that

$$\begin{aligned} f(t, S_t) &= f(0, S_0) + \int_0^t \frac{\partial f}{\partial x}(u, S_u) S_u dB_u \\ &\quad + \int_0^t \frac{\partial f}{\partial u}(u, S_u) du + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(u, S_u) S_u^2 du \\ &= f(0, S_0) + \int_0^t \frac{\partial f}{\partial x}(u, S_u) S_u dB_u \end{aligned}$$

for $t \in (0, T)$ a.s. Without proof, we remark that

$$\psi(S_T) = f(0, S_0) + \int_0^T \frac{\partial f}{\partial x}(u, S_u) S_u dB_u \text{ a.s.}$$

by $t \uparrow T$ where the integrand of the stochastic integral is defined to be zero for $u = T$ and one has that

$$\int_0^T \mathbb{E} \left| \frac{\partial f}{\partial x}(u, S_u) S_u \right|^2 du < \infty.$$

Example 3.2.16 (Computation of moments) ([1], [2]) The following example is of importance in Stochastic Finance when the Brownian motion is replaced by the geometric Brownian motion. For simplicity we use the Brownian motion. Let $\psi : \mathbb{R} \rightarrow \mathbb{R}$ be a Borel function such that $\mathbb{E}\psi(B_T)^2 < \infty$ for some $T > 0$. Similarly to Example 3.2.14 we can define

$$f(t, x) := \mathbb{E}\psi(x + W_{T-t})$$

and get, for some $\varepsilon > 0$, a function $f \in C^\infty((-\varepsilon, T) \times \mathbb{R})$ which solves the PDE

$$\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} = 0.$$

Again, by ITÔ's formula

$$\psi(S_T) = f(0, S_0) + \int_0^T \frac{\partial f}{\partial x}(u, B_u) dB_u \text{ a.s.}$$

where

$$\int_0^T \mathbb{E} \left| \frac{\partial f}{\partial x}(u, B_u) \right|^2 du < \infty.$$

We are interested in

$$\mathbb{E} \left| \int_a^b \left[\frac{\partial f}{\partial x}(u, B_u) - \frac{\partial f}{\partial x}(a, B_a) \right] dB_u \right|^2$$

for $0 \leq a < b < T$, which can be interpreted as the quadratic one-step error if $\int_a^b \frac{\partial f}{\partial x}(u, B_u) dB_u$ is approximated by $\frac{\partial f}{\partial x}(a, B_a)(B_b - B_a)$. To compute the error we proceed formally as follows: by the ITÔ-isometry we have

$$\begin{aligned} \mathbb{E} \left| \int_a^b \left[\frac{\partial f}{\partial x}(u, B_u) - \frac{\partial f}{\partial x}(a, B_a) \right] dB_u \right|^2 \\ = \int_a^b \mathbb{E} \left[\frac{\partial f}{\partial x}(u, B_u) - \frac{\partial f}{\partial x}(a, B_a) \right]^2 du. \end{aligned}$$

Now we rewrite the expression under the integral by ITÔ's formula. We let

$$\mathcal{A} = \frac{\partial}{\partial t} + \frac{1}{2} \frac{\partial^2}{\partial x^2}$$

and get by ITO's formula that

$$\begin{aligned} \mathbb{E} \left[\frac{\partial f}{\partial x}(u, B_u) - \frac{\partial f}{\partial x}(a, B_a) \right]^2 &= \mathbb{E} \left[\frac{\partial f}{\partial x}(u, B_u) - \frac{\partial f}{\partial x}(a, B_a) \right]^2 \Big|_{u=a} \\ &\quad + \mathbb{E} \int_a^v \frac{\partial}{\partial x} \left[\frac{\partial f}{\partial x}(v, B_v) - \frac{\partial f}{\partial x}(a, B_a) \right]^2 dB_v \\ &\quad + \mathbb{E} \int_a^u \mathcal{A} \left[\frac{\partial f}{\partial x}(v, B_v) - \frac{\partial f}{\partial x}(a, B_a) \right]^2 dv \\ &= \mathbb{E} \int_a^u \mathcal{A} \left[\frac{\partial f}{\partial x}(v, B_v) - \frac{\partial f}{\partial x}(a, B_a) \right]^2 dv \\ &= \mathbb{E} \int_a^u \left[\frac{\partial^2 f}{\partial x^2}(v, B_v) \right]^2 dv \end{aligned}$$

since

$$\begin{aligned} &\frac{1}{2} \mathcal{A}_{(v,x)} \left[\frac{\partial f}{\partial x}(v, x) - \frac{\partial f}{\partial x}(a, y) \right]^2 \\ &= \left[\frac{\partial f}{\partial x}(v, x) - \frac{\partial f}{\partial x}(a, y) \right] \frac{\partial^2 f}{\partial x \partial t}(v, x) \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \frac{\partial}{\partial x} \left(\left[\frac{\partial f}{\partial x}(v, x) - \frac{\partial f}{\partial x}(a, y) \right] \frac{\partial^2 f}{\partial x^2}(v, B_v) \right) \\
& = \left[\frac{\partial f}{\partial x}(v, x) - \frac{\partial f}{\partial x}(a, y) \right] \frac{\partial^2 f}{\partial x \partial t}(v, x) \\
& \quad + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(v, x) \frac{\partial^2 f}{\partial x^2}(v, B_v) + \frac{1}{2} \left[\frac{\partial f}{\partial x}(v, x) - \frac{\partial f}{\partial x}(a, y) \right] \frac{\partial^3 f}{\partial x^3}(v, B_v) \\
& = \left[\frac{\partial f}{\partial x}(v, x) - \frac{\partial f}{\partial x}(a, y) \right] \left(\frac{\partial^2 f}{\partial x \partial t}(v, x) + \frac{1}{2} \frac{\partial^3 f}{\partial x^3}(v, B_v) \right) \\
& \quad + \frac{1}{2} \left(\frac{\partial^2 f}{\partial x^2}(v, x) \frac{\partial^2 f}{\partial x^2}(v, B_v) \right) \\
& = \frac{1}{2} \left(\frac{\partial^2 f}{\partial x^2}(v, x) \right)^2.
\end{aligned}$$

Summarizing the computations gives

$$\begin{aligned}
\mathbb{E} \left[\frac{\partial f}{\partial x}(u, B_u) - \frac{\partial f}{\partial x}(a, B_a) \right]^2 & = \mathbb{E} \int_a^b \int_a^u \mathbb{E} \left(\frac{\partial^2 f}{\partial x^2}(v, x) \right)^2 dv du \\
& = \mathbb{E} \int_a^b (b-u) \mathbb{E} \left(\frac{\partial^2 f}{\partial x^2}(u, B_u) \right)^2 du.
\end{aligned}$$

Now take a net $0 = t_0 < t_1 < \dots < t_n = T$ one can show that

$$\begin{aligned}
\mathbb{E} \left| \int_0^T \frac{\partial f}{\partial x}(u, B_u) dB_u - \sum_{k=1}^n \frac{\partial f}{\partial x}(t_{k-1}, B_{t_{k-1}}) (B_{t_k} - B_{t_{k-1}}) \right|^2 \\
= \sum_{k=1}^n \int_{t_{k-1}}^{t_k} (t_k - u) \mathbb{E} \left(\frac{\partial^2 f}{\partial x^2}(u, B_u) \right)^2 du.
\end{aligned}$$

Taking for example $\psi(x) := \chi_{[K, \infty)(x)}$ one can compute that

$$\mathbb{E} \left(\frac{\partial^2 f}{\partial x^2}(u, B_u) \right)^2 \sim \frac{1}{(T-u)^{\frac{3}{2}}}.$$

3.3 Proof of ITÔ's formula in a simple case

Throughout this section we assume an ITÔ process such that $L \in \mathcal{L}_2$. Before we start to prove ITÔ's formula we need the following lemmata:

Lemma 3.3.1 *Let $Y_n \rightarrow 0$ a.s. and $Z_n \rightarrow Z$ in probability. Then*

$$Y_n Z_n \rightarrow_{\mathbb{P}} 0.$$

The lemma will be an exercise.

Lemma 3.3.2 *Let $Y = (Y_t)_{t \geq 0}$ be a continuous and adapted, and assume that*

$$\sup_{t \geq 0, \omega \in \Omega} |Y_t(\omega)| < \infty.$$

Then, for $t_i^n := \frac{i}{n}t$, one has that

$$\sum_{i=1}^n Y_{t_{i-1}^n} (X_{t_i^n} - X_{t_{i-1}^n}) \xrightarrow{\mathbb{P}} \int_0^t Y_u dX_u, \quad (3.5)$$

$$\sum_{i=1}^n Y_{t_{i-1}^n} (X_{t_i^n} - X_{t_{i-1}^n})^2 \xrightarrow{\mathbb{P}} \int_0^t Y_u L_u^2 du. \quad (3.6)$$

The proof of this lemma is indicated at the end of the section.

Proof of Proposition 3.2.9. We shall prove ITÔ's formula in the case that $f^{(3)}$ exists and is continuous, and satisfies

$$\sup_{x \in \mathbb{R}, k=0,1,2,3} |f^{(k)}(x)| < \infty.$$

We fix $t > 0$ and let

$$t_i^n := \frac{i}{n}t$$

for $i = 0, \dots, n$ be the equidistant time-net on $[0, t]$. Then, a.s.,

$$\begin{aligned} & f(X_t) - f(X_0) \\ &= \sum_{i=1}^n [f(X_{t_i^n}) - f(X_{t_{i-1}^n})] \\ &= \sum_{i=1}^n \left[f'(X_{t_{i-1}^n})(X_{t_i^n} - X_{t_{i-1}^n}) + \frac{1}{2} f''(X_{t_{i-1}^n})(X_{t_i^n} - X_{t_{i-1}^n})^2 \right. \\ & \quad \left. + \frac{1}{6} f^{(3)}(\tilde{X}_{t_{i-1}^n})(X_{t_i^n} - X_{t_{i-1}^n})^3 \right] \\ &= \sum_{i=1}^n f'(X_{t_{i-1}^n})(X_{t_i^n} - X_{t_{i-1}^n}) + \sum_{i=1}^n \frac{1}{2} f''(X_{t_{i-1}^n})(X_{t_i^n} - X_{t_{i-1}^n})^2 \\ & \quad + \sum_{i=1}^n \frac{1}{6} f^{(3)}(\tilde{X}_{t_{i-1}^n})(X_{t_i^n} - X_{t_{i-1}^n})^3 \\ &=: I_1^n + \frac{1}{2} I_2^n + \frac{1}{6} I_3^n \end{aligned}$$

where we used TAYLOR's formula with the LAGRANGE remainder. Applying Lemma 3.3.2 we get, in probability, that

$$\lim_n I_1^n = \int_0^t f'(X_u) dX_u,$$

$$\lim_n I_2^n = \int_0^t f''(X_u) L_u^2 du.$$

To get

$$\lim_n I_3^n =_{\mathbb{P}} 0$$

we observe that

$$\begin{aligned} & \left| \sum_{i=1}^n f^{(3)}(\tilde{X}_{t_{i-1}^n})(X_{t_i^n} - X_{t_{i-1}^n})^3 \right| \\ & \leq \left[\sup_{x \in \mathbb{R}} |f^{(3)}(x)| \sup_{|u-v| \leq t/n} |X_u - X_v| \right] \left[\sum_{i=1}^n (X_{t_i^n} - X_{t_{i-1}^n})^2 \right] \\ & =: Y_n Z_n \end{aligned}$$

with $Y_n \rightarrow 0$ a.s. and $Z_n \rightarrow_{\mathbb{P}} \int_0^t L_u^2 du$ according to Lemma 3.3.2. \square

Proof of Lemma 3.3.2. (a) First we consider (3.5) and get

$$\sum_{i=1}^n Y_{t_{i-1}^n} (X_{t_i^n} - X_{t_{i-1}^n}) = \sum_{i=1}^n Y_{t_{i-1}^n} \int_{t_{i-1}^n}^{t_i^n} L_u dB_u + \sum_{i=1}^n Y_{t_{i-1}^n} \int_{t_{i-1}^n}^{t_i^n} a_u du.$$

By standard calculus the second term converges for all $\omega \in \Omega$ to

$$\int_0^t Y_u(\omega) a_u(\omega) du.$$

To consider the first term we let

$$K_u^{(n)} := Y_{t_{i-1}^n} L_u \quad \text{for } u \in (t_{i-1}^n, t_i^n]$$

and otherwise zero. Then

$$\begin{aligned} & \mathbb{E} \left| \sum_{i=1}^n Y_{t_{i-1}^n} \int_{t_{i-1}^n}^{t_i^n} L_u dB_u - \int_0^t Y_u L_u dB_u \right|^2 \\ & = \mathbb{E} \left| \sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} (Y_{t_{i-1}^n} L_u) dB_u - \int_0^t Y_u L_u dB_u \right|^2 \\ & = \mathbb{E} \left| \int_0^t (K_u^{(n)} - Y_u L_u) dB_u \right|^2 \\ & = \mathbb{E} \int_0^t |K_u^{(n)} - Y_u L_u|^2 du \\ & \rightarrow_n 0 \end{aligned}$$

by dominated convergence since

$$|K_u^{(n)} - Y_u L_u| \leq [2 \sup_{t \geq 0, \omega \in \Omega} |Y_t(\omega)|] |L_u|$$

and $\lim_n |K_u^{(n)} - Y_u L_u| = 0$ for all $\omega \in \Omega$.

(b) Now we consider (3.6) and prove this statement for $a_u \equiv 0$ and $|L_u(\omega)| \leq c$ for all $u \geq 0$ and $\omega \in \Omega$. We get

$$\begin{aligned} & \sum_{i=1}^n Y_{t_{i-1}^n} (X_{t_i^n} - X_{t_{i-1}^n})^2 - \int_0^t Y_u L_u^2 du \\ &= \sum_{i=1}^n Y_{t_{i-1}^n} \left[(X_{t_i^n} - X_{t_{i-1}^n})^2 - \int_{t_{i-1}^n}^{t_i^n} L_u^2 du \right] + \sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} (Y_{t_{i-1}^n} - Y_u) L_u^2 du \end{aligned}$$

where the second converges to zero for all $\omega \in \Omega$. To treat the first sum we let

$$d_i := (X_{t_i^n} - X_{t_{i-1}^n})^2 - \int_{t_{i-1}^n}^{t_i^n} L_u^2 du$$

and get a martingale difference sequence since

$$\mathbb{E} \left((X_{t_i^n} - X_{t_{i-1}^n})^2 | \mathcal{F}_{t_{i-1}^n} \right) = \mathbb{E} \left(\int_{t_{i-1}^n}^{t_i^n} L_u^2 du | \mathcal{F}_{t_{i-1}^n} \right) \quad a.s.$$

Consequently,

$$\begin{aligned} & \mathbb{E} \left| \sum_{i=1}^n Y_{t_{i-1}^n} \left[(X_{t_i^n} - X_{t_{i-1}^n})^2 - \int_{t_{i-1}^n}^{t_i^n} L_u^2 du \right] \right|^2 \\ &= \sum_{i=1}^n \mathbb{E} Y_{t_{i-1}^n}^2 d_i^2 \leq \sup_{\omega, u} Y_u(\omega)^2 \sum_{i=1}^n \mathbb{E} d_i^2. \end{aligned}$$

Finally,

$$\mathbb{E} d_i^2 = \mathbb{E} (X_{t_i^n} - X_{t_{i-1}^n})^4 - 2 \mathbb{E} (X_{t_i^n} - X_{t_{i-1}^n})^2 \int_{t_{i-1}^n}^{t_i^n} L_u^2 du + \mathbb{E} \left(\int_{t_{i-1}^n}^{t_i^n} L_u^2 du \right)^2$$

where

$$\int_{t_{i-1}^n}^{t_i^n} L_u^2 du \leq c^2 \frac{t}{n}.$$

Since

$$\sum_{i=1}^n \mathbb{E} (X_{t_i^n} - X_{t_{i-1}^n})^2 = \mathbb{E} \int_0^t L_u^2 du < \infty$$

it remains to show that

$$\sum_{i=1}^n \mathbb{E} (X_{t_i^n} - X_{t_{i-1}^n})^4 \rightarrow_n 0.$$

But this follows from the Burkholder-Davis-Gundy inequality

$$\mathbb{E} |X_{t_i^n} - X_{t_{i-1}^n}|^4 \leq c_4 \mathbb{E} \left(\int_{t_{i-1}^n}^{t_i^n} L_u^2 du \right)^2 \leq c_4 \left(\frac{t}{n} \right)^2 c^4.$$

□

Chapter 4

Stochastic differential equations

Stochastic differential equations (SDE's) play an important role in stochastic modeling. For example, in Economics solutions of the SDE's considered below are used to model share prices. In Biology solutions of stochastic partial differential equations (not considered here) describe sizes of populations.

4.1 What is a stochastic differential equation?

Stochastic differential equations are (for us) a formal abbreviation of *integral* equations as described now. For the rest of this chapter we fix a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_t)_{t \geq 0})$ which satisfies the usual assumptions and an $(\mathcal{F}_t)_{t \geq 0}$ -Brownian motion $B = (B_t)_{t \geq 0}$.

Definition 4.1.1 Let $x_0 \in \mathbb{R}$, $D \subseteq \mathbb{R}$ be an open set, and $\sigma, a : [0, \infty) \times D \rightarrow \mathbb{R}$ be continuous. A continuous and adapted stochastic process $X = (X_t)_{t \geq 0}$ is a solution of the *stochastic differential equation* (SDE)

$$dX_t = \sigma(t, X_t)dB_t + a(t, X_t)dt \quad \text{with} \quad X_0 = x_0 \quad (4.1)$$

provided that the following conditions are satisfied:

- (i) $X_t(\omega) \in D$ for all $t \geq 0$ and $\omega \in \Omega$.
- (ii) $X_0 \equiv x_0$.
- (iii) $X_t = x_0 + \int_0^t \sigma(u, X_u)dB_u + \int_0^t a(u, X_u)du$ for $t \geq 0$ a.s.

Let us give some examples of SDE's.

Example 4.1.2 (Brownian motion) A solution of

$$dX_t = dB_t \quad \text{and} \quad X_0 = 0$$

is the Brownian motion itself $B = (B_t)_{t \geq 0}$ since $B_t = \int_0^t 1dB_u$. We can take $D = \mathbb{R}$.

Example 4.1.3 (Geometric Brownian motion with drift) Letting $X_t := x_0 e^{cB_t + bt}$ with $x_0, b, c \in \mathbb{R}$ we obtain by Itô's formula that, a.s.,

$$\begin{aligned} X_t &= x_0 + \int_0^t cX_u dB_u + \int_0^t bX_u du + \frac{1}{2} \int_0^t c^2 X_u du \\ &= x_0 + \int_0^t cX_u dB_u + \int_0^t \left[b + \frac{1}{2}c^2 \right] X_u du \\ &= x_0 + \int_0^t \sigma X_u dB_u + \int_0^t aX_u du \end{aligned}$$

with

$$\begin{aligned} \sigma &:= c, \\ a &:= b + \frac{1}{2}c^2. \end{aligned}$$

Going the other way round by starting with a and σ , we get that

$$\begin{aligned} c &= \sigma, \\ b &= a - \frac{1}{2}\sigma^2. \end{aligned}$$

Consequently, the SDE

$$dX_t = \sigma X_t dB_t + aX_t dt \quad \text{with } X_0 = x_0$$

is solved by

$$X_t = x_0 e^{\sigma B_t + (a - \frac{1}{2}\sigma^2)t}.$$

We may use $D = \mathbb{R}$ for $\sigma(t, x) := \sigma x$ and $a(t, x) := ax$.

The following examples only provide the formal SDE's. We do not discuss solvability at this point.

Example 4.1.4 (Ornstein-Uhlenbeck process) Here one considers the SDE

$$dX_t = -cX_t dt + \sigma dB_t \quad \text{with } X_0 = x_0.$$

We close by some examples from Stochastic Finance.

Example 4.1.5 (Vasicek interest rate model) Here one considers that

$$dr_t = a(b - r_t)dt + \sigma dB_t \quad \text{with } r_0 \geq 0,$$

$\sigma \geq 0$, and $a, b > 0$ models an interest rate in Stochastic Finance. The problem with this model is that r_t might be negative if $\sigma > 0$. If $\sigma = 0$, then one gets as one solution

$$r_t = r_0 e^{-at} + b(1 - e^{-at})$$

so that the meaning of a and b become more clear: the interest rate moves from its initial value r_0 to the value b as $t \rightarrow \infty$ with a speed determined by the parameter a . If $\sigma > 0$ one tries to add a random perturbation to that.

Both, the Ornstein-Uhlenbeck process and the process in the Vasicek interest rate model are Gaussian processes since the diffusion coefficient is not random. Moreover, the Vasicek interest rate model process is a generalization of the Ornstein-Uhlenbeck process.

The drawback of a negative interest rate in the Vasicek model can be removed by the following model:

Example 4.1.6 (Cox-Ingersoll-Ross Model) For $a, b > 0$ and $\sigma \geq 0$ one proposes the SDE

$$dr_t = a(b - r_t)dt + \sigma\sqrt{r_t}dB_t \quad \text{with } r_0 > 0.$$

The difference to the VASICEK interest rate model is that the factor $\sqrt{r_t}$ is added in the diffusion part. This guarantees that the fluctuation is getting smaller if r_t is close to zero. In fact, the parameters can be adjusted that the trajectories stay positive (which should be surprising).

Instead of considering the interest rate r_0 as initial condition one can take into the account the whole interest curve as anticipated by the market at time $t = 0$ as initial condition. This yields to a considerably more complicated model, the HEATH-JARROW-MORTON model.

Example 4.1.7 (Heath-Jarrow-Morton model) We assume that $f(s, t)$ stands for the instantaneous interest rate at time t as anticipated by the market at time s with $0 \leq s \leq t < \infty$. In particular, $r_t = f(t, t)$ is the interest rate at time t . Now one considers the equation

$$f(t, u) = f(0, u) + \int_0^t \alpha(v, u)dv + \int_0^t \sigma(f(v, u))dB_v$$

with $f(0, u) = \Phi(u)$.

4.2 Strong Uniqueness of SDE's

We shall start with a beautiful lemma, the GRONWALL lemma.

Lemma 4.2.1 (Gronwall) *Let $A, B, T \geq 0$ and $f : [0, T] \rightarrow [0, \infty)$ be a continuous function such that*

$$f(t) \leq A + B \int_0^t f(s) ds$$

for all $t \in [0, T]$. Then one has that $f(T) \leq Ae^{BT}$.

Proof. Letting $g(t) := e^{-Bt} \int_0^t f(s) ds$ we deduce

$$\begin{aligned} g'(t) &= -Be^{-Bt} \int_0^t f(s) ds + e^{-Bt} f(t) \\ &= e^{-Bt} \left(f(t) - B \int_0^t f(s) ds \right) \leq Ae^{-Bt} \end{aligned}$$

and

$$g(T) = \int_0^T g'(t) dt \leq A \int_0^T e^{-Bt} dt \leq \frac{A}{B} (1 - e^{-BT}).$$

Consequently,

$$\begin{aligned} f(T) &\leq A + B \int_0^T f(t) dt = A + Be^{BT} g(T) \\ &\leq A + Be^{BT} \frac{A}{B} (1 - e^{-BT}) = Ae^{BT}. \end{aligned}$$

□

Proposition 4.2.2 (Strong uniqueness) *Suppose that for all $n = 1, 2, \dots$ there is a constant $c_n > 0$ such that*

$$|\sigma(t, x) - \sigma(t, y)| + |a(t, x) - a(t, y)| \leq c_n |x - y|$$

for $|x| \leq n$, $|y| \leq n$, and $t \geq 0$. Assume that $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ are solutions of the SDE (4.1). Then

$$\mathbb{P}(X_t = Y_t, t \geq 0) = 1.$$

Proof. We use the stopping times

$$\sigma_n := \inf \{t \geq 0 : |X_t| \geq n\} \quad \text{and} \quad \tau_n := \inf \{t \geq 0 : |Y_t| \geq n\}$$

where we assume that $n > |x_0|$. Letting $\rho_n := \min\{\sigma_n, \tau_n\}$ we obtain, a.s., that

$$\begin{aligned} & X_{t \wedge \rho_n} - Y_{t \wedge \rho_n} \\ &= \int_0^{t \wedge \rho_n} [a(u, X_u) - a(u, Y_u)] du + \int_0^{t \wedge \rho_n} [\sigma(u, X_u) - \sigma(u, Y_u)] dB_u. \end{aligned}$$

Hence

$$\begin{aligned} \mathbb{E} |X_{t \wedge \rho_n} - Y_{t \wedge \rho_n}|^2 &\leq 2\mathbb{E} \left| \int_0^{t \wedge \rho_n} [a(u, X_u) - a(u, Y_u)] du \right|^2 \\ &\quad + 2\mathbb{E} \left| \int_0^{t \wedge \rho_n} [\sigma(u, X_u) - \sigma(u, Y_u)] dB_u \right|^2 \\ &\leq 2t\mathbb{E} \int_0^{t \wedge \rho_n} |a(u, X_u) - a(u, Y_u)|^2 du \\ &\quad + 2\mathbb{E} \int_0^{t \wedge \rho_n} [\sigma(u, X_u) - \sigma(u, Y_u)]^2 du \\ &\leq (2t + 2)c_n^2 \mathbb{E} \int_0^{t \wedge \rho_n} |X_u - Y_u|^2 du \\ &\leq (2t + 2)c_n^2 \mathbb{E} \int_0^t |X_{u \wedge \rho_n} - Y_{u \wedge \rho_n}|^2 du. \end{aligned}$$

Now fix $T > 0$. The above computation gives

$$\mathbb{E} |X_{t \wedge \rho_n} - Y_{t \wedge \rho_n}|^2 \leq (2T + 2)c_n^2 \int_0^t \mathbb{E} |X_{u \wedge \rho_n} - Y_{u \wedge \rho_n}|^2 du$$

for $t \in [0, T]$. For

$$f(t) := \mathbb{E} |X_{t \wedge \rho_n} - Y_{t \wedge \rho_n}|^2$$

we may apply GRONWALL's lemma. The function f is continuous since for $t_k \rightarrow t$ one gets

$$\begin{aligned} \lim_k f(t_k) &= \lim_k \mathbb{E} |X_{t_k \wedge \rho_n} - Y_{t_k \wedge \rho_n}|^2 \\ &= \mathbb{E} \lim_k |X_{t_k \wedge \rho_n} - Y_{t_k \wedge \rho_n}|^2 \\ &= \mathbb{E} |X_{t \wedge \rho_n} - Y_{t \wedge \rho_n}|^2 \\ &= f(t) \end{aligned}$$

by dominated convergence as a consequence of (for example)

$$\mathbb{E} \sup_{t \in [0, T]} |X_{t \wedge \rho_n}|^2 \leq n^2$$

and the continuity of the processes X and Y . Exploiting GRONWALL's lemma with $A := 0$ and $B := (2T + 2)c_n$ yields

$$f(T) \leq Ae^{BT} = 0 \quad \text{and} \quad \mathbb{E} |X_{t \wedge \rho_n} - Y_{t \wedge \rho_n}|^2 = 0.$$

Since

$$\lim_n \rho_n = \infty$$

because X and Y are continuous processes, we get by FATOU's lemma that

$$\mathbb{E} |X_t - Y_t|^2 = \mathbb{E} \liminf_n |X_{t \wedge \rho_n} - Y_{t \wedge \rho_n}|^2 \leq \liminf_n \mathbb{E} |X_{t \wedge \rho_n} - Y_{t \wedge \rho_n}|^2 = 0.$$

Hence $\mathbb{P}(X_t = Y_t) = 1$ and, by the continuity of X and Y ,

$$\mathbb{P}(X_t = Y_t, t \geq 0) = 1.$$

□

Sometimes the assumptions of the above criteria are too strong. There is a nice extension:

Proposition 4.2.3 (Yamada-Tanaka) *Suppose that*

$$\sigma, a : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$$

are continuous such that

$$\begin{aligned} |\sigma(t, x) - \sigma(t, y)| &\leq h(|x - y|), \\ |a(t, x) - a(t, y)| &\leq K(|x - y|) \end{aligned}$$

for $x, y \in \mathbb{R}$, where $h : [0, \infty) \rightarrow [0, \infty)$ is strictly increasing with $h(0) = 0$ and $K : [0, \infty) \rightarrow \mathbb{R}$ is strictly increasing and concave with $K(0) = 0$, such that

$$\int_0^\varepsilon \frac{du}{K(u)} = \int_0^\varepsilon \frac{du}{h(u)^2} = \infty$$

for all $\varepsilon > 0$. Then any two solutions of (4.1) are indistinguishable.

Example 4.2.4 *One can take $h(x) := x^\alpha$ for $\alpha \geq \frac{1}{2}$.*

However, there is also the following example:

Example 4.2.5 *Let $\sigma : \mathbb{R} \rightarrow [0, \infty)$ be continuous such that*

- (i) $\sigma(x_0) = 0$,
- (ii) $\int_{x_0-\varepsilon}^{x_0+\varepsilon} \frac{dx}{\sigma^2(x)} < \infty$ and $\sigma(x) \geq 1$ if $|x - x_0| > \varepsilon$ for some $\varepsilon > 0$.

Then the SDE

$$dX_t = \sigma(X_t)dB_t \quad \text{with} \quad X_0 = x_0$$

has infinitely many solutions.

4.3 Existence of strong solutions of SDE's

The result we want to prove is

Proposition 4.3.1 *Suppose that $\sigma, a : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous such that*

$$\begin{aligned} |\sigma(t, x) - \sigma(t, y)| + |a(t, x) - a(t, y)| &\leq K|x - y|, \\ |\sigma(t, x)| + |a(t, x)| &\leq K(1 + |x|) \end{aligned}$$

for all $t \geq 0$, $x, y \in \mathbb{R}$, and some $K > 0$. Then there exists a solution of the SDE (4.1).

Proof. (a) We define a sequence of processes $X^{(k)} = (X_t^{(k)})_{t \geq 0}$ which converges to our solution:

$$\begin{aligned} X_t^{(0)} &:= x_0, \\ X_t^{(k+1)} &:= x_0 + \int_0^t \sigma(u, X_u^{(k)}) dB_u + \int_0^t a(u, X_u^{(k)}) du. \end{aligned}$$

(b) Let us fix $T > 0$. By induction we show that

$$\sup_{t \in [0, T]} \mathbb{E}|X_t^{(k)}|^2 =: A_k < \infty.$$

For $k = 0$ this is clear so let us consider the step from k to $k + 1$ where we get that

$$\begin{aligned} &\mathbb{E}|X_t^{(k+1)}|^2 \\ &\leq 3 \left[|x_0|^2 + \mathbb{E} \int_0^t \sigma(u, X_u^{(k)})^2 du + \mathbb{E} \left| \int_0^t a(u, X_u^{(k)}) du \right|^2 \right] \\ &\leq 3 \left[|x_0|^2 + \int_0^t \mathbb{E} K^2 (1 + |X_u^{(k)}|)^2 du + T \int_0^t \mathbb{E} K^2 (1 + |X_u^{(k)}|)^2 du \right] \\ &=: A_{k+1} < \infty \end{aligned}$$

since $\mathbb{E}(1 + |X_u^{(k)}|)^2 \leq 2(1 + A_k)$.

(c) Now we show some kind of CAUCHY sequence property for the sequence of processes $X^{(k)}$ to obtain an appropriate limit. First we decompose the difference $X_t^{(k+1)} - X_t^{(k)}$ almost surely as

$$\begin{aligned} X_t^{(k+1)} - X_t^{(k)} &= \int_0^t [\sigma(u, X_u^{(k)}) - \sigma(u, X_u^{(k-1)})] dB_u \\ &\quad + \int_0^t [a(u, X_u^{(k)}) - a(u, X_u^{(k-1)})] du \\ &=: M_t + C_t. \end{aligned}$$

Now

$$\mathbb{E} \sup_{t \in [0, T]} |C_t|^2 \leq TK^2 \int_0^T \mathbb{E} |X_u^{(k)} - X_u^{(k-1)}|^2 du$$

and

$$\begin{aligned} \mathbb{E} \sup_{t \in [0, T]} |M_t|^2 &\leq 4\mathbb{E}M_T^2 \\ &= 4\mathbb{E} \int_0^T |\sigma(u, X_u^{(k)}) - \sigma(u, X_u^{(k-1)})|^2 du \\ &\leq 4K^2 \int_0^T \mathbb{E} |X_u^{(k)} - X_u^{(k-1)}|^2 du \end{aligned}$$

by DOOB's inequality and ITO's isometry. Consequently,

$$\mathbb{E} \sup_{t \in [0, T]} |X_t^{(k+1)} - X_t^{(k)}|^2 \leq L \int_0^T \mathbb{E} |X_u^{(k)} - X_u^{(k-1)}|^2 du$$

for $L := 2(4K^2 + TK^2)$. We iterate the last equation and get

$$\begin{aligned} \mathbb{E} \sup_{t \in [0, T]} |X_t^{(k+1)} - X_t^{(k)}|^2 &\leq L^2 \int_0^T \int_0^{u_1} \mathbb{E} |X_v^{(k-1)} - X_v^{(k-2)}|^2 dv du_1 \\ &\leq L^k \int_0^T \int_0^{u_{k-1}} \cdots \int_0^{u_1} \mathbb{E} |X_v^1 - x_0|^2 dv du_1 \cdots du_{k-1} \\ &\leq L^k \frac{T^k}{k!} \sup_{v \in [0, T]} \mathbb{E} |X_v^{(1)} - x_0|^2 \\ &= c \frac{(LT)^k}{k!}. \end{aligned}$$

Using CHEBYCHEV's inequality we continue to

$$\mathbb{P} \left(\sup_{t \in [0, T]} |X_t^{(k+1)} - X_t^{(k)}| > \frac{1}{2^{k+1}} \right) \leq 4^{k+1} c \frac{(LT)^k}{k!} = (4c) \frac{(4LT)^k}{k!}.$$

Now we use the BOREL-CANTELLI lemma: letting

$$A_k := \left\{ \sup_{t \in [0, T]} |X_t^{(k+1)} - X_t^{(k)}| > \frac{1}{2^{k+1}} \right\}$$

we get

$$\sum_{k=1}^{\infty} \mathbb{P}(A_k) < \infty \quad \text{so that} \quad \mathbb{P} \left(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n \right) = 0$$

or

$$\mathbb{P} \left(\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n^c \right) = 1.$$

Assuming

$$\omega \in \Omega_0^T := \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n^c$$

we have

$$\sup_{t \in [0, T]} \left| X_t^{(n+1)}(\omega) - X_t^{(n)}(\omega) \right| \leq \frac{1}{2^{n+1}}$$

for $n \geq k(\omega)$. Define

$$Y_t^{(T)}(\omega) := \begin{cases} \lim_k X_t^{(k)}(\omega) & : \omega \in \Omega_0^T \\ 0 & : \omega \notin \Omega_0^T \end{cases}.$$

For this process one can show that it satisfies

$$Y_t^{(T)} = x_0 + \int_0^t \sigma(u, Y_u^{(T)}) dB_u + \int_0^t a(u, Y_u^{(T)}) du$$

on $[0, T]$. By the uniqueness argument for the strong solutions we also get that

$$\mathbb{P}(Y_t^{(T_1)} = Y_t^{(T_2)}) = 1$$

for $t \in [0, \min \{T_1, T_2\}]$. Hence, as already carried out earlier we may find a continuous and adapted process $X = (X_t)_{t \geq 0}$ such that

$$\mathbb{P}(X_t = Y_t^{(n)}) = 1 \quad \text{for all } t \in [0, n]$$

which turns out to be our solution. \square

Remark 4.3.2 From the above proof it follows that we obtain a Gaussian process in case of $\sigma(t, x) = \sigma(t)$ as the approximating processes $X^{(k)}$ are Gaussian and the L_2 -limit of Gaussian random variables is Gaussian as well.

4.4 Solutions of SDE's by a transformation of drift

To explain this method we need two fundamental theorems. The first one is a characterization of the Brownian motion by the quadratic variation. To introduce the quadratic variation we need

Proposition 4.4.1 *Let $M = (M_t)_{t \geq 0} \in \mathcal{M}_{\text{loc}}^{c,0}$ be a continuous local martingale starting in zero. Then there exists a continuous and adapted process $\langle M \rangle = (\langle M \rangle_t)_{t \geq 0}$, unique up to indistinguishableness, such that*

- (i) $0 = \langle M \rangle_0 \leq \langle M \rangle_s \leq \langle M \rangle_t$ for all $0 \leq s \leq t < \infty$,
- (ii) $\lim_n \left[\sum_{i=1}^n \left(M_{t_i^n} - M_{t_{i-1}^n} \right)^2 \right] = \langle M \rangle_t$ in probability for all $0 = t_0^n \leq \dots \leq t_n^n = t$ such that $\lim_n \sup_{i=1, \dots, n} |t_i^n - t_{i-1}^n| = 0$.

Definition 4.4.2 The process $\langle M \rangle$ is called *quadratic variation* of the local martingale M .

Proposition 4.4.3 For $L = (L_u)_{u \geq 0} \in \mathcal{L}_2^{\text{loc}}$ one has that

$$\left\langle \int_0^\cdot L_u dB_u \right\rangle_t = \int_0^t L_u^2 du \quad \text{for } t \geq 0 \text{ a.s.}$$

In particular, we have

Example 4.4.4 For the Brownian motion $B = (B_t)_{t \geq 0}$ one has that $\langle B \rangle_t = t$, $t \geq 0$, a.s.

The converse is true as well. For the following we assume the usual conditions for the stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$ although we work on the compact time interval $[0, T]$ only.

Proposition 4.4.5 (P. Lévy) Let $M = (M_t)_{t \geq 0}$ be a continuous adapted process such that $M_0 \equiv 0$. Then the following assertions are equivalent:

- (i) M is an $(\mathcal{F}_t)_{t \geq 0}$ -Brownian motion.
- (ii) $M \in \mathcal{M}_{\text{loc}}^{c,0}$ and $\langle M \rangle_t = t$, $t \geq 0$, a.s.

Now we turn to our second fundamental theorem:

Proposition 4.4.6 (Girsanov) Let $L = (L_t)_{t \geq 0} \in \mathcal{L}_2$ and assume that the process $(\mathcal{E}_t)_{t \geq 0}$ defined by

$$\mathcal{E}_t := \exp \left(- \int_0^t L_u dB_u - \frac{1}{2} \int_0^t L_u^2 du \right)$$

is a martingale. Let $T > 0$ and

$$dQ_T := \mathcal{E}_T d\mathbb{P}.$$

Then $(W_t)_{t \geq 0}$ with

$$W_t := B_t + \int_0^t \chi_{[0, T]}(u) L_u du$$

defines a Brownian motion $(W_t)_{t \geq 0}$ with respect to $(\Omega, \mathcal{F}, Q_T, (\mathcal{F}_t)_{t \geq 0})$.

Lemma 4.4.7 *Let $0 \leq t \leq T < \infty$.*

- (i) *The measures Q_t and Q_T coincide on \mathcal{F}_t .*
- (ii) *Assume that $Z : \Omega \rightarrow \mathbb{R}$ is \mathcal{F}_T -measurable such that $\mathbb{E}_{Q_T}|Z| < \infty$.
Then*

$$\mathbb{E}_{Q_T}(Z|\mathcal{F}_t) = \frac{\mathbb{E}(Z\mathcal{E}_T|\mathcal{F}_t)}{\mathcal{E}_t} \quad a.s.$$

Proof. (i) For $B \in \mathcal{F}_t$ one has

$$Q_T(B) = \int_B \mathcal{E}_T d\mathbb{P} = \int_B \mathbb{E}(\mathcal{E}_T|\mathcal{F}_t) d\mathbb{P} = \int_B \mathcal{E}_t d\mathbb{P} = Q_t(B)$$

where we have used that $(\mathcal{E}_t)_{t \geq 0}$ is a martingale.

(ii) We show that

$$\mathbb{E}\mathcal{E}_t \mathbb{E}_{Q_T}(|Z||\mathcal{F}_t) < \infty \quad \text{and} \quad \int_B \mathcal{E}_t \mathbb{E}_{Q_T}(Z|\mathcal{F}_t) d\mathbb{P} = \int_B Z \mathcal{E}_T d\mathbb{P}$$

for all $B \in \mathcal{F}_t$, which follows from

$$\begin{aligned} \infty &> \int_{\Omega} |Z| dQ_T \\ &= \int_{\Omega} \mathbb{E}_{Q_T}(|Z||\mathcal{F}_t) dQ_T \\ &= \int_{\Omega} \mathbb{E}_{Q_T}(|Z||\mathcal{F}_t) dQ_t \\ &= \int_{\Omega} \mathcal{E}_t \mathbb{E}_{Q_T}(|Z||\mathcal{F}_t) d\mathbb{P}, \end{aligned}$$

and, the other way round, by

$$\begin{aligned} \int_B \mathcal{E}_t \mathbb{E}_{Q_T}(Z|\mathcal{F}_t) d\mathbb{P} &= \int_B \mathbb{E}_{Q_T}(Z|\mathcal{F}_t) dQ_t \\ &= \int_B \mathbb{E}_{Q_T}(Z|\mathcal{F}_t) dQ_T \\ &= \int_B Z dQ_T \\ &= \int_B Z \mathcal{E}_T d\mathbb{P}. \end{aligned}$$

□

Proof of Proposition 4.4.6. We restrict ourselves to $L_t \equiv \mu \in \mathbb{R}$. In this case we have that $W_t = B_t + \mu t$ which is a Brownian motion with a deterministic drift. Moreover, using the same argument as in Example 3.1.5 we derive that $(\mathcal{E}_t)_{t \geq 0}$ is a martingale.

(a) First we show that $(W_t)_{t \in [0, T]}$ is a Q_T -martingale, that means $\mathbb{E}_{Q_T} |W_t| < \infty$ and $\mathbb{E}_{Q_T}(W_T | \mathcal{F}_t) = W_t$ a.s. for $0 \leq t \leq T$. The integrability is a consequence of

$$\begin{aligned} \mathbb{E}_{Q_T} |W_t| &= \int_{\Omega} |B_t + \mu t| \mathcal{E}_T d\mathbb{P} \\ &= \int_{\Omega} |B_t + \mu t| e^{-\mu B_T - \frac{\mu^2}{2} T} d\mathbb{P} \\ &\leq \int_{\Omega} |B_t| e^{-\mu B_T - \frac{\mu^2}{2} T} d\mathbb{P} + \mu t \int_{\Omega} e^{-\mu B_T - \frac{\mu^2}{2} T} d\mathbb{P} \\ &= \int_{\Omega} \sqrt{t} |g_1| e^{-\mu \sqrt{t} g_1 - \mu \sqrt{T-t} g_2 - \frac{\mu^2}{2} T} d\mathbb{P} + \mu t \\ &\leq \sqrt{t} \int_{\Omega} |g_1| e^{-\mu \sqrt{t} g_1} d\mathbb{P} \int_{\Omega} e^{-\mu \sqrt{T-t} g_2} d\mathbb{P} e^{-\frac{\mu^2}{2} T} + \mu t \end{aligned}$$

where g_1 and g_2 are independent standard Gaussian random variables. So we have to estimate, for an integer $n \geq 0$ and $c \in \mathbb{R}$, that

$$\int_{\Omega} |g|^n e^{cg} d\mathbb{P} = \int_{\mathbb{R}} |\xi|^n e^{c\xi} e^{-\frac{\xi^2}{2}} \frac{d\xi}{\sqrt{2\pi}} < \infty$$

which follows by an easy computation. Now we turn to the martingale property. Using Lemma 4.4.7 we need to show that

$$\frac{\mathbb{E}(W_T \mathcal{E}_T | \mathcal{F}_t)}{\mathcal{E}_t} = W_t \text{ a.s.}$$

which is equivalent to

$$\mathbb{E} \left((W_T - W_t) \frac{\mathcal{E}_T}{\mathcal{E}_t} | \mathcal{F}_t \right) + W_t \mathbb{E} \left(\frac{\mathcal{E}_T}{\mathcal{E}_t} | \mathcal{F}_t \right) = W_t \text{ a.s.}$$

Since

$$\mathbb{E} \left(\frac{\mathcal{E}_T}{\mathcal{E}_t} | \mathcal{F}_t \right) = 1 \text{ a.s.}$$

because $(\mathcal{E}_t)_{t \geq 0}$ is a martingale, we end up by checking that

$$\mathbb{E} \left((W_T - W_t) \frac{\mathcal{E}_T}{\mathcal{E}_t} | \mathcal{F}_t \right) = 0 \text{ a.s.}$$

Again, by independence,

$$\mathbb{E} \left((W_T - W_t) \frac{\mathcal{E}_T}{\mathcal{E}_t} | \mathcal{F}_t \right) = \mathbb{E} \left((W_T - W_t) \frac{\mathcal{E}_T}{\mathcal{E}_t} \right) \text{ a.s.}$$

Finally, we observe, for $s := T - t$, that

$$\mathbb{E} \left((W_T - W_t) \frac{\mathcal{E}_T}{\mathcal{E}_t} \right) = \int_{\mathbb{R}} (\sqrt{s}\xi + \mu s) e^{-\mu \sqrt{s}\xi - \frac{s\mu^2}{2}} e^{-\frac{\xi^2}{2}} \frac{d\xi}{\sqrt{2\pi}} = 0.$$

(b) Now we compute the quadratic variation of W . It should be the same as the quadratic variation of the Brownian motion B , since the only difference is a process of 1-variation. Let us take a sequence of nets

$$0 = t_0^n \leq \dots \leq t_n^n = t$$

with

$$\lim_n \max_{1 \leq i \leq n} |t_i^n - t_{i-1}^n| = 0.$$

We find a subsequence $(n_k)_{k=1}^\infty$ such that

$$\lim_k \sum_{i=1}^{n_k} |W_{t_i^{n_k}} - W_{t_{i-1}^{n_k}}|^2 = \langle W \rangle_t \quad \text{Q-a.s.}$$

Since $\mathbb{P} \sim Q$ we may replace Q-a.s. by \mathbb{P} -a.s. Finally, we have that

$$\begin{aligned} & \left(\sum_{i=1}^n |B_{t_i^n} - B_{t_{i-1}^n}|^2 \right)^{\frac{1}{2}} - \mu \left(\sum_{i=1}^n |t_i^n - t_{i-1}^n|^2 \right)^{\frac{1}{2}} \\ & \leq \left(\sum_{i=1}^n |W_{t_i^n} - W_{t_{i-1}^n}|^2 \right)^{\frac{1}{2}} \\ & \leq \left(\sum_{i=1}^n |B_{t_i^n} - B_{t_{i-1}^n}|^2 \right)^{\frac{1}{2}} + \mu \left(\sum_{i=1}^n |t_i^n - t_{i-1}^n|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Since

$$\begin{aligned} \lim_n \left(\sum_{i=1}^n |t_i^n - t_{i-1}^n|^2 \right) & \leq \lim_n \left[\sup_i |t_i^n - t_{i-1}^n| \right] \left(\sum_{i=1}^n |t_i^n - t_{i-1}^n| \right) \\ & = \lim_n t \sup_i |t_i^n - t_{i-1}^n| = 0 \end{aligned}$$

it follows that, a.s.,

$$\lim_k \left(\sum_{i=1}^{n_k} |B_{t_i^{n_k}} - B_{t_{i-1}^{n_k}}|^2 \right) = \lim_k \left(\sum_{i=1}^{n_k} |W_{t_i^{n_k}} - W_{t_{i-1}^{n_k}}|^2 \right).$$

Because of $\langle B \rangle_t = t$, $t \geq 0$, a.s., this implies that

$$\langle W \rangle_t = t, t \geq 0, \text{ a.s.}$$

Applying Proposition 4.4.5 we can conclude the proof. \square

Now we turn to the NOVIKOV condition, an important condition to decide whether $(\mathcal{E}_t)_{t \geq 0}$ is a martingale.

Proposition 4.4.8 *Assume that $M = (M_t)_{t \in [0, T]}$ is a continuous and local martingale such that*

$$\mathbb{E}e^{\frac{1}{2}\langle M \rangle_T} < \infty.$$

Then $\mathcal{E} = (\mathcal{E}_t)_{t \geq 0}$ with

$$\mathcal{E}_t := e^{M_t - \frac{1}{2}\langle M \rangle_t}$$

is a martingale.

Now we come back to our SDE's and show how the method of the transformation of drift works.

Proposition 4.4.9 (Transformation of drift) . *Let $\sigma, a : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous such that*

$$\begin{aligned} |\sigma(t, x) - \sigma(t, y)| + |a(t, x) - a(t, y)| &\leq K|x - y| \\ |\sigma(t, x)| + |a(t, x)| &\leq K(1 + |x|) \end{aligned}$$

for all $x, y \in \mathbb{R}$ and $t \geq 0$. Let $X = (X_t)_{t \geq 0}$ be the unique strong solution of

$$dX_t = \sigma(t, X_t)dB_t + a(t, X_t)dt$$

with $X_0 \equiv x_0 \in \mathbb{R}$. Let $T > 0$ and $L : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous such that

$$\mathbb{E}e^{\frac{1}{2}\int_0^T L(u, X_u)^2 du} < \infty$$

and let

$$W_t := B_t + \int_0^t L(u, X_u)du$$

for $u \in [0, T]$. Then, under Q_T with

$$dQ_T := \mathcal{E}_T d\mathbb{P} \quad \text{where} \quad \mathcal{E}_t := e^{-\int_0^t L_u B_u - \frac{1}{2}\int_0^t L_u^2 du},$$

one has that

$$dX_t = \sigma(t, X_t)dW_t + [a(t, X_t) - \sigma(t, X_t)L(t, X_t)] dt \quad \text{for } t \in [0, T].$$

What is our philosophy in this case? We wish to solve

$$dX_t = \sigma(t, X_t)dW_t + [a(t, X_t) - \sigma(t, X_t)L(t, X_t)] dt \quad \text{for } t \in [0, T].$$

For this purpose we construct a specific Brownian motion $W = (W_t)_{t \in [0, T]}$ on an appropriate stochastic basis $(\Omega, \mathcal{F}, Q_T; (\mathcal{F}_t)_{t \in [0, T]})$ so that this problem has the solution $X = (X_t)_{t \in [0, T]}$ which is called *weak solution*.

Proof of Proposition 4.4.9. By Propositions 4.2.2 and 4.3.1 there is a unique strong solution $X = (X_t)_{t \geq 0}$. Setting

$$M_t := \int_0^t (-L_u) dB_u$$

we get that $(\mathcal{E}_t)_{t \geq 0}$ is a martingale by NOVIKOV's condition (Proposition 4.4.8). The GIRSANOV Theorem (Proposition 4.4.6) gives that $(W_t)_{t \in [0, T]}$ is a Brownian motion with respect to Q_T . And finally (and also a bit formally)

$$\begin{aligned} dX_t &= \sigma(t, X_t) dB_t + a(t, X_t) dt \\ &= \sigma(t, X_t) (dB_t + L(t, X_t) dt) - \sigma(t, X_t) L(t, X_t) dt + a(t, X_t) dt \\ &= \sigma(t, X_t) dW_t + (a(t, X_t) - \sigma(t, X_t) L(t, X_t)) dt. \end{aligned}$$

□

Example 4.4.10 Let $\sigma(t, x) = x$, $a \equiv 0$, $x_0 = 1$, $S_t = e^{Bt - \frac{t}{2}}$, and

$$\mathbb{E} e^{\frac{1}{2} \int_0^T L(u, S_u)^2 du} < \infty.$$

Then

$$dS_t = S_t dW_t - S_t L(t, S_t) dt, \quad t \in [0, T], \quad \text{under } Q_T.$$

4.5 Weak solutions

In the previous section we already indicated the principle of weak solutions: we do not start with a stochastic basis, we construct a particular basis to our problem. The formal definition is as follows:

Definition 4.5.1 Assume that $\sigma, a : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous. A pair $(X_t, W_t)_{t \geq 0}$ is a *weak solution* of

$$dX_t = \sigma(t, X_t) dB_t + a(t, X_t) dt \quad \text{with } X_0 \equiv x_0$$

if there exists a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$ such that

- (i) $(\Omega, \mathcal{F}, \mathbb{P})$ is complete,
- (ii) all null-sets of \mathcal{F} belong to \mathcal{F}_0 ,
- (iii) the filtration is right-continuous, i.e. $\mathcal{F}_t = \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}$,
- (iv) $(W_t)_{t \geq 0}$ is an $(\mathcal{F}_t)_{t \geq 0}$ -Brownian motion,
- (v) $X_t = x_0 + \int_0^t \sigma(u, X_u) dW_u + \int_0^t a(u, X_u) du$, $t \geq 0$, a.s., and $X_0 \equiv x_0$.

To explain the usage of weak solutions we also introduce the notion of *path-wise uniqueness*.

Definition 4.5.2 The SDE

$$dX_t = \sigma(t, X_t)dB_t + a(t, X_t)dt \quad \text{with} \quad X_0 \equiv x_0,$$

$t \geq 0$, a.s., $X_0 \equiv x_0$, satisfies the *path-wise uniqueness* if any two solutions with respect to the same stochastic basis and Brownian motion are indistinguishable.

The application of this concept consists in

Proposition 4.5.3 *The existence of weak solutions together with the path-wise uniqueness implies the existence of strong solutions.*

This concept can be applied to the COX-INGERSOLL-ROSS SDE

$$dX_t = (a - bX_t)dt + \sigma\sqrt{X_t}dB_t \quad \text{with} \quad X_0 \equiv x_0 > 0 \quad (4.2)$$

on $[0, \tau]$ where $a, \sigma > 0$, $b \in \mathbb{R}$, and

$$\tau(\omega) := \inf \{t \geq 0 : X_t(\omega) = 0\}.$$

Proposition 4.5.4 *There exists a unique continuous and adapted solution of the SDE (4.2).*

We are not in a position to prove this proposition. The uniqueness can be deduced by exploiting the uniqueness criteria of YAMADA-TANAKA presented in Proposition 4.2.3 since

$$|\sqrt{|x|} - \sqrt{|y|}| \leq h(|x - y|)$$

with $h(x) := \sqrt{x}$ and

$$\int_0^\infty \frac{1}{h(x)^2} dx = \int_0^\infty \frac{1}{x} dx = \infty.$$

What we can do in more detail is to study the quantitative behavior of this equation.

Proposition 4.5.5 *One has the following:*

- (i) *If $a \geq \frac{\sigma^2}{2}$, then $\mathbb{P}(\tau = \infty) = 1$,*
- (ii) *if $0 < a < \frac{\sigma^2}{2}$ and $b \geq 0$, then $\mathbb{P}(\tau = \infty) = 0$,*
- (iii) *if $0 < a < \frac{\sigma^2}{2}$ and $b < 0$, then $\mathbb{P}(\tau = \infty) \in (0, 1)$.*

Proof. For $x, M > 0$ we let $(X_t^x)_{t \geq 0}$ be the solution of the COX-INGERSOLL-ROSS SDE starting in $x > 0$ and

$$\tau_M^x(\omega) := \inf \{t \geq 0 : X_t^x(\omega) = M\}.$$

(a) Define

$$s(x) := \int_1^x e^{\frac{2by}{\sigma^2}} y^{-\frac{2a}{\sigma^2}} dy.$$

Then

$$\frac{\sigma^2}{2} x s''(x) + (a - bx) s'(x) = 0$$

by an easy computation.

(b) Let $0 < \varepsilon < x < M$ and $\tau_{\varepsilon, M}^x := \tau_\varepsilon^x \wedge \tau_M^x$. By ITÔ's formula

$$\begin{aligned} s(X_{t \wedge \tau_{\varepsilon, M}^x}^x) &= s(x) + \int_0^{t \wedge \tau_{\varepsilon, M}^x} s'(X_s^x) dX_s^x + \frac{1}{2} \int_0^{t \wedge \tau_{\varepsilon, M}^x} s''(X_s^x) \sigma^2 X_s^x ds \\ &= s(x) + \int_0^{t \wedge \tau_{\varepsilon, M}^x} s'(X_s^x) \sigma \sqrt{X_s^x} dB_s \\ &\quad + \int_0^{t \wedge \tau_{\varepsilon, M}^x} \left[(a - bX_s^x) s'(X_s^x) + \frac{1}{2} s''(X_s^x) \sigma^2 X_s^x \right] ds \\ &= s(x) + \int_0^{t \wedge \tau_{\varepsilon, M}^x} s'(X_s^x) \sigma \sqrt{X_s^x} dB_s. \end{aligned}$$

(c) Since $X_{t \wedge \tau_{\varepsilon, M}^x}^x \in [\varepsilon, M]$ for all $t \geq 0$ we have that

$$\begin{aligned} \mathbb{E} \int_0^{t \wedge \tau_{\varepsilon, M}^x} s'(X_s^x)^2 \sigma^2 X_s^x ds &= \mathbb{E} \left| s \left(X_{t \wedge \tau_{\varepsilon, M}^x}^x \right) - s(x) \right|^2 \\ &\leq 4 \sup_{y \in [\varepsilon, M]} |s(y)|^2 \\ &=: c \\ &< \infty. \end{aligned}$$

Letting $t \rightarrow \infty$ gives that

$$\mathbb{E} \int_0^{\tau_{\varepsilon, M}^x} s'(X_s^x)^2 X_s^x \sigma^2 ds < \infty.$$

Since $X_s^x \geq \varepsilon$ for $s \in [0, \tau_{\varepsilon, M}^x]$ by definition and since

$$s'(x) = e^{\frac{2bx}{\sigma^2}} x^{-\frac{2a}{\sigma^2}} \geq e^{-2\frac{|b|M}{\sigma^2}} M^{-2\frac{a}{\sigma^2}} =: d > 0$$

we get that

$$\mathbb{E} \int_0^{\tau_{\varepsilon, M}^x} ds < \infty$$

so that $\mathbb{E} \tau_{\varepsilon, M}^x < \infty$ and $\tau_{\varepsilon, M}^x < \infty$ a.s.

(d) Now

$$s(x) = \mathbb{E} \left(s(X_{\tau_{\varepsilon, M}^x \wedge t}^x) - \int_0^{\tau_{\varepsilon, M}^x \wedge t} s'(X_s^x) \sigma \sqrt{X_s^x} dB_s \right)$$

and the boundedness of the integrand of the stochastic integral on $[0, \tau_{\varepsilon, M}^x \wedge t]$ yields that

$$s(x) = \mathbb{E} s(X_{\tau_{\varepsilon, M}^x \wedge t}^x).$$

By $t \rightarrow \infty$, dominated convergence, and the fact that $\tau_{\varepsilon, M}^x$ is almost surely finite, we conclude that

$$s(x) = \mathbb{E} s(X_{\tau_{\varepsilon, M}^x}^x) = s(M) \mathbb{P}(\tau_M^x < \tau_{\varepsilon}^x) + s(\varepsilon) \mathbb{P}(\tau_M^x > \tau_{\varepsilon}^x).$$

(e) Now we can prove our assertion.

Case (i) with $a \geq \frac{\sigma^2}{2}$: First we observe that

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} s(\varepsilon) &= \lim_{\varepsilon \downarrow 0} \int_1^{\varepsilon} e^{\frac{2by}{\sigma^2}} y^{-\frac{2a}{\sigma^2}} dy \\ &= - \lim_{\varepsilon \downarrow 0} \int_{\varepsilon}^1 \left[e^{2b\frac{y}{\sigma^2}} \right] y^{-\theta} dy \\ &= -\infty \end{aligned}$$

since $\theta = 2\frac{a}{\sigma^2} \geq 1$. Moreover, $\mathbb{P}(\tau_{\varepsilon}^x < \tau_M^x)$ decreases as $\varepsilon \downarrow 0$. Assume that

$$\lim_{\varepsilon \downarrow 0} \mathbb{P}(\tau_{\varepsilon}^x < \tau_M^x) = \delta > 0.$$

Then

$$s(x) \leq |s(M)| - \lim_{\varepsilon \downarrow 0} s(\varepsilon) \delta = -\infty$$

which is a contradiction. Hence

$$\lim_{\varepsilon \downarrow 0} \mathbb{P}(\tau_{\varepsilon}^x < \tau_M^x) = 0.$$

Since τ_{ε}^x is monotone in ε we conclude that

$$\mathbb{P}(\tau_0^x < \tau_M^x) \leq \mathbb{P} \left(\bigcap_{1/N < x}^{\infty} \left\{ \tau_{\frac{1}{N}}^x < \tau_M^x \right\} \right) = 0$$

Letting $M \rightarrow \infty$ gives $\tau_M^x(\omega) \uparrow \infty$ so that

$$\mathbb{P}(\tau_0^x < \infty) = \lim_{M \uparrow \infty} \mathbb{P}(\tau_0^x < \tau_M^x) = 0.$$

Hence $\mathbb{P}(\tau_0^x = \infty) = 1$.

Case (ii) and (iii): The condition $0 < a < \frac{\sigma^2}{2}$ gives that $\theta = 2\frac{a}{\sigma^2} < 1$ and

$$\lim_{\varepsilon \downarrow 0} s(\varepsilon) = - \int_0^1 e^{2b\frac{y}{\sigma^2}} y^{-\theta} dy \in \mathbb{R}$$

which is denoted by $s(0)$. Hence

$$s(x) = s(M) \lim_{N \rightarrow \infty} \mathbb{P}(\tau_M^x < \tau_{\frac{1}{N}}^x) + s(0) \lim_{N \rightarrow \infty} \mathbb{P}(\tau_{\frac{1}{N}}^x < \tau_M^x).$$

Next we claim (without proof!) that

$$\lim_{N \rightarrow \infty} \mathbb{P}(\tau_M^x < \tau_{\frac{1}{N}}^x) = \mathbb{P}(\tau_M^x < \tau_0^x) \text{ and } \lim_{N \rightarrow \infty} \mathbb{P}(\tau_{\frac{1}{N}}^x < \tau_M^x) = \mathbb{P}(\tau_0^x < \tau_M^x)$$

so that

$$\mathbb{P}(\tau_M^x < \tau_0^x) + \mathbb{P}(\tau_0^x < \tau_M^x) = 1$$

and

$$s(x) = s(M)\mathbb{P}(\tau_M^x < \tau_0^x) + s(0)\mathbb{P}(\tau_0^x < \tau_M^x).$$

If $b \geq 0$, then we have

$$\lim_{M \rightarrow \infty} s(M) = \infty$$

so that

$$\lim_{M \rightarrow \infty} \mathbb{P}(\tau_M^x < \tau_0^x) = 0 \text{ and } \lim_{M \rightarrow \infty} \mathbb{P}(\tau_M^x > \tau_0^x) = 1$$

(note that $\mathbb{P}(\tau_M^x = \tau_0^x) = 0$) and

$$\mathbb{P} \left(\bigcup_{M>0} \{\tau_M^x > \tau_0^x\} \right) = 1.$$

Because $\lim_M \tau_M^x(\omega) = \infty$ for all ω , this implies

$$\mathbb{P}(\tau_0^x < \infty) = 1.$$

If $b < 0$, then we have

$$\lim_{M \rightarrow \infty} s(M) =: s(\infty) \in (0, \infty)$$

and

$$s(x) = s(\infty)\mathbb{P}(\tau_0^x = \infty) + s(0)\mathbb{P}(\tau_0^x < \infty),$$

and $\mathbb{P}(\tau_0^x = \infty) \in (0, 1)$ as well as $\mathbb{P}(\tau_0^x < \infty) \in (0, 1)$.

□

Bibliography

- [1] S. Geiss. Quantitative approximation of certain stochastic integrals. *Stochastics and Stochastics Reports*, 73:241–270, 2002.
- [2] S. Geiss and M. Hujo. Interpolation and approximation in $L_2(\gamma)$. Submitted. 2004.
- [3] S. Geiss. *Stochastic Processes*.