

# **Models in Financial Mathematics 1** **-discrete time-**

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# Chapter 1

## Introduction

### a History of financial contracts

Financial contracts have been made already centuries ago. For example to have more security, farmers wanted to have a contract about how much they will get for their crop long before the harvest. They feared that

good harvest  $\Rightarrow$  prices go down.

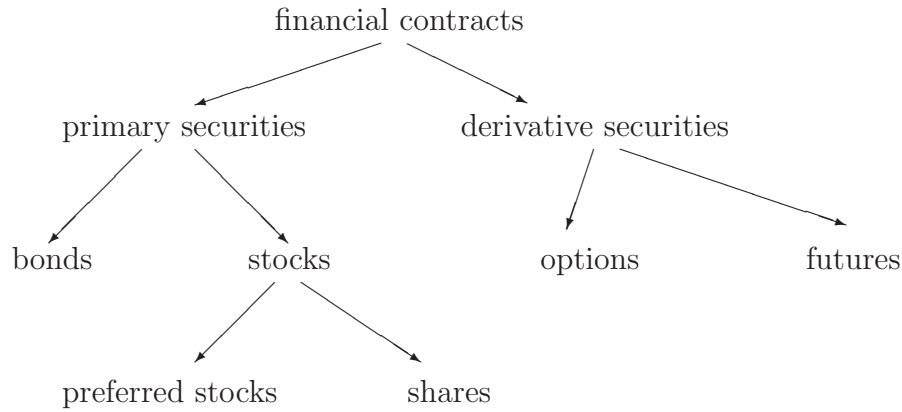
The customers of the farmers worried that

bad harvest  $\Rightarrow$  prices go up

so they also wanted an early agreement about the price. This led to *forward contracts/future contracts*.

Another idea was to create a joint ownership in order to finance the building of ships, starting a trade or a production of goods. For example if someone wanted to build or enlarge a factory but did not have enough money, he borrowed the money from others with the promise to share the profit with them. This in turn led to *shares*.

## b Types of financial contracts



**a security** is a piece of paper representing a promise

**bonds** are certificates issued by a government or a public company promising to repay a fixed interest rate at a specified time

**a share** (or stock) is a security representing partial ownership of a company and/or makes dividend payments according to the profits. Shares are traded on a *stock exchange*

**preferred stocks** are entitled to a fixed dividend

**an asset** (in Finance) is anything owned, whether in possession or by right to take possession, by a person or a company, the value of which can be expressed in monetary terms

stock }  
cash } current assets

land }  
building } fixed assets  
machinery }

goodwill }  
copyrights } intangible assets  
education }  
... }

**a forward contract** is an agreement between two parties to buy or sell an asset (which can be of any kind) at a pre-agreed future point in time.

**a futures contract** is a forward contract that has been standardized:

- amount to be traded: for example a fixed number of barrels of oil
- currency (US dollar often)
- quality
- delivery month
- last trading date

Future contracts are traded on a *futures exchange*

**an option** gives the holder of the option the right to buy (or sell) a security (shares) at a predefined time (or timeperiod) in the future and for a pre-determined amount.

- Types of options:
- stock options
  - foreign exchange options
  - interest rate options (=largest derivatives market in the world)
  - warrants
  - options on bonds
  - swaptions
  - ...

**long position** someone agrees to buy the asset

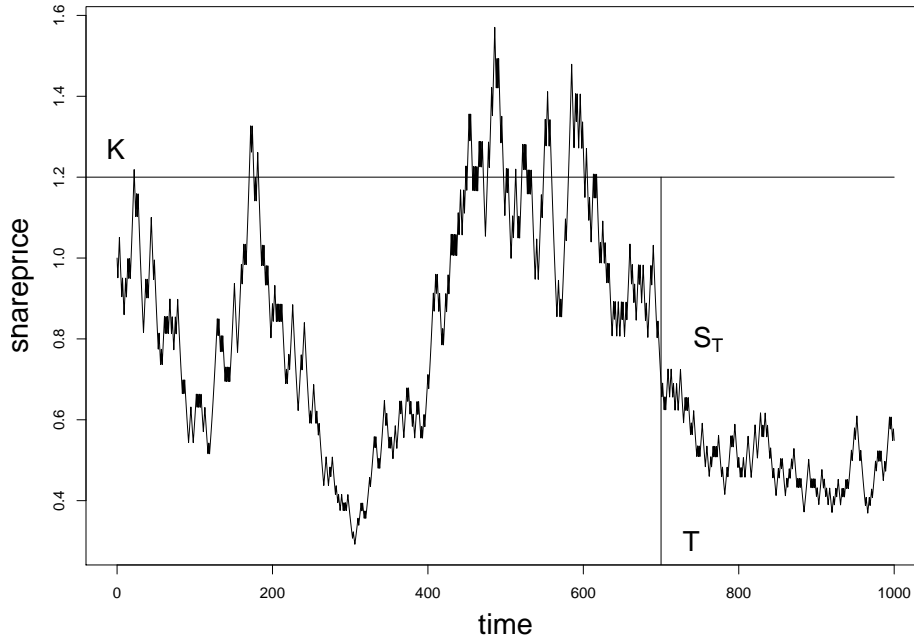
**short position** someone agrees to sell the asset

One purpose of derivatives is as a form of insurance to move risk from someone who cannot afford a major loss to someone who could absorb the loss, or is able to hedge against the risk by buying some other derivative.

The central topic of Financial Mathematics is the fair valuation of derivatives. One key equation used to value derivatives is the Black-Scholes-Equation (published 1973), (Fischer Black and Myron Scholes received the Nobel Prize in Economics for this) after 1973 trading with options increased rapidly.

## c Example: the European call option

Someone buys at time 0 a "European call option". Then he can (but does not have to) buy a given number of shares (1 share here) for a fixed price  $K$  (= the so called "strike price") at a fixed time  $T$ .



If  $S_T > K$  then he will buy the shares for the price  $K$  and if he sells them immediately his  
*gain* =  $S_T - K$  - price of the option

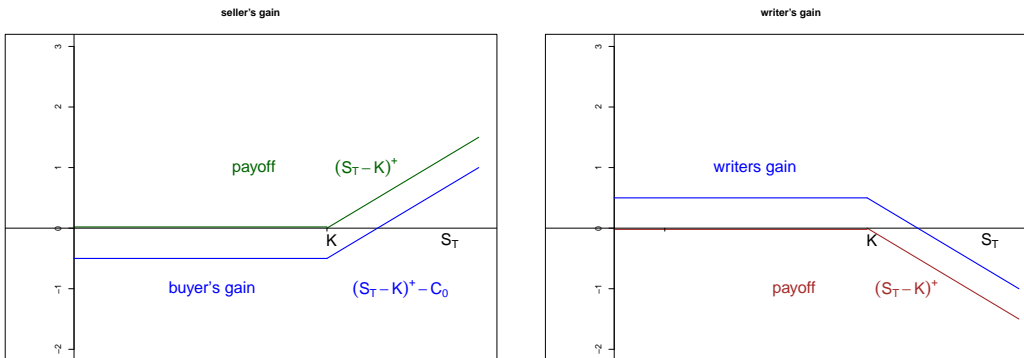
If  $S_T \leq K$  he will not buy and his  
*loss* = price of the option

**Question: How to determine a "fair" price for an option?**

1. If the price would be 0: then the *option holder* (= the one who bought the option) could make a riskless profit: this is against the "rules of the market"
2. if the option price is too high and if there is no sign that the share price  $S_T$  will be much higher than the strike price  $K$ , nobody will buy this option.

**Summary: European call-option;  $C_0$  :=option price**

The "gain" (outcome) of the	
option holder (= buyer)	writer (=seller of the option)
$\begin{cases} S_T - K - C_0 & \text{if } S_T > K \\ -C_0 & \text{if } S_T \leq K \end{cases}$	$\begin{cases} K - S_T + C_0 & \text{if } S_T > K \\ C_0 & \text{if } S_T \leq K \end{cases}$



**the purpose of a European call option:**

1. The writer reduces the risk in case  $S_t$  will go down: he gets  $C_0$ .
2. The buyer hopes that  $S_T > K + C_0$  and takes the risk that  $S_t$  will go down. In this case he loses the price  $C_0$  of the option.

*purpose:* it is a form of insurance (for the writer)

**What we will do in this course:**

- an easy mathematical model for the share price process  $(S_t)_{t=0}^T$ : (discrete-time) geometric Brownian motion  
determine the "fair" price for an option
- "hedging"

## A fair price of an European call option

$$f(S_T) = (S_T - K)^+ \quad \text{(Example)}$$

A fair price of  $f(S_T)$  would be a price where both the writer and the buyer could not make riskless profit. We consider the following example:

Assume 2 trading dates: 0 and T.

at time 0 share price  $S_0 = 20$  (dollar)

at time T  $S_T = \begin{cases} 20 \text{ (dollar) with probability } p \\ 7.5 \text{ (dollar) with probability } 1 - p \end{cases}$  ( $0 < p < 1$ ).

Let the strike price be  $K = 15$  (dollar).

$\Rightarrow$  the option writer has to pay  $\begin{cases} 5 \text{ (dollar) if } S_T = 20 \\ \text{nothing if } S_T = 7.5 \end{cases}$

We can do the following: "hedging" (=counterbalancing action to protect oneself from losing). Let us assume here for simplicity that the interest rate  $r = 0$ . That means one can borrow from the bank without paying interest. The writer sells the option, so he gets  $C_0$ . He borrows ( $-\varphi_0$ ) dollar from the bank and can buy

$$\varphi_1 = \frac{C_0 - \varphi_0}{10} \quad (S_0 = 10)$$

shares at time 0.

The portfolio  $(\varphi_0, \varphi_1)$  is correctly chosen if

$$\left. \begin{aligned} \varphi_0 1 + \varphi_1 20 &= 5 \\ \varphi_0 1 + \varphi_1 7.5 &= 0 \end{aligned} \right\}$$

We get

$$\begin{aligned} \varphi_0 &= -7.5\varphi_1 \\ 12.5\varphi_1 &= 5 \\ \varphi_1 &= \frac{5}{12.5} = 0.4 \end{aligned}$$

and

$$\begin{aligned} \varphi_0 &= -7.5 * 0.4 = -3 \\ C_0 &= 10\varphi_1 + \varphi_0 = 4 - 3 = 1 \end{aligned}$$

is the fair price for the option.

Hence at the time 0 the writer gets 1 dollar for the option and borrows 3 dollars from the bank. With these 4 dollars he can buy 0.4 shares.

*Case 1:*  $S_T = 20$ . The option is exercised at a cost of 5 dollars. The writer repays the loan (cost 3 dollars) and sells the shares (gain  $0.4 * 20 = 8$ )  
balance of trade:  $8 - 5 - 3 = 0$

*Case 2:*  $S_T = 7.5$  The option is not exercised (cost = 0). The writer repays the loan (cost 3 dollars) and sells the shares (gain  $0.4 * 7.5 = 3$ )  
balance of trade:  $3 - 3 = 0$

If  $C_0 > 1$ , then the writer can make (by hedging like above) the riskless profit  $C_0 - 1$ .

If  $C_0 < 1$  the option holder can make a riskless profit by the following procedure: buy the option (cost  $-C_0$ ), sell 0.4 shares (gain: 4) and put  $4 - C_0$  to the bank account. Then, at time T

$$\begin{cases} S_T = 20 : 4 - C_0 + 5 - 0.4 * 20 = 1 - C_0 \\ S_T = 7.5 : 4 - C_0 - 0.4 * 7.5 = 1 - C_0 \end{cases}$$

Where the 5 is from the option and  $1 - C_0$  is riskless profit.

### Summary

What we did in this example was:

1. found the hedge portfolio: we solved the equation

$$\varphi_0 + \varphi S_T = f(S_T) \quad \text{here } (= (S_t - K)^+)$$

$\Rightarrow$  one gets  $(\varphi_0, \varphi_1)$

2. found the "fair price": how much money does a trader need at time 0 to have the amount  $f(S_T)$  at time T?

He needs  $\varphi_0 + \varphi_1 S_0 =$  "fair price".

### Remark

One can compute the fair price of an option also by using a "martingale-measure". For this we need probability theory.



# Chapter 2

## Basics of Probability theory

### 1 Finite probability spaces

**Definition 1.1** Let  $\Omega = \{\omega_1, \dots, \omega_N\}$  be a finite set. Assume

$$\mathbb{P}(\{\omega_i\}) = p_i > 0, i = 1, \dots, N$$

$$\sum_{i=1}^N p_i = 1$$

Then  $\mathbb{P}$  is a *probability measure*. Let  $A \subseteq \Omega$ , then  $\mathbb{P}(A) := \sum_{\omega_i \in A} \mathbb{P}(\{\omega_i\})$

**Example 1.2** Rolling a die

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

$$\mathbb{P}(\{\omega\}) = \frac{1}{6}, \omega \in \Omega$$

$A :=$  "rolling an odd number"

$\mathbb{P}(A) = ?$

It follows from the definition that

$$\mathbb{P}(\Omega) = \sum_{i=1}^N \mathbb{P}(\{\omega_i\}) = \sum_{i=1}^N p_i = 1.$$

$$\mathbb{P}(\emptyset) = \sum_{\omega_i \in \emptyset} \mathbb{P}(\{\omega_i\}) = 0.$$

Let  $F := 2^\Omega$  be the *power set* of  $\Omega$   
= the set of *all* subsets of  $\Omega$

**Example 1.3**  $\Omega = \{1, 2\}, 2^\Omega = \{\{1, 2\}, \{1\}, \{2\}, \emptyset\}$  (it has  $2^{\#\Omega}$  elements)

**Definition 1.4** [  $\sigma$  - field,  $\sigma$  - algebra ]

Let  $\Omega$  be a non-empty set. A system  $\mathcal{F}$  of subsets  $A \subseteq \Omega$  is a  $\sigma$ -field or  $\sigma$ -algebra on  $\Omega$  if

1.  $\emptyset, \Omega \in \mathcal{F}$ ,
2.  $A \in \mathcal{F} \Rightarrow A^C := \Omega \setminus A \in \mathcal{F}$ ,
3.  $A_1, A_2, \dots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$ .

**Remark 1.5** If  $\Omega$  is finite, it is enough to check in (3) that  $A_1, \dots, A_M \in \mathcal{F}$  implies  $\bigcup_{i=1}^M A_i \in \mathcal{F}$ , for any  $M = 2, 3, \dots$

**Examples**

1.  $2^\Omega$  is a  $\sigma$ -field
2. Let  $\Omega$  be a set and assume  $A_1, \dots, A_M$  is a *finite partition* of  $\Omega$  i.e.  $A_1, \dots, A_M$  are mutually disjoint:

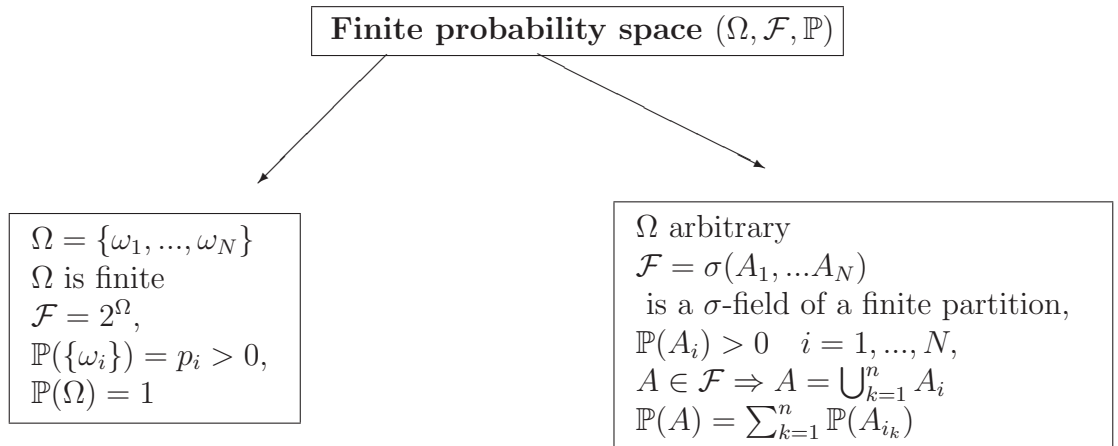
$$A_i \cap A_j = \emptyset, \forall i \neq j$$

and

$$\bigcup_{i=1}^M A_i = \Omega$$

$$\begin{aligned} \text{Then } \mathcal{F} &= \{ \bigcup_{i=1}^N A_{i_k} : \{i_1, \dots, i_n\} \subseteq \{1, \dots, N\}, n = 0, \dots, M \} \\ &= \{ \emptyset, A_1, \dots, A_M, A_1 \cup A_2, A_1 \cup A_3, \dots, \Omega \} \end{aligned}$$

is a  $\sigma$ -field. We say  $\mathcal{F}$  is *generated* by  $A_1, \dots, A_N$  and use the notation  $\sigma(A_1, \dots, A_N) := \mathcal{F}$ .



# Chapter 3

## The Cox-Ross-Rubinstein model (CRR-model, binomial tree model)

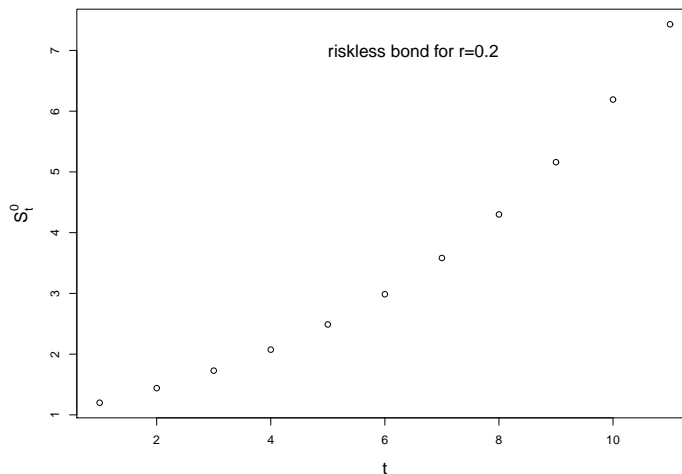
We want to model the time-development of shares and bonds with a simple model:

Assume  $\mathbb{T} = \{0, 1, \dots, T\}$  are trading dates ( $T =$  trading horizon).

$S^0 = (S_0^0, S_1^0, \dots, S_T^0)$  is a *riskless bond* (or *bank account*)

$S^1 := S = (S_1, \dots, S_T)$  is a risky (i.e. random) *stock*.

We assume a constant interest rate  $r > 0$ , i.e. if  $S_0^0 = 1$ , then  $S_1^0 = 1 + r$ ,  
 $S_k^0 = (1 + r)^k, k = 0, 1, \dots, T$



We model the random behavior of the stock  $S$  as follows:  $0 < p < 1$  fixed.

$$S_{n+1} = \begin{cases} S_n(1+a) & \text{with probability } 1-p \\ S_n(1+b) & \text{with probability } p \end{cases}$$

$$-1 < a < b$$

If we choose

$$\Omega := \{\omega = (\epsilon_1, \dots, \epsilon_T) : \epsilon_i \in \{1+a, 1+b\}\}$$

then

$$S_t(\omega) = S_0 \epsilon_1 \epsilon_2 \dots \epsilon_t, \quad t \in \mathbb{T}$$

Hence each  $\omega \in \Omega$  corresponds to one "possible case" of a stock development. We can also compute the probability of each case:

$$\mathbb{P}((\epsilon_1, \dots, \epsilon_T)) = p^k (1-p)^{T-k}$$

where

$$k := \#\{i : \epsilon_i = 1+b\}.$$

is the Binomial distribution.  $\mathbb{P}$  is clearly a probability measure on  $\Omega$  when:

$$\mathbb{P}(\omega) > 0 \quad \forall \omega \in \Omega$$

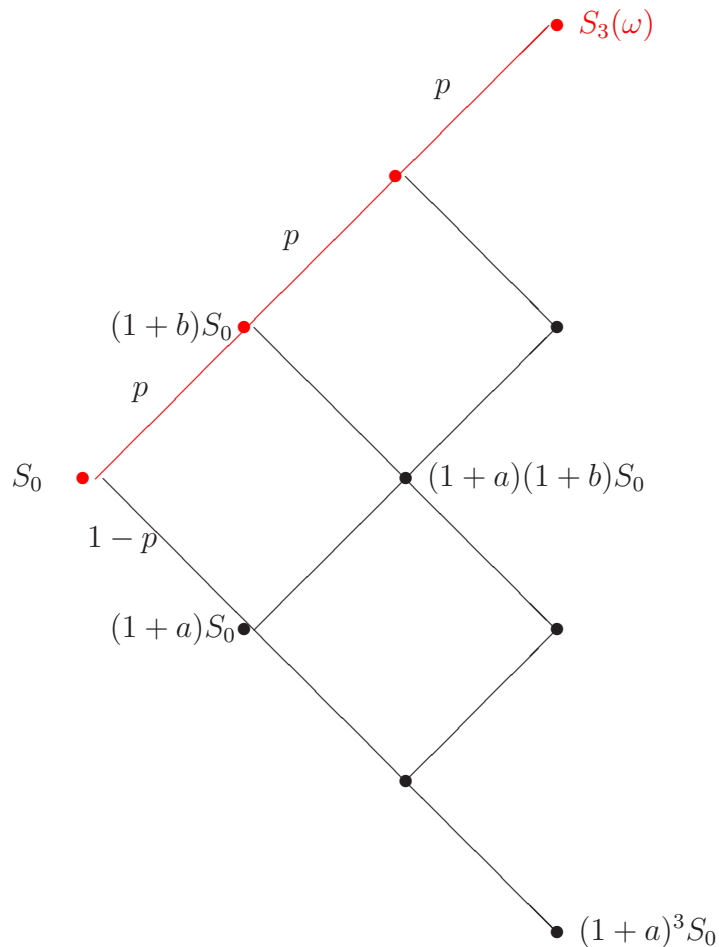
We have to check that

$$\mathbb{P}(\Omega) = 1$$

$$\begin{aligned} \mathbb{P}(\Omega) &= \sum_{\epsilon_i \in \{1+a, 1+b\}; i=1, \dots, T} \mathbb{P}((\epsilon_1, \dots, \epsilon_T)) \\ &= \sum_{k=0}^T \sum_{\epsilon_i \in \{1+a, 1+b\}} p^k (1-p)^{T-k} \\ &\quad \text{such that } k = \#\{i, \epsilon_i = 1+b\} \text{ hence:} \\ &= \sum_{k=0}^T \binom{T}{k} p^k (1-p)^{T-k} \quad \binom{T}{k} = \frac{T!}{k!(T-k)!} \\ &= (p + (1-p))^T = 1. \end{aligned}$$

**Example 0.6**  $\mathbb{T} = \{0, 1, 2, 3\}$

$$\omega = (1 + b, 1 + b, 1 + b)$$



## 1 Filtration

The investor does not know at time 0 how the values of  $S_t, t = 1, \dots, T$  will be. At time  $t > 0$  he knows all about  $S_0, S_1, \dots, S_t$  but nothing about  $S_{t+1}, \dots, S_T$ . We model the situation using a filtration.

**Definition 1.1** A *filtration* is an increasing sequence of  $\sigma$ -fields:

$$\{\emptyset, \Omega\} = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \dots \subseteq \mathcal{F}_T$$

**Definition 1.2** Assume  $f : \Omega \rightarrow \{m_1, \dots, m_N\}$ , and  $\mathcal{G}$  is a  $\sigma$ -field on  $\Omega$ . Then

$$f \text{ is } \mathcal{G}\text{-measurable} \Leftrightarrow f^{-1}(m_i) = \{\omega \in \Omega : f(\omega) = m_i\} \in \mathcal{G} \forall m_i$$

If we have functions  $f_1, f_2, \dots, f_l : \Omega \rightarrow \{m_1, \dots, m_N\}$  then  $\mathcal{G} = \sigma(f_1, \dots, f_l)$  denotes the smallest  $\sigma$ -field, such that all functions  $f_1, \dots, f_l$  are  $\mathcal{G}$ -measurable.

**Example 1.3** CRR model:

We assume  $\mathcal{F}_t = \sigma\{S_0, \dots, S_t\}$  is the information which the investor has till time  $t$ .

$$\mathbb{T} = \{0, 1, 2, \dots\} \quad S_0 := 1.$$

$$\Omega = \{\omega = (\epsilon_1, \epsilon_2) : \epsilon_i \in \{1+a, 1+b\}\}$$

$$\mathcal{F}_1 := \sigma\{S_0, S_1\} \quad S_0 \equiv 1$$

$$S_1(\omega) = 1+a \Leftrightarrow \begin{aligned} \omega &= (1+a, 1+a) \\ \text{or } \omega &= (1+a, 1+b) \end{aligned}$$

$$S_1(\omega) = 1+b \Leftrightarrow \begin{aligned} \omega &= (1+b, 1+a) \\ \text{or } \omega &= (1+b, 1+b) \end{aligned}$$

$$\Rightarrow \mathcal{F}_1 = \{\emptyset, \Omega, \{(1+a, 1+a), (1+a, 1+b)\}, \{(1+b, 1+a), (1+b, 1+b)\}\}$$

$$\begin{aligned} \text{but } S_2(\omega) &= (1+a)^2 \Leftrightarrow \omega = (1+a, 1+a) \\ S_2(\omega) &= (1+a)(1+b) \Leftrightarrow \omega = (1+a, 1+b) \text{ or } \omega = (1+b, 1+a) \\ S_2(\omega) &= (1+b)^2 \Leftrightarrow \omega = (1+b, 1+b) \end{aligned}$$

$$\Rightarrow S_2 \text{ is not } \mathcal{F}_1\text{-measurable.}$$

We say that  $(f_n)_{n=0}^T$  ( $f_n : \Omega \rightarrow \mathbb{R}$ ) is *adapted* to  $(\mathcal{F}_n)_{n=0}^T$  if it holds  $f_n$  is  $\mathcal{F}_n$ -measurable  $\forall n$ . If  $f_n$  is  $\mathcal{F}_{n-1}$ -measurable  $\forall n$  we say  $(f_n)_{n=0}^T$  is *predictable*.

## 2 Martingales and conditional expectation

We assume we have a finite probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . That means we can find a partition  $A_1, \dots, A_n$  of  $\Omega$  with  $\mathcal{F} = \sigma(A_1, \dots, A_n)$ .

Let  $f : \Omega \rightarrow \mathbb{R}$  be  $\mathcal{F}$ -measurable.

Then  $f$  can always be written like

$$f(\omega) = \sum_{i=1}^N a_i \mathbb{1}_{A_i}(\omega) \quad \text{with } a_i \in \mathbb{R}$$

using *indicator functions* which are defined by

$$\mathbb{1}_A(\omega) := \begin{cases} 1 & \omega \in A \\ 0 & \omega \in A^c \end{cases}$$

We define the *expectation of  $f$*  by

$$\mathbb{E}f := \sum_{i=1}^N a_i \mathbb{P}(A_i)$$

**Remark 2.1** Let  $\Omega = \{\omega_1, \dots, \omega_N\}$ . Then:

$$\mathbb{E}f := \sum_{i=1}^N f(\omega_i) \mathbb{P}(\{\omega_i\})$$

**Example 2.2** Rolling a die:

$$\begin{aligned} f(\omega_i) &= \omega_i \\ \Rightarrow &= 1 * \frac{1}{6} + 2 * \frac{1}{6} + \dots + 6 * \frac{1}{6} \\ &= \frac{1+\dots+6}{6} = 3,5 \end{aligned}$$

**Definition 2.3** Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a finite probability space and  $f : \Omega \rightarrow \mathbb{R}$  an  $\mathcal{F}$ -measurable function. Let  $\mathcal{G} \subseteq \mathcal{F}$  be a sub- $\sigma$ -field of  $\mathcal{F}$ . If

1.  $g : \Omega \rightarrow \mathbb{R}$  is  $\mathcal{G}$ -measurable and
- 2.

$$\mathbb{E}g \mathbb{1}_G = \mathbb{E}f \mathbb{1}_G \forall G \in \mathcal{G} \tag{3.1}$$

We say  $g$  is the *conditional expectation* of  $f$  with respect to  $\mathcal{G}$  and write  $g := \mathbb{E}[f|\mathcal{G}]$

**Example 2.4** Let  $\Omega = \{1, 2, \dots, 2^N\}$

$$\mathcal{G}_N = 2^\Omega \quad , \quad f(\omega) = \omega \quad , \quad \mathbb{P}(\{\omega\}) = \frac{1}{2^N}$$

As sub- $\sigma$ -field we choose

$$\mathcal{G}_{N-1} = \sigma\{\{1, 2\}\{3, 4\}, \dots, \{2^N - 1, 2^N\}\}$$

We want to compute  $\mathbb{E}[f|\mathcal{G}_{N-1}]$ . Clearly, if (3.1) holds for all sets

$$G = \{2k - 1, 2k\} \quad k = 1, \dots, 2^{N-1}$$

then (3.1) holds for all sets  $G \in \mathcal{G}_{N-1}$ .

By definition, if  $g := \mathbb{E}[f|\mathcal{G}_{N-1}]$ , then

$$g(2k - 1, 2k) = g(2k) \quad , \quad \forall k$$

$$\Rightarrow \mathbb{E}g\mathbb{1}_{\{2k-1, 2k\}} = \mathbb{E}f\mathbb{1}_{\{2k-1, 2k\}}$$

$$\begin{aligned} \mathbb{E}g\mathbb{1}_{\{2k-1, 2k\}} &= g(2k - 1)\mathbb{E}\mathbb{1}_{\{2k-1, 2k\}} \\ &= g(2k - 1)\mathbb{P}(\{2k - 1, 2k\}) = \frac{2}{2^N}g(2k - 1) \end{aligned}$$

On the other hand

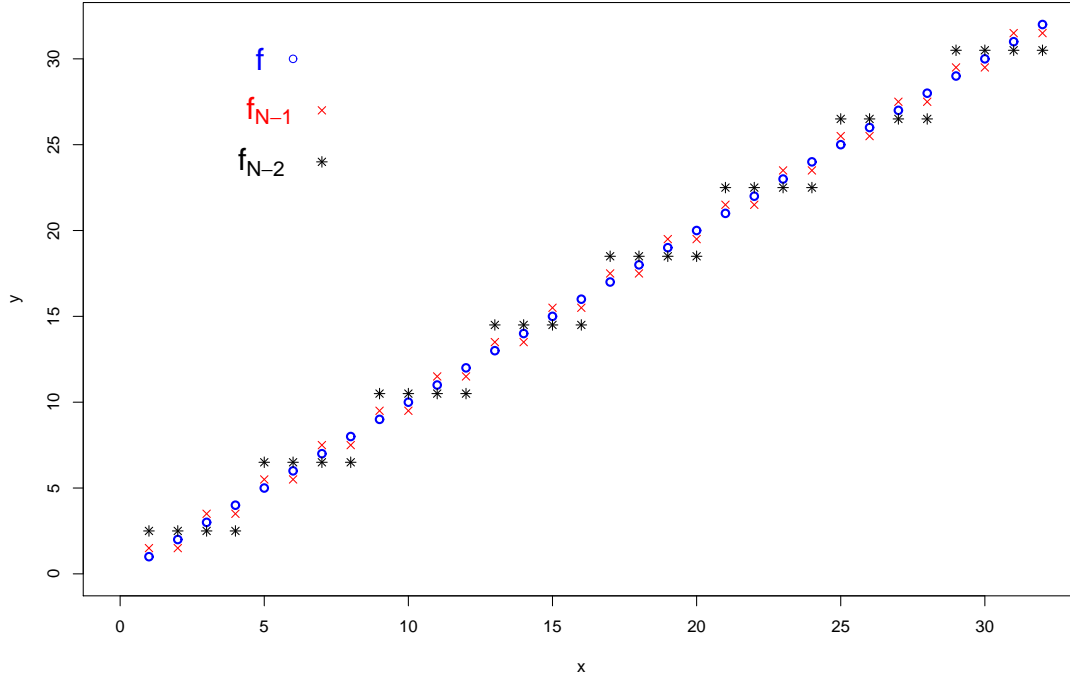
$$\begin{aligned} \mathbb{E}f\mathbb{1}_{\{2k-1, 2k\}} &= f(2k - 1)\mathbb{P}(\{2k - 1\}) + f(2k)\mathbb{P}(2k) \\ &= \frac{2k-1+2k}{2^N} \end{aligned}$$

$$\Rightarrow g(2k - 1) = g(2k) = \frac{2k - 1 + 2k}{2}$$

Iteration:

$$\begin{aligned} \mathbb{G}_{N-2} &:= \sigma\{\{1, 2, 3, 4\}, \dots, \{2^N - 3, 2^N - 2, 2^N - 1, 2^N\}\} \\ \mathbb{G}_0 &= \{\emptyset, \Omega\} \end{aligned}$$

$$\begin{aligned} \mathbb{E}[f|\mathcal{G}_{N-1}] &=: f_{N-1} \\ \mathbb{E}[f|\mathcal{G}_{N-2}] &=: f_{N-2} \\ \mathbb{E}f &=: f_0 \end{aligned}$$



**Remark 2.5**  $\mathcal{G}_0 \subseteq \mathcal{G}_1 \subseteq \dots \subseteq \mathcal{G}_N$  is a filtration.

$$f_k := \mathbb{E}[f | \mathcal{G}_k], \quad (f_k)_{k=0}^N$$

is an adapted sequence. Moreover it holds

$$\mathbb{E}[f_{k+1} | \mathcal{G}_k] = f_k \quad \forall k$$

**Definition 2.6** Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a finite probability space. An  $(\mathcal{F}_n)_{n=0}^T$  adapted process  $(M_n)_{n=0}^T$  is

1. a *martingale* if  $\mathbb{E}[M_{n+1} | \mathcal{F}_n] = M_n, \quad \forall 0 \leq n < T,$
2. a *supermartingale* if  $\mathbb{E}[M_{n+1} | \mathcal{F}_n] \leq M_n, \quad \forall 0 \leq n < T,$
3. a *submartingale* if  $\mathbb{E}[M_{n+1} | \mathcal{F}_n] \geq M_n, \quad \forall 0 \leq n < T,$



# Chapter 4

## Finite market models and non-arbitrage pricing

### 1 The market model

$(\Omega, \mathcal{F}, \mathbb{P})$  finite probability space (remember the convention  $\mathbb{P}(A) > 0 \quad \forall A \in \mathcal{F}, A \neq \emptyset$  "every event is possible").

**Trading dates** :  $\mathbb{T} = \{0, 1, \dots, T\}$

**The information** available to the investors at time  $t$  we model by the  $\sigma$ -field  $\mathcal{F}_t$  where we assume

$$\{\emptyset, \Omega\} = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \dots \subseteq \mathcal{F}_T = \mathcal{F}$$

**The securities** (assets) are modelled by a stochastic process in  $\mathbb{R}^{d+1}$ :

$$(S_t^0, S_t^1, \dots, S_t^d)_{t \in \mathbb{T}}$$

$S_t^0$ : bond (or bank account) nonrandom

$S_t^1, \dots, S_t^d$ : share prices at time  $t$  for  $d$  different shares: *random* (=depend on  $\omega$ )

We want that  $S^i$  is  $(\mathcal{F}_t)$ -adapted ,  $i = 1, \dots, d$ .

This we can achieve by setting

$$\mathcal{F}_t := \sigma(S_u^1, \dots, S_u^d : 0 \leq u \leq t)$$

The tuple  $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{T}, (\mathcal{F}_t), (S_t^0, \dots, S_t^d))$  is the (securities) market model.

## 2 Strategies

**Example 2.1** A *trading strategy* is a predictable process

$$\varphi = (\varphi_t^0, \dots, \varphi_t^d)_{t=1}^T$$

$\varphi_t^i =$  number of shares of asset  $i$  the investor has at time  $t$ .

$t$  fixed:  $(\varphi_t^0, \dots, \varphi_t^d)$  is called the *portfolio* at time  $t$ .

The *value process* (wealth process)  $V_t(\varphi)$  is given by

$$V_0(\varphi) = \varphi_1 \cdot S_0 \quad \text{the investors initial wealth}$$

$$V_t(\varphi) = \varphi_t \cdot S_t = \sum_{i=0}^d \varphi_t^i S_t^i \quad \forall t \in \mathbb{T}, t \geq 1$$

The investor trades at time  $t - 1 \Rightarrow$  portfolio  $\varphi_t$ . At time  $t$  he will have  $\varphi_t \cdot S_t = V_t(\varphi)$ . If he uses exactly his wealth  $V_t(\varphi)$  to trade at time  $t$ , then it must hold

$$V_t(\varphi) = \varphi_t \cdot S_t = \varphi_{t+1} \cdot S_t.$$

where  $\varphi_t S_t$  is the wealth which comes out from choosing  $\varphi_t$  at time  $t - 1$  and  $\varphi_{t+1} S_t$  the needed wealth to buy the portfolio  $\varphi_{t+1}$  at time  $t$ .

We call  $\varphi$  *self-financing* if

$$\varphi_t S_t = \varphi_{t+1} S_t, \quad t = 1, \dots, t - 1$$

Let us introduce *discounted prices*:  $S^0$  models the bond, i.e. for example,

$$S_t^0 = (1 + r)^t$$

if we assume a constant interest rate, but always it holds

$$S_t^0 > 0, \quad t = 0, \dots, T$$

Then

$$\tilde{S}_t = \left(1, \frac{S_t^1}{S_t^0}, \dots, \frac{S_t^d}{S_t^0}\right)$$

is the vector of *discounted prices*. (In case of interest rate  $r = 0$  the discounted price and the share price are equal). Now

$$\tilde{V}_t(\varphi) = \frac{1}{S_t^0}(\varphi_t \cdot S_t) = \varphi_t \cdot \tilde{S}_t$$

is the *discounted value* of  $\varphi$  at  $t$ .

**Proposition 2.2** The following assertions are equivalent.

1.  $\varphi$  is self-financing

2.  $V_t(\varphi) = V_0(\varphi) + \sum_{k=1}^t \varphi_k \cdot (S_k - S_{k-1}), \quad 1 \leq t \leq T$
3.  $\tilde{V}_t(\varphi) = V_0(\varphi) + \sum_{k=1}^t \varphi_k \cdot (\tilde{S}_k - \tilde{S}_{k-1}), \quad 1 \leq t \leq T$

*Proof* (1)  $\Leftrightarrow$  (2) : We know that

$$V_t(\varphi) = \varphi_t \cdot S_t = \sum_{i=0}^d \varphi_t^i S_t^i$$

and this gives

$$\begin{aligned} V_t(\varphi) &= (V_t(\varphi) - V_{t-1}(\varphi)) + \cdots + (V_1(\varphi) - V_0(\varphi)) + V_0(\varphi) \\ &= (\varphi_t \cdot S_t - \varphi_{t-1} \cdot S_{t-1}) + \cdots + (\varphi_1 \cdot S_1 - \varphi_1 \cdot S_0) + \varphi_1 \cdot S_0 \\ &= \varphi_t \cdot (S_t - S_{t-1}) + \cdots + \varphi_1 \cdot (S_1 - S_0) + V_0(\varphi) \end{aligned}$$

which is self financing, hence:

$$\varphi_{t-1} \cdot S_{t-1} = \varphi_t \cdot S_{t-1}$$

$$(1) \Leftrightarrow (3) : \varphi_t \cdot S_t = \varphi_{t+1} \cdot S_t \Leftrightarrow \varphi_t \cdot \frac{S_t}{S_0} = \varphi_{t+1} \cdot \frac{S_t}{S_0}$$

□

**Example 2.3** a self financing strategy  $\varphi$

	bank account	first share	second share
time	$S^0$	$S^1$	$S^2$
0	1	20	50
1	$1+0.05$	25	40
2	$(1+0.05)^2$	23	45

**Day 0:** investors money:  $V_0(\varphi) = 300\$$ . The portfolio chosen at time 0

$$\begin{aligned} \varphi_1 &= (\varphi_1^0, \varphi_1^1, \varphi_1^2) \\ &= (100, 5, 2) \end{aligned}$$

$$V_0(\varphi) = \varphi_1 \cdot S_0 = 100 * 1 + 5 * 20 + 2 * 50 = 300$$

**Day 1:** investors value of  $\varphi_1 : V_1(\varphi) = \varphi_1 \cdot S_1$

$$V_1(\varphi) = 105 + 5 * 25 + 2 * 40 = 310$$

which is the amount that can be used for the new portfolio  $\varphi_2$ . It is *self-financing*:

$$\varphi_1 \cdot S_1 = 310 \stackrel{!}{=} \varphi_2 S_1$$

If  $\varphi_2 = (\frac{70}{1,05}, 8, 1)$ , then

$$\varphi_2 \cdot S_1 = 70 + 8 * 25 + 1 * 40 = 310$$

**Day 2:**

$$V_2(\varphi) = \frac{70}{1,05} * (1,05)^2 + 8 * 23 + 45 = 302,5$$

**Proposition 2.4** For any predictable process  $(\varphi_t^1, \dots, \varphi_t^d)_{t=1}^T$  and for any  $V_0 \in \mathbb{R}$ , there exists a unique predictable process  $(\varphi_t^0)_{t=1}^T$  such that the strategy  $\varphi = (\varphi^0, \varphi^1, \dots, \varphi^d)$  is self-financing and  $V_0(\varphi) = V_0$ .

*Proof* if  $\varphi$  is self-financing we get by Proposition 2.2 (3)

$$\begin{aligned} \tilde{V}_t(\varphi) &= V_0(\varphi) + \sum_{k=1}^t \varphi_k \cdot (\tilde{S}_k - \tilde{S}_{k-1}) \\ &= V_0(\varphi) + \sum_{k=1}^t \varphi_k^0 (\tilde{S}_k^0 - \tilde{S}_{k-1}^0) + \varphi_k^1 (\tilde{S}_k^1 - \tilde{S}_{k-1}^1) + \dots \\ &\quad + \varphi_k^d (\tilde{S}_k^d - \tilde{S}_{k-1}^d). \end{aligned} \quad (4.1)$$

On the other hand,

$$\tilde{V}_t(\varphi) = \varphi_t \cdot \tilde{S}_t = \varphi_t^0 + \varphi_t^1 \tilde{S}_t^1 + \dots + \varphi_t^d \tilde{S}_t^d \quad (4.2)$$

Now we set (4.1) = (4.2) and get

$$\begin{aligned} \varphi_t^0 &= V_0(\varphi) + \sum_{k=1}^t \sum_{j=1}^d \varphi_k^j (\tilde{S}_k^j - \tilde{S}_{k-1}^j) - \sum_{j=1}^d \varphi_t^j \tilde{S}_t^j \\ &= V_0 + \sum_{k=1}^{t-1} \sum_{j=1}^d \varphi_k^j (\tilde{S}_k^j - \tilde{S}_{k-1}^j) - \sum_{j=1}^d \varphi_t^j \tilde{S}_{t-1}^j \end{aligned}$$

By this we get that  $(\varphi_t^0)_{t=1}^T$  is uniquely defined and predictable.  $\square$

Predictable means that  $\varphi_t^0$  is  $\mathcal{F}_{t-1}$ -measurable. This is true because:

1.  $V_0$  is a constant  $\Leftrightarrow \mathcal{F}_0 \subseteq \mathcal{F}_{t-1}$  measurable
2.  $\varphi_t^j$  is  $\mathcal{F}_{t-1}$ -measurable  $j = 1, \dots, d$
3.  $S_{t-1}$  is  $\mathcal{F}_{t-1}$ -measurable
4. plus (+) and times (\*) keeps measurability

### 2.0.1 Questions for the future

1. How can we get market models

$$(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{T}, (\mathcal{F}_t), (S_t^0, S_t^1, \dots, S_t^d))$$

where riskless profit is not possible?

2. Is there always a self-financing strategy  $\varphi$  to hedge the pay-off:  $V_T(\varphi) = f(S_T)$ ?
3. How to compute the fair price for an option?

## 3 Conditional expectation: Properties

**Proposition 3.1** Assume  $\Omega$  is a set and  $A_1, \dots, A_N$  is a partition of  $\Omega$ . Set  $\mathcal{F} = \sigma(A_1, \dots, A_N)$

1. A function  $f : \Omega \rightarrow \mathbb{R}$  is  $\mathcal{F}$ -measurable
  - $\Leftrightarrow$   $f$  is constant on  $A_1, \dots, A_N$ , i.e.  $f$  can be represented by
 
$$f(\omega) = \sum_{i=1}^N a_i \mathbb{1}_{A_i}(\omega), \quad a_i \in \mathbb{R}, \omega \in \Omega.$$
2. If  $f_1$  and  $f_2$  are  $\mathcal{F}$ -measurable, then  $af_1 + bf_2$  is  $\mathcal{F}$ -measurable ( $a, b \in \mathbb{R}$ ), and  $f_1 \times f_2$  is  $\mathcal{F}$ -measurable.
3.  $\mathbb{E}(af_1 + bf_2) = a\mathbb{E}f_1 + b\mathbb{E}f_2$ , for  $f_1, f_2$   $\mathcal{F}$ -measurable and  $a, b \in \mathbb{R}$

*Proof* (1) " $\Leftarrow$ " Assume

$$f = \sum_{i=1}^N a_i \mathbb{1}_{A_i}, \quad a_i \in \mathbb{R}$$

If all the  $a_i$ 's are different, then

$$f^{-1}(a_i) = A_i \in \mathcal{F}, \quad i = 1, \dots, N$$

If some  $a_i$ 's are equal, we have

$$f = \sum_{i=1}^n b_j \mathbb{1}_{B_j}, \quad b_j \text{'s different}$$

and  $B_1, \dots, B_n$  is a partition on  $\Omega$ , all  $B_j$ 's are unions of some  $A_i$ 's

$$\Rightarrow B_j \in \mathcal{F} \quad \forall j$$

Since  $f^{-1}(b_j) = B_j$ ,  $\Rightarrow f$  is  $\mathcal{F}$ -measurable.

" $\Rightarrow$ " Assume  $f$  is *not* constant on all  $A_1, \dots, A_N$ . We will show that then  $f$  is not  $\mathcal{F}$ -measurable. There exists  $A_j : f$  that is not constant on  $A_j$ :

$$\begin{aligned} \Rightarrow \omega_1, \omega_2 \in A_i \quad a &= f(\omega_1) \neq f(\omega_2) = b \\ \Rightarrow \omega_1 \in f^{-1}(a) \\ \omega_2 &\in f^{-1}(b) \end{aligned}$$

Because  $f$  is a function we have

$$f^{-1}(a) \cap f^{-1}(b) = \emptyset$$

But  $\mathcal{F}$  consists only of unions of  $A_1, \dots, A_N$ , that means for any set  $A \in \mathcal{F}$  it holds

$$\begin{aligned} \text{either } \{\omega_1, \omega_2\} &\subseteq A \\ \text{or } \{\omega_1, \omega_2\} &\subseteq A^c \\ \Rightarrow f^{-1}(a), f^{-1}(b) &\in \mathcal{F} \\ A, B &\in \mathcal{F} \end{aligned}$$

□

**Example 3.2** If  $f_1 = \mathbb{1}_A$ ,  $f_2 = \mathbb{1}_B$  Then

$$\begin{aligned} f_1 + f_2 &= \mathbb{1}_A + \mathbb{1}_B \\ &= \mathbb{1}_{A \cap B} + \mathbb{1}_{A \cup B} \\ &= \mathbb{1}_{A \setminus B} + 2\mathbb{1}_{A \cap B} + \mathbb{1}_{B \setminus A} + 0\mathbb{1}_{(A \cup B)^c} \end{aligned}$$

$$f_1 f_2 = \mathbb{1}_A \mathbb{1}_B = \mathbb{1}_{A \cap B}$$

$$A \cap B = (A^c \cup B^c)^c \in \mathcal{F}$$

$$A \setminus B = A \cap B^c \in \mathcal{F}$$

**Recall:**

Assume  $\mathbb{P}$  is a probability measure on  $(\Omega, \mathcal{F})$ :

$$\mathbb{P} : A \mapsto \mathbb{P}(A) \quad \forall A \in \mathcal{F} := \sigma\{A_1, \dots, A_N\}$$

with the known properties of  $\mathbb{P}$  If

$$f = \sum_{i=1}^N a_i \mathbb{1}_{A_i}$$

the *expectation* of  $f$  is

$$\mathbb{E}f = \sum_{i=1}^N a_i \mathbb{P}(A_i).$$

**Proposition 3.3**  $\mathcal{F} = \sigma(A_1, \dots, A_N)$  like above.

1. If  $\mathcal{G}$  is a  $\sigma$ -field with  $\mathcal{G} \subseteq \mathcal{F}$  and  $f$  is  $\mathcal{G}$ -measurable, then

$$\mathbb{E}[f|\mathcal{G}] = f$$

2. "tower-property":  $f$  is  $\mathcal{F}$ -measurable,  $\mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \mathcal{F}$ ,  $\sigma$ -fields, then

$$\begin{aligned} \mathbb{E}[\mathbb{E}[f|\mathcal{G}_1]|\mathcal{G}_2] &= \mathbb{E}[\mathbb{E}[f|\mathcal{G}_2]|\mathcal{G}_1] \\ &= \mathbb{E}[f|\mathcal{G}_1]. \end{aligned}$$

3. If  $g$  is  $\mathcal{G}$ -measurable,  $\mathcal{G} \subseteq \mathcal{F}$ , then

$$\mathbb{E}[fg|\mathcal{G}] = g\mathbb{E}[f|\mathcal{G}].$$

*Proof Exercise*

**Example 3.4**  $A_1, A_2, A_3$  partition of  $\Omega$

$$\mathbb{P}(A_1) = \frac{1}{10}, \quad \mathbb{P}(A_2) = \frac{7}{10}, \quad \mathbb{P}(A_3) = \frac{2}{10}$$

$$\mathcal{F} = \sigma(A_1, A_2, A_3)$$

$$f = a_1 \mathbb{1}_{A_1} + a_2 \mathbb{1}_{A_2} + a_3 \mathbb{1}_{A_3}$$

$$\mathbb{E}f = \frac{a_1}{10} + \frac{7a_2}{10} + \frac{2a_3}{10}$$

$$\mathcal{G} = \sigma(A_1 \cup A_2, A_3) \subseteq \sigma(A_1, A_2, A_3)$$

$H = \mathbb{E}[fg|\mathcal{G}]$  is by definition  $\mathcal{G}$ -measurable,

$$h = b_1 \mathbb{1}_{A_1 \cup A_2} + b_2 \mathbb{1}_{A_3}$$

*Question:*  $b_1 = ?$   $b_2 = ?$

By definition:

$$\mathbb{E}(h \mathbb{1}_B) \stackrel{!}{=} \mathbb{E}(f \mathbb{1}_B) \quad \forall B \in \mathcal{G}$$

It is sufficient to test only with  $B \in \{A_1 \cup A_2, A_3\}$ . The reason will be given in the fact below. We start with the condition

$$\mathbb{E}[(b_1 \mathbb{1}_{A_1 \cup A_2} + b_2 \mathbb{1}_{A_3}) \mathbb{1}_{A_1 \cup A_2}] = \mathbb{E}(f \mathbb{1}_{A_1 \cup A_2})$$

From the left hand side we get:

$$\mathbb{E}b_1 \mathbb{1}_{A_1 \cup A_2} = b_1 \mathbb{P}(A_1 \cup A_2)$$

From the right hand side:

$$\mathbb{E}(a_1 \mathbb{1}_{A_1} + a_2 \mathbb{1}_{A_2}) = a_1 \mathbb{P}(A_1) + a_2 \mathbb{P}(A_2)$$

This implies

$$b_1 = \frac{a_1 + 7a_2}{10} * \frac{10}{8} = \frac{a_1 + 7a_2}{8}$$

For  $B = A_3$  we get from

$$\mathbb{E}(b_1 \mathbb{1}_{A_1 \cup A_2} + b_2 \mathbb{1}_{A_3}) \mathbb{1}_{A_3} = \mathbb{E}(f \mathbb{1}_{A_3})$$

that it should hold

$$\mathbb{E}b_2 \mathbb{1}_{A_3} = \mathbb{E}a_3 \mathbb{1}_{A_3}$$

which implies

$$b_2 = a_3$$

Hence

$$\mathbb{E}[f|\mathcal{G}] = \frac{a_1 + 7a_2}{8} \mathbb{1}_{A_1 \cup A_2} + a_3 \mathbb{1}_{A_3}$$

**Fact:**

Let  $\mathcal{F}$  be a  $\sigma$ -field,  $f$  a  $\mathcal{F}$ -measurable function and  $\mathcal{G} = \sigma(B_1, \dots, B_n)$  a partition of  $\Omega$ . Assume

$$\mathbb{E}(h \mathbb{1}_{B_j}) = \mathbb{E}(f \mathbb{1}_{B_j}) \quad \forall B_j, j = 1, \dots, n$$

$$\Rightarrow \mathbb{E}(h \mathbb{1}_B) = \mathbb{E}(f \mathbb{1}_B) \quad \forall B \in \mathcal{G}.$$

*Proof*  $\mathbb{1}_B = \mathbb{1}_{\cup_{k=1}^e B_{j_k}} = \mathbb{1}_{B_{j_1}} + \dots + \mathbb{1}_{B_{j_k}}$

$$\begin{aligned} \mathbb{E}(h \mathbb{1}_B) &= \mathbb{E}(h(\mathbb{1}_{B_{j_1}} + \dots + \mathbb{1}_{B_{j_k}})) = \mathbb{E}(h \mathbb{1}_{B_{j_1}}) + \dots + \mathbb{E}(\mathbb{1}_{B_{j_e}}) \\ &= \dots = \mathbb{E}f \mathbb{1}_B \end{aligned}$$

□

## 4 Admissible strategies and arbitrage

If  $\varphi_t^0 < 0$ , we have borrowed the amount  $|\varphi_t^0|$  from the bank (at time  $t - 1$ ). If  $\varphi_t^i < 0$  for  $i \in \{1, \dots, d\}$  we say that we are *short* a number  $\varphi_t^i$  of assets (shares)  $i$ . Borrowing and short-selling is allowed as long as the value of the portfolio  $V_t(\varphi)$  is always non-negative.

Remember: A strategy is a predictable process

$$(\varphi_t)_{t=1}^T, \quad \varphi_t : \Omega \rightarrow \mathbb{R}^{d+1}$$

**Definition 4.1** 1. A strategy  $\varphi$  is *admissible* if it is self-financing and if  $V_t(\varphi) \geq 0 \quad \forall t \in \mathbb{T}$ .

2. An *arbitrage opportunity* is an admissible strategy  $\varphi$  such that

$$V_0(\varphi) = 0 \text{ and } \mathbb{E}V_T(\varphi) > 0$$

(”arbitrage means possibility of riskless profit: ”free lunch”)

3. The market is *viable* if it does not contain any arbitrage opportunities, i.e. if it holds

$$V_0(\varphi) = 0 \Rightarrow V_T(\varphi) = 0 \quad \forall \text{ admissible } \varphi$$

Let us assume in the following that  $\Omega = \{\omega_1, \dots, \omega_N\}$ . We can identify the space of all functions  $f : \Omega \rightarrow \mathbb{R}$  with  $\mathbb{R}^N$ :

$$f \leftrightarrow (f(\omega_1), \dots, f(\omega_N)) \in \mathbb{R}^N$$

Define

$$C := \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : x_i > 0, i = 1, \dots, N \text{ and there exists } i : x_i > 0\}$$

$C$  is a *convex cone*.

**Definition 4.2** A subset  $C$  of a vector space is a *convex cone* if it holds:

$$x, y \in C \Rightarrow x + y \in C$$

$$x \in C, a > 0 \Rightarrow ax \in C$$

Define  $\Psi_a :=$  set of admissible strategies.

Recall:  $\varphi$  is self-financing if and only if

$$\tilde{V}_t(\varphi) = V_0(\varphi) + \sum_{k=1}^t \varphi_k \cdot (\tilde{S}_k - S_{k-1}).$$

Discounted gains process will be defined by

$$\tilde{G}_t(\varphi) := \sum_{k=1}^t \varphi_k \cdot (\tilde{S}_k - S_{k-1}).$$

**Lemma 4.3** If the market  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), (S_t))$  is *viable* (does not admit any arbitrage opportunities) then it holds

$$\tilde{G}_T(\varphi) \notin C \quad \forall \text{ predictable } (\varphi_t^1, \dots, \varphi_t^d)_{t=1}^T.$$

*Proof* Assume  $\tilde{G}_T(\varphi) \in C$ . We will show that the market is not viable. Here we prove that if:

$$\tilde{G}_T(\varphi) \geq 0, \quad t = 0, \dots, T$$

it follows that the market is not viable. Notice:

$$\begin{aligned} \tilde{G}_T(\varphi) &= \sum_{k=1}^t \varphi_k \cdot (\tilde{S}_k - \tilde{S}_{k-1}) \\ &= \sum_{k=1}^t \varphi_k^0 (\tilde{S}_k^0 - \tilde{S}_{k-1}^0) + \varphi_k^1 (\tilde{S}_k^1 - \tilde{S}_{k-1}^1) + \dots + \varphi_k^d (\tilde{S}_k^d - \tilde{S}_{k-1}^d). \end{aligned}$$

Hence  $\tilde{G}_T(\varphi)$  does not depend on  $\varphi^0$ . From Proposition 2.4: Given  $(\varphi^1, \dots, \varphi^d)$  which is predictable and  $V_0 = 0$  implies the existence of a predictable and self-financing  $\varphi$  such that

$$\tilde{V}_t(\varphi) = V_0 + \tilde{G}_t(\varphi).$$

So we conclude that

$$\tilde{V}_0(\varphi) = 0, \quad \tilde{V}_t(\varphi) \geq 0 \quad t = 0, \dots, T$$

But  $\tilde{G}_T(\varphi) \in C$  means  $\tilde{G}_T(\varphi)(\omega_i) \geq 0 \quad i = 1, \dots, N$  and there exists  $i_0$  with  $\tilde{G}_T(\varphi)(\omega_{i_0}) > 0$ .

Hence:

$$\begin{aligned} \tilde{G}_T(\varphi) &= \sum_{i=1}^N \tilde{G}_T(\varphi)(\omega_i) \mathbb{P}(\{\omega_i\}) \\ &\geq \tilde{G}_T(\varphi)(\omega_{i_0}) \mathbb{P}(\{\omega_{i_0}\}) > 0 \end{aligned}$$

So there exists an arbitrage opportunity and the market is not viable.

Now the general case:

$\tilde{G}_T(\varphi)$  can have negative values.

$$t_0 = \sup\{t : \mathbb{P}(\{\omega : \tilde{G}_t(\varphi) < 0\}) > 0\}$$

Clearly:

1.  $t_0 \leq T - 1 \quad (\tilde{G}_T(\varphi) \in C)$ ,
2.  $\mathbb{P}(\{\tilde{G}_{t_0}(\varphi) < 0\}) > 0$ ,
3.  $\tilde{G}_t(\varphi) \geq 0, \quad \forall t = t_0 + 1, \dots, T$

We define a new strategy: for  $i = 1, \dots, d$

$$\psi_t^i(\omega) := \begin{cases} 0 & \text{if } t \leq t_0 \\ \mathbb{1}_A(\omega) \varphi_t^i(\omega) & \text{if } t > t_0 \end{cases}$$

where  $A = \{\omega : \tilde{G}_{t_0}(\varphi) < 0\} \in \mathcal{F}_{t_0}$  and hence  $\psi_t$  is predictable.

$$\tilde{G}_t(\psi) = \sum_{k=1}^t \psi_k \cdot (\tilde{S}_k - \tilde{S}_{k-1}) = \begin{cases} 0 & t \leq t_0 \\ \mathbb{1}_A(\tilde{G}_t(\varphi) - \tilde{G}_{t_0}(\varphi)) & t > t_0 \end{cases}$$

By construction we have  $\tilde{G}_t(\varphi) \geq 0$  and  $-\tilde{G}_{t_0}(\varphi) > 0$  on  $A$ .

$$\Rightarrow \tilde{G}_t(\psi) \geq 0 \quad t = 0, \dots, T, \quad \tilde{G}_T(\varphi) > 0 \text{ on } A$$

Hence

$$\begin{aligned} \mathbb{E}\tilde{G}_T(\psi) &= \sum_{i=1}^N G_T(\psi)(\omega_i)\mathbb{P}(\{\omega_i\}) \\ &\geq \sum_{\omega_i \in A} G_T(\psi)(\omega_i)\mathbb{P}(\{\omega_i\}) > 0 \end{aligned}$$

So the market is not viable which means  $G_T(\psi) \notin C$ . □

### Remark

About the assumptions on our "market": in contrary to reality we always assume here:

- a "frictionless" market: no transaction costs
- short sale and borrowing without any limit ( $\varphi_t^i \in \mathbb{R}$ )
- the securities are perfectly divisible:  $S_t^i \in [0, \infty)$



# Chapter 5

## The fundamental theorem of asset pricing

### 1 Separation of convex sets in $\mathbb{R}^N$

**Theorem 1.1** Let  $C \in \mathbb{R}^N$  be a closed convex set and  $(0, \dots, 0) \notin C$ . Then there exists a real linear functional  $\xi : \mathbb{R}^N \rightarrow \mathbb{R}$  and  $\alpha > 0$  such that

$$\xi(x) \geq \alpha \quad \forall x \in C.$$

*Proof* Let  $B(0, r) = \{x \in \mathbb{R}^N : \|x\| := (x_1^2 + \dots + x_N^2)^{\frac{1}{2}} \leq r\}$  which equals to a closed ball of radius  $r$  and center at the origin. Choose  $r > 0$  such that

$$C \cap B(0, r) \neq \emptyset$$

The map  $x \mapsto \|x\|$  is a continuous function and  $C \cap B(0, r)$  is closed and bounded. Let  $m_0 := \min_{x \in C \cap B(0, r)} \|x\|$ . Then there exists an  $x_0 \in C \cap B(0, r)$  with  $\|x_0\| = m_0$ .

Indeed: take  $(x_n)_{n=1}^\infty$  with  $\|x_n\|_{n \rightarrow \infty} \rightarrow m_0$  then there exists a subsequence  $(x_{n_k})$  such that  $x_{n_k} \rightarrow x_0$  for  $k \rightarrow \infty$ . The the claim follows from

$$\|x_0\| \leq \|x_0 - x_{n_k}\| + \|x_{n_k}\|$$

because  $\|x_0 - x_{n_k}\| \rightarrow 0$  and  $\|x_{n_k}\| \rightarrow m_0$ .

Hence,

$$\|x\| \geq \|x_0\| \quad \forall x \in C \quad (x \notin B(0, r) \Rightarrow \|x\| > r)$$

$$x \in C \Rightarrow \lambda x + (1 - \lambda)x_0 \in C$$

for  $\lambda \in [0, 1]$  since  $C$  is convex.

$$\begin{aligned} & \|\lambda x + (1 - \lambda)x_0\| \geq \|x_0\| \\ \Rightarrow & \lambda^2 x \cdot x + 2\lambda(1 - \lambda)x \cdot x_0 + (1 - \lambda)^2 x_0 \cdot x_0 \geq x_0 \cdot x_0 \end{aligned}$$

$$\begin{aligned} &\Rightarrow \lambda x \cdot x + 2(1 - \lambda)x \cdot x_0 - 2x_0 \cdot x_0 + \lambda x_0 \cdot x_0 \geq 0 \\ &\Rightarrow 2x \cdot x_0 + \lambda(x \cdot x - 2x \cdot x_0 + x_0 \cdot x_0) \geq 2x_0 \cdot x_0 \quad \forall \lambda \in [0, 1] \end{aligned}$$

For  $\lambda \rightarrow 0$  this inequality is only true if

$$x \cdot x_0 \geq x_0 \cdot x_0 = \|x_0\|^2 > 0$$

If we define  $\xi(x) := x_0 \cdot x$  we get a linear functional

$$\xi(x) \geq m_0^2 = \alpha \quad x \in C.$$

□

**Theorem 1.2** *Let  $K$  be a compact convex subset in  $\mathbb{R}^N$  and  $V$  a linear subspace of  $\mathbb{R}^N$ . If  $V \cap K = \emptyset$ , then there exists a linear functional*

$$\xi : \mathbb{R}^N \rightarrow \mathbb{R}$$

such that

1.  $\xi(x) > 0, \quad \forall x \in K,$
2.  $\xi(x) = 0, \quad \forall x \in V.$

Therefore, the subspace  $V$  is included in a hyperplane that does not intersect  $K$ .

*Proof*

The set

$$C := K - V = \{x \in \mathbb{R}^N : \exists(k, v) \in K \times V, x = k - v\}$$

is convex since for  $x_1, x_2 \in C$  we have

$$\begin{aligned} \lambda x_1 + (1 - \lambda)x_2 &= \lambda(k_1 - v_1) + (1 - \lambda)(k_2 - v_2) \\ &= \lambda k_1 + (1 - \lambda)k_2 - (\lambda v_1 + (1 - \lambda)v_2). \end{aligned}$$

Now  $\lambda k_1 + (1 - \lambda)k_2 \in K$  and  $\lambda v_1 + (1 - \lambda)v_2 \in V$  and its difference is in  $C$ . The set  $C$  is closed because  $V$  is closed and  $K$  is compact. We have  $(0, \dots, 0) \notin C$  since  $V \cap K = \emptyset$ . Hence we can apply Theorem 1.1 and find a linear functional  $\xi : \mathbb{R}^N \rightarrow \mathbb{R}$  and a constant  $\alpha > 0$  with

$$\xi(x) \geq \alpha \quad \forall x \in C.$$

This implies

$$\xi(k - v) = \xi(k) - \xi(v) \geq \alpha \quad \forall k \in K, v \in V.$$

Especially, it holds for fixed  $k_0 \in K$  and  $v_0 \in V$  and all  $\lambda \in \mathbb{R}$

$$\xi(k_0) - \xi(\lambda v_0) \geq \alpha$$

and because  $\xi$  is linear also

$$\xi(k_0) - \lambda \xi(v_0) \geq \alpha.$$

Consequently,  $\xi(v) = 0$  for all  $v \in V$  and  $\xi(k) \geq \alpha$  for all  $k \in K$ . □

## 2 Martingale transforms

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a finite probability space. Recall from Proposition 3.3 the following properties of the conditional expectation.

1. If  $\mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \mathcal{F}$  are sub- $\sigma$ -fields

$$\mathbb{E}[f|\mathcal{G}_1] = \mathbb{E}[\mathbb{E}[f|\mathcal{G}_2]|\mathcal{G}_1]$$

2. If  $g$  is  $\mathcal{G}$ -measurable ( $\mathcal{G} \subseteq \mathcal{F}$ ) Then

$$\mathbb{E}[fg|\mathcal{G}] = g\mathbb{E}[f|\mathcal{G}]$$

Linearity also holds:

$$\mathbb{E}[\alpha f + \beta g|\mathcal{G}] = \alpha\mathbb{E}[f|\mathcal{G}] + \beta\mathbb{E}[g|\mathcal{G}] \quad \alpha, \beta \in \mathbb{R}.$$

**Lemma 2.1** Let  $(\mathcal{F}_n)_{n=0}^T$  be a filtration  $(\varphi_n)_{n=1}^T$  a predictable sequence and  $(M_n)_{n=0}^T$  a martingale. Then the process

$$X_0 := 0$$

$$X_n := \varphi_1(M_1 - M_0) + \varphi_2(M_2 - M_1) + \dots + \varphi_n(M_n - M_{n-1})$$

is a martingale with respect to  $(\mathcal{F}_n)_{n=0}^T$ .  $(X_n)_{n=0}^T$  is called a *martingale transform* of  $(M_n)$  by  $(\varphi_n)$ .

*Proof*

$X_n$  is  $\mathcal{F}_n$ -measurable,  $n = 0, \dots, T$ . We check the martingale property:

$$\begin{aligned} \mathbb{E}[X_{n+1}|\mathcal{F}_n] &= \mathbb{E}\left[\sum_{t=1}^{n+1} \varphi_t(M_t - M_{t-1})|\mathcal{F}_n\right] \\ &= \sum_{t=1}^n \varphi_t(M_t - M_{t-1}) + \mathbb{E}[\varphi_{n+1}(M_{n+1} - M_n)|\mathcal{F}_n] \\ &= X_n \end{aligned}$$

where we used

$$\begin{aligned} \mathbb{E}[\varphi_{n+1}(M_{n+1} - M_n)|\mathcal{F}_n] &= \varphi_{n+1}\mathbb{E}[(M_{n+1} - M_n)|\mathcal{F}_n] \\ &= \varphi_{n+1}\mathbb{E}[M_{n+1}|\mathcal{F}_n] - \varphi_{n+1}M_n \end{aligned}$$

because  $(\varphi_n)$  is predictable and  $(M_n)$  adapted. □

**Theorem 2.2** An adapted real-value process  $(M_n)_{n=0}^T$  is a martingale if and only if

$$\mathbb{E} \sum_{n=1}^t \varphi_n (M_n - M_{n-1}) = 0 \quad \forall t = 1, \dots, T \quad (5.1)$$

for all predictable processes  $(\varphi_n)_{n=1}^T$ .

*Proof* "  $\Rightarrow$  "

If  $(M_n)_{n=0}^T$  is a martingale,  $X_t = \sum_{n=1}^t \varphi_n (M_n - M_{n-1})$  is a martingale transform. Hence by the previous Lemma

$$\mathbb{E} X_t = 0 \quad \forall t = 1, \dots, T.$$

"  $\Leftarrow$  " Assume (5.1) holds. Let  $A \in \mathcal{F}_{n_0}$  and define

$$\begin{aligned} \varphi_n(\omega) &:= \begin{cases} 0 & n \neq n_0 + 1 \\ \mathbb{1}_A(\omega) & n = n_0 + 1 \end{cases} \\ \Rightarrow \mathbb{E} X_T &= \mathbb{E} \mathbb{1}_A (M_{n_0+1} - M_{n_0}) = 0 \quad \forall A \in \mathcal{F}_{n_0} \\ \Rightarrow \mathbb{E}[M_{n_0+1} | \mathcal{F}_{n_0}] &= M_{n_0} \quad n_0 = 0, \dots, T. \end{aligned}$$

□

**Definition 2.3** (Independence)

1. The sets  $A, B \in \mathcal{F}$  are called *independent*

$$:\Leftrightarrow \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

2. The  $\sigma$ -fields  $\mathcal{G}_1, \mathcal{G}_2 \subseteq \mathcal{F}$  are called *independent*

$$:\Leftrightarrow \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B) \quad \forall A \in \mathcal{G}_1, B \in \mathcal{G}_2$$

(every set of  $\mathcal{G}_1$  is independent of every set of  $\mathcal{G}_2$ ).

3. If  $f_1, \dots, f_n : \Omega \rightarrow \{a_1, \dots, a_M\}$  ( $a_i \in \mathbb{R}$ ) are  $\mathcal{F}$ -measurable then  $f_1, \dots, f_n$  are called *independent (random variables)* : $\Leftrightarrow$

$$\mathbb{P}(\{\omega : f_1(\omega) = x_1, \dots, f_n(\omega) = x_n\}) = \prod_{k=1}^n \mathbb{P}(\{\omega : f_k(\omega) = x_k\})$$

$\forall x_i \in \{a_1, \dots, a_M\}$ . In other words all the pre-images of  $f_1, \dots, f_n$  are independent sets.

4. An  $\mathcal{F}$ -measurable function  $f$  is called *independent* from a  $\sigma$ -field  $\mathcal{G}$  ( $\mathcal{G} \subseteq \mathcal{F}$ ) : $\Leftrightarrow$

$$f \text{ and } \mathbb{1}_G \text{ are independent } \forall G \in \mathcal{G}.$$

**Remark to (3)**

$$\begin{aligned} \{\omega : f_1(\omega) = x_1, \dots, f_N(\omega) = x_n\} &= \{\omega : f_1(\omega) = x_1 \text{ and } \dots \text{ and } f_n(\omega) = x_n\} \\ &= \bigcap_{k=1}^n f_k^{-1}(\{x_k\}). \end{aligned}$$

**Example 2.4** 1. Tossing a coin 2 times:

$$\mathbb{P}(\text{1st toss} = \text{"heads"} \text{ and } \text{2nd toss} = \text{"tails"}) = \mathbb{P}(\text{1st toss})\mathbb{P}(\text{2nd toss})$$

	CRR model	Tossing a coin T-times
	$\Omega = \{\omega = (\epsilon_1, \dots, \epsilon_T), \epsilon_i \in \{(1+a), (1+b)\}\}$	Write for each toss
2.	$\mathbb{P}(\{\omega\}) = p^k(1-p)^{T-k}$	$\begin{cases} 1+a \text{ if "tails"} \\ 1+b \text{ if "heads"} \end{cases}$
	if $\omega$ contains $k$ times $1+b$ and $T-k$ times $1+a$	$\begin{aligned} \mathbb{P}(\text{tossing "heads"}) &= p \\ \Rightarrow \mathbb{P}(\text{tossing "tails"}) &= 1-p \end{aligned}$

$$S_t(\omega) = S_0 \epsilon_1 \epsilon_2 \cdots \epsilon_t$$

$$\Rightarrow \frac{S_{t+1}(\omega)}{S_t(\omega)} = \epsilon_{t+1}$$

The functions

$$\frac{S_1}{S_0}, \frac{S_2}{S_1}, \dots, \frac{S_T}{S_{T-1}}$$

are independent: for  $x_i \in \{(1+a), (1+b)\}$ 

$$\begin{aligned} \mathbb{P}\left(\left\{\omega : \frac{S_1(\omega)}{S_0(\omega)} = x_1, \dots, \frac{S_T(\omega)}{S_{T-1}(\omega)} = x_T\right\}\right) \\ &= \mathbb{P}(\{\omega = (\epsilon_1, \dots, \epsilon_T) : \epsilon_1 = x_1, \dots, \epsilon_T = x_T\}) \\ &= p^k(1-p)^{T-k} \\ &= \prod_{t=1}^T \mathbb{P}\left(\left\{\omega : \frac{S_t}{S_{t-1}} = \epsilon_t = x_t\right\}\right) \end{aligned}$$

if  $k$  of the  $x_i$ 's are  $1+b$  and  $T-k$  are equal to  $1+a$ . We have for  $t = 1, \dots, T$ 

$$\mathbb{P}(\{\omega = (\epsilon_1, \dots, \epsilon_T) : \epsilon_t = 1+b\}) = p$$

where  $p$  is the probability that one tosses "heads" the  $t$ -th time if one tosses  $T$ -times alltogether. The outcome of the other times does not influence that of time  $t$  (independence).

**Theorem 2.5** Let  $f, g$  be  $\mathcal{F}$ -measurable.1. If  $f$  and  $g$  are independent then

$$\mathbb{E}fg = \mathbb{E}f\mathbb{E}g$$

2. If  $f$  is independent from the  $\sigma$ -field  $\mathcal{G}(\mathcal{G} \subseteq \mathcal{F})$  then  $\mathbb{E}[f|\mathcal{G}] = \mathbb{E}f$

*Proof*

1. Let

$$f = \sum_{i=1}^n x_i \mathbb{1}_{F_i}, \quad g = \sum_{j=1}^m y_j \mathbb{1}_{G_j}$$

where we assume that all  $x_i$ 's are different and all  $y_j$ 's are different.

$$\begin{aligned} \mathbb{E}fg &= \mathbb{E} \sum_{i=1}^n \sum_{j=1}^m x_i y_j \mathbb{1}_{F_i} \mathbb{1}_{G_j} \\ &= \mathbb{E} \sum_{i=1}^n \sum_{j=1}^m x_i y_j \mathbb{1}_{F_i \cap G_j} \\ &= \sum_{i=1}^n \sum_{j=1}^m x_i y_j \mathbb{P}(F_i \cap G_j) \\ &= \left( \sum_{i=1}^n x_i \mathbb{P}(F_i) \right) \left( \sum_{j=1}^m y_j \mathbb{P}(G_j) \right) = \mathbb{E}f \mathbb{E}g \end{aligned}$$

where we used  $\mathbb{P}(F_i \cap G_j) = \mathbb{P}(\{\omega : f(\omega) = x_i\} \cap \{\omega : g(\omega) = y_j\}) = \mathbb{P}(\{\omega : f(\omega) = x_i\}) \mathbb{P}(\{\omega : g(\omega) = y_j\}) = \mathbb{P}(F_i) \mathbb{P}(G_j)$

2. Exercise -5-

□

**Proposition 2.6** Assume  $f_1, \dots, f_n$  are independent and  $\mathcal{F}$ -measurable. Let  $\mathcal{F}_k = \sigma(f_1, \dots, f_k)$ .

1. Then  $f_l, l > k$  is independent from  $\mathcal{F}_k$
2. If  $\mathbb{E}f_k = 0$ , for  $k = 1, \dots, n$  then  $(M_t)$  with  $M_t := \sum_{k=1}^t f_k$  for  $t \geq 1$  and  $M_0 = 0$  is a martingale with respect to  $(\mathcal{F}_t)$ .
3. If  $\mathbb{E}f_k = 1$ , for  $k = 1, \dots, n$  then  $(N_t)$  with  $N_t := \prod_{k=1}^t f_k$  for  $t \geq 1$  and  $N_0 = 1$  is a martingale with respect to  $(\mathcal{F}_t)$ .

*Proof*

1. Idea:  $G \in \mathcal{F}_k$ ;  $G = \{f_1 = x_1, \dots, f_k = x_k\}$

$$\mathbb{P}(f_l = x_l, \mathbb{1}_G = x) = \mathbb{P}(f_l = x_l) \mathbb{P}(\mathbb{1}_G = x)$$

2. 3. Exercise

**Remark**

One can show that

$$\sigma\left(\frac{S_1}{S_0}, \dots, \frac{S_t}{S_{t-1}}\right) = \sigma(S_1, \dots, S_t) = \mathcal{F}_t.$$

From (3) it follows now that  $(S_t)_{t=0}^T$  with  $S_t = S_0 \frac{S_1}{S_0} \frac{S_2}{S_1} \cdots \frac{S_t}{S_{t-1}}$  is a martingale

$$\Leftrightarrow \mathbb{E} \frac{S_t}{S_{t-1}} = 1 \quad \forall t.$$

### 3 The fundamental theorem of asset pricing

With the results of the previous sections we get a characterization of the "no arbitrage" condition.

**Remark:**

Assume  $\Omega$  and a  $\sigma$ -field  $\mathcal{F}$  on  $\Omega$  and  $\mathbb{P}, \mathbb{Q} : \mathcal{F} \rightarrow [0, 1]$  probability measures. One says that  $\mathbb{P}$  is equivalent to  $\mathbb{Q}$  (notation  $\mathbb{P} \sim \mathbb{Q}$ ) if and only if

$$\mathbb{P}(A) = 0 \Leftrightarrow \mathbb{Q}(A) = 0 \quad \text{for any } A \in \mathcal{F}.$$

We assume  $\Omega := \{\omega_1, \dots, \omega_N\}$ ,  $\mathcal{F} := 2^\Omega$  and  $\mathbb{P}(\{\omega_i\}) > 0$ ,  $i = 1, \dots, N$ . So  $\mathbb{Q} \sim \mathbb{P}$  if  $\mathbb{Q}(\{\omega_i\}) > 0$ ,  $i = 1, \dots, N$ .

**Theorem 3.1** [Fundamental Theorem of Asset pricing]

The market  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), (S_t))$  is viable if and only if there exists a probability measure  $\mathbb{Q} \sim \mathbb{P}$  such that  $\tilde{S}_t^i = \frac{1}{S_0^i} S_t^i$ ,  $i = 1, \dots, d$  are  $\mathbb{Q}$ -martingales ( $\mathbb{Q}$  is called the *equivalent martingale measure: EMM*).

*Proof* "  $\Leftarrow$  "

Assume  $\mathbb{Q} \sim \mathbb{P}$  and  $\tilde{S}_t^i$ ,  $i = 1, \dots, d$  are  $\mathbb{Q}$ -martingales. By Proposition 2.2(3) we have, if  $\varphi$  is self-financing that

$$\tilde{V}_t(\varphi) = V_0(\varphi) + \sum_{k=1}^t \varphi_k \cdot (\tilde{S}_k - S_{k-1}).$$

We denote by  $\mathbb{E}_{\mathbb{Q}}$  the expectation with respect to  $\mathbb{Q}$ . Hence by Theorem 2.2

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}} \tilde{V}_T(\varphi) &= \mathbb{E}_{\mathbb{Q}} \tilde{V}_0(\varphi) + \mathbb{E}_{\mathbb{Q}} \sum_{k=1}^T \varphi_k \cdot (\tilde{S}_k - S_{k-1}) \\ &= \mathbb{E}_{\mathbb{Q}} \tilde{V}_0(\varphi) + \mathbb{E}_{\mathbb{Q}} \sum_{i=1}^d \sum_{k=1}^T \varphi_k^i (\tilde{S}_k^i - S_{k-1}^i) \end{aligned}$$

$$\begin{aligned}
&= \mathbb{E}_{\mathbb{Q}} \tilde{V}_0(\varphi) + \sum_{i=1}^d \mathbb{E}_{\mathbb{Q}} \sum_{k=1}^T \varphi_k^i (\tilde{S}_k^i - S_{k-1}^i) \\
&= \mathbb{E}_{\mathbb{Q}} \tilde{V}_0(\varphi)
\end{aligned} \tag{5.2}$$

If  $V_0(\varphi) = 0 \Rightarrow \mathbb{E}_{\mathbb{Q}} \tilde{V}_T(\varphi) = 0$ . Now assume that

$$\mathbb{E}_{\mathbb{Q}} \tilde{V}_T(\varphi) = \sum_{i=1}^N \tilde{V}_T(\varphi)(\omega_i) \mathbb{Q}(\{\omega_i\}) = 0. \tag{5.3}$$

If  $\varphi$  is admissible, then  $\tilde{V}_T(\varphi)(\omega_i) \geq 0$ ,  $i = 1, \dots, N$ . So (5.3) implies that  $\tilde{V}_T(\varphi)(\omega_i) = 0$ ,  $i = 1, \dots, N$ .

Consequently,

$$V_0(\varphi)(\omega_i) = 0, \quad i = 1, \dots, N \Rightarrow V_T(\varphi)(\omega_i) = 0, \quad i = 1, \dots, N$$

for all admissible  $\varphi$ . Hence the market does not admit arbitrage opportunities and is not viable. □

”  $\Rightarrow$  ”

By Lemma 4.3 we have :

The market is viable

$$\Rightarrow \tilde{G}_T(\varphi) \notin C = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N, x_i \geq 0, i = 1, \dots, N, \exists i x_i > 0\} \\
\forall (\varphi^1, \dots, \varphi^d) \text{ predictable}$$

where

$$\tilde{G}_t(\varphi) = \sum_{k=1}^t (\varphi_k^1 (\tilde{S}_k^1 - S_{k-1}^1) + \dots + \varphi_k^d (\tilde{S}_k^d - S_{k-1}^d))$$

is the discounted gains process. We define

$$V := \{\tilde{G}_T(\varphi) : (\varphi^1, \dots, \varphi^d) \text{ predictable}\}$$

$V$  is a linear subspace of  $\mathbb{R}^N$ :

$$(\tilde{G}_T(\varphi)(\omega_1), \dots, \tilde{G}_T(\varphi)(\omega_N)) \in \mathbb{R}^N$$

By the Lemma we have  $V \cap C = \emptyset$ . We define

$$K := \{f = (f(\omega_1), \dots, f(\omega_N)) \in C : \sum_{i=1}^N f(\omega_i) = 1\}$$

$$V \cap K = \emptyset$$

$K$  is convex: If  $f, g \in K$  then

1.  $\lambda f + (1 - \lambda)g \in C$  (since  $C$  is convex)
2.  $\sum_{i=1}^N (\lambda f(\omega_i) + (1 - \lambda)g(\omega_i)) = \lambda + (1 - \lambda) = 1$

$K$  is compact because it is bounded ( $\|f\| = \sum_{i=1}^N |f(\omega_i)| = 1$ ) and closed. Therefore, by Theorem 1.2 there exists a linear functional  $\xi(x) = \xi_1 x_1 + \dots + \xi_N x_N$  with

1.  $\sum_{i=1}^N \xi_i f(\omega_i) > 0 \quad \forall f \in K$
2.  $\sum_{i=1}^N \xi_i \tilde{G}_T(\omega_i) = 0 \quad \forall (\varphi^1, \dots, \varphi^d)$  predictable.

Now, if  $f := (0, \dots, 0, 1, 0, \dots, 0)$ , then  $f \in K$  and "1." implies  $\xi_i > 0 \Leftrightarrow \xi_i > 0$ ,  $i = 1, \dots, N$ .

We define

$$\mathbb{Q}(\{\omega_i\}) := \frac{\xi_i}{\sum_{i=1}^N \xi_i}$$

Then  $\mathbb{Q}$  is a probability measure,  $\mathbb{Q} \sim \mathbb{P}$  and by "2."

$$\mathbb{E}_{\mathbb{Q}} \tilde{G}_T(\varphi) = \sum_{i=1}^N \tilde{G}_T(\varphi)(\omega_i) \mathbb{Q}(\{\omega_i\}) = 0 \quad \forall (\varphi_1, \dots, \varphi^d) \text{ predictable.}$$

In other words,

$$\mathbb{E}_{\mathbb{Q}} \sum_{t=1}^T \sum_{i=1}^d \varphi_t^i (\tilde{S}_t^i - \tilde{S}_{t-1}^i) = 0$$

or in short form

$$\mathbb{E}_{\mathbb{Q}} \sum_{i=1}^d \varphi_t \cdot (\tilde{S}_t - \tilde{S}_{t-1}) = 0 \quad \forall i = 1, \dots, d \text{ } (\varphi_t^i) \text{ predictable.}$$

Hence by Theorem 2.2 we have that  $(\tilde{S}^1, \dots, \tilde{S}^d)$  are  $\mathbb{Q}$ -martingales.  $\square$

### Remark

The *scalar product* (or inner product) on a vector space  $V (= \mathbb{R}_N)$  is a function

$$(\cdot, \cdot) : V \times V \rightarrow \mathbb{R} \text{ such that } \forall \alpha, \beta \in \mathbb{R}$$

and  $v_1, v_2, v \in V$

1.  $(\alpha v_1 + \beta v_2, v) = \alpha(v_1, v) + \beta(v_2, v)$  linearity
2.  $(v_1, v) = (v, v_1)$  symmetry
3.  $(v, v) \geq 0$  and  $(v, v) = 0 \Leftrightarrow v = 0$  positive definite.

If  $(x_1, \dots, x_N) \in \mathbb{R}_N$ ,  $c, w \in V$   
 $(v, w) := \sum_{i=1}^N v_i w_i x_i$  is a scalar product

$$\Leftrightarrow x_i > 0 \quad \forall i = 1, \dots, N$$

One can see this assuming  $x_1 = 0 \Leftrightarrow v = (1, 0, \dots, 0)$  implies  $(v, v) = 0$  which is a contadiction.

If all  $x_i < 0 \Leftrightarrow (v, v) \leq 0$  which leads again to a contradiction.

If some  $x_i < 0$ , some  $x_j > 0$  them there exists  $v \neq 0$  with  $(v, v) = 0$  which is a contadiction.

Orthogonality: We define  $V \perp W$  orthogonal  $\Leftrightarrow (v, w) = 0$ .

## 4 Complete markets and option pricing

Let us assume the market model  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), (S_t))$ . We already know the European call-option  $H = (S_T^1 - K)^+$  and the European put-option  $H = (K - S_T^1)^+$ .

Options can also depend on the whole path of the underlying security. For example,

$$H = \left( S_T^1 - \frac{S_1^1 + S_2^1 + \dots + S_T^1}{T} \right)$$

would be one type of a so-called *Asian option*. In general we define a European option ( or a *contingent claim*) to be a non-negative function  $H : \Omega \rightarrow [0, \infty)$  which is  $\mathcal{F}$ -measurable. We say the contingent claim  $H$  is *attainable* if there exists an admissible strategy  $\varphi$  with

$$H = V_T(\varphi).$$

If the market is viable, then there exists a  $\mathbb{Q} \sim \mathbb{P}$  such that  $(\tilde{S}_t)_{t=0}^T$  is a  $\mathbb{Q}$ -martingale and if we find a self-financing strategy  $\varphi$  such that

$$H = V_T(\varphi) \quad (\text{resp. } \frac{H}{S_T^0} = \tilde{V}_T(\varphi)).$$

It follows

$$\mathbb{E}_{\mathbb{Q}} \tilde{V}_T(\varphi) = V_0(\varphi)$$

is a no-arbitrage price. This implies that

$$\mathbb{E}_{\mathbb{Q}} H / S_T^0$$

is a no-arbitrage price if  $H$  is attainable and  $\mathbb{Q} \sim \mathbb{P}$ .

By the way, it holds

$$\tilde{V}_t(\varphi) = \mathbb{E}_{\mathbb{Q}} \left[ \frac{H}{S_T^0} \middle| \mathcal{F}_t \right]$$

discounted no-arbitrage price at time  $t$ .

**Definition 4.1** The market is *complete* if every contingent claim is attainable; i.e. for any  $\mathcal{F}_T$ -measurable  $H \geq 0$  there exists an admissible strategy  $\varphi$  such that  $H = V_T(\varphi)$ .

**Remark:**

Completeness is a restrictive assumption: a lot of market models are not complete. But there is a nice mathematical characterization of completeness.

**Theorem 4.2** A viable market is complete if and only if there exists a *unique*  $\mathbb{Q} \sim \mathbb{P}$  such that  $(\tilde{S}^1, \dots, \tilde{S}^d)$  are  $\mathbb{Q}$ -martingales.

*Proof* ”  $\Rightarrow$  ”

Assume the market is viable and complete. Let  $H$  be  $\mathcal{F}_T$ -measurable and  $H \geq 0$ . Complete so that there exists an admissible strategy such that

$$H = V_T(\varphi) \tag{5.4}$$

$\varphi$  is self-financing so that:

$$\tilde{V}_T(\varphi) = V_0(\varphi) + \sum_{t=1}^T \varphi_t(\tilde{S}_t - \tilde{S}_{t-1})$$

Viable so that there exists  $\mathbb{Q} \sim \mathbb{P}$  such that  $(\tilde{S}_t)_t$  are  $\mathbb{Q}$ -martingales. We have to show that  $\mathbb{Q}$  is unique. Let  $\hat{\mathbb{Q}}$  be another probability measure such that  $\hat{\mathbb{Q}} \sim \mathbb{P}$  and  $(\tilde{S}_t)_t$  are  $\hat{\mathbb{Q}}$ -martingales. Then

$$\mathbb{E}_{\mathbb{Q}} \tilde{V}_T(\varphi) = V_0(\varphi) = \mathbb{E}_{\hat{\mathbb{Q}}} \tilde{V}_T(\varphi).$$

Hence

$$\mathbb{E}_{\mathbb{Q}} \frac{H}{S_T^0} = V_0(\varphi) = \mathbb{E}_{\hat{\mathbb{Q}}} \frac{H}{S_T^0}$$

By assumption  $H$  is attainable, so the arbitrage price is the *same* for any  $\mathbb{Q}$ . Choose

$$\begin{aligned} H &= \mathbb{1}_A S_T^0 \text{ for } A \in \mathcal{F}_T \\ \Rightarrow \mathbb{Q}(A) &= \hat{\mathbb{Q}}(A) \quad \forall A \in \mathcal{F}_T = F \\ \Rightarrow \mathbb{Q} &= \hat{\mathbb{Q}}. \end{aligned}$$

□

”  $\Leftarrow$  ”

Assume the market is viable and incomplete. Then there exists  $H \geq 0$ , and  $H$  is  $\mathcal{F}_T$ -measurable and not attainable.

$$V = \left\{ V_0 + \sum_{t=1}^T \varphi_t (\tilde{S}_t - \tilde{S}_{t-1}), \quad V_0 \in \mathbb{R}, (\varphi^1, \dots, \varphi^d) \text{ predictable} \right\}$$

$$\Rightarrow \frac{H}{S_T^0} \notin V.$$

$$W = \{f = (f(\omega_1), \dots, f(\omega_N))\} = \mathbb{R}^N$$

Hence we get:

$$V \not\subseteq W.$$

We introduce the scalar product

$$(f, g) := \mathbb{E}_{\mathbb{Q}} fg = \sum_{i=1}^N f(\omega_i) g(\omega_i) \mathbb{Q}(\{\omega_i\})$$

We take a basis  $v_1, \dots, v_M \in V$ ,  $x := \frac{H}{S_T^0} \notin V$

$$\Rightarrow \hat{x} := x - \sum_{i=1}^M (x, v_i) v_i \perp V$$

Indeed: take any  $v = \sum_{k=1}^M \alpha_k v_k \in V$

$$\Rightarrow (\hat{x}, v) = \sum_{k=1}^M \alpha_k (x, v_k) - \sum_{n=1}^M \alpha_n (x, v_n) = 0$$

Define:

$$\hat{\mathbb{Q}}(\{\omega\}) = \left(1 + \frac{\hat{x}(\omega)}{2 \sup_{\tilde{\omega}} |\hat{x}(\tilde{\omega})|}\right) \mathbb{Q}(\{\omega\})$$

Obviously,

$$\hat{\mathbb{Q}}(\{\omega\}) > 0 \quad \forall \omega$$

and

$$\hat{\mathbb{Q}}(\Omega) = \sum_{i=1}^N \hat{\mathbb{Q}}(\omega_i) + \sum_{i=1}^N \frac{\hat{x}(\omega_i)}{2 \sup_{\tilde{\omega}} |\hat{x}(\tilde{\omega})|} \mathbb{Q}(\{\omega_i\}) = 1.$$

Indeed since  $1 \in V$  we have

$$(\hat{x}, 1) = \mathbb{E}_{\mathbb{Q}} \hat{x} = \sum_{i=1}^N \hat{x}(\omega_i) \mathbb{Q}(\{\omega_i\}) = 0$$

Hence  $\hat{\mathbb{Q}}$  is a probability measure and  $\hat{\mathbb{Q}} \sim \mathbb{P}$  and  $\hat{\mathbb{Q}} \neq \mathbb{Q}$  by definition. Finally; we show that  $(\tilde{S}_t)$  is also a  $\mathbb{Q}$ -martingale

$$\begin{aligned}
\mathbb{E}_{\hat{\mathbb{Q}}} \sum_{t=1}^N \varphi_t(\tilde{S}_t - \tilde{S}_{t-1}) &= \sum_{i=1}^N v(\omega_i) \hat{\mathbb{Q}}(\{\omega_i\}) \\
&= \sum_{i=1}^N v(\omega_i) \left(1 + \frac{\hat{x}(\omega_i)}{2 \sup_{\omega} |x\omega|}\right) \mathbb{Q}(\{\omega_i\}) \\
&= \mathbb{E}_{\mathbb{Q}} v + \frac{1}{2 \sup_{\omega} |\hat{x}\hat{\omega}|} \mathbb{E}_{\mathbb{Q}} v \hat{x} \\
&= 0 \quad \forall (\varphi^1, \dots, \varphi^d) \text{ predictable}
\end{aligned}$$

Hence by the Theorem 2.2  $(\tilde{S}_t)_{t=0}^T$  is a  $\hat{\mathbb{Q}}$ -martingale.

□



# Chapter 6

## American Options

### 1 Stopping Times

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a finite probability space,  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t=0}^T, (S_t)_{t=0}^T)$  a market model as before.

An American option can be exercised **at any time**  $t \in \{0, 1, \dots, T\} =: \mathbb{T}$ .

For example, the American call option with strike price  $K$ :

$Z_t = (S_t^1 - K)^+$ ,  $t = 0, 1, \dots, T$  is then a sequence adapted  $(\mathcal{F}_T)$ .  $Z_t$  = profit made by exercising the option at time  $t$ .

For the decision to exercise or not at time  $t$  the trader can only use the information available at time  $\mathcal{F}_t$ . We describe this using stopping times.

**Definition 1.1** A random variable  $\tau: \Omega \rightarrow \mathbb{T}$  is a **stopping time** if

$$\{\tau = t\} \in \mathcal{F}_t \quad \forall t = 0, \dots, T.$$

$$(\{\tau = t\} = \{\omega \in \Omega : \tau(\omega) = t\})$$

**Remark 1.2** It holds

$$\begin{aligned} \{\tau = t\} &= \{\tau \leq t\} \setminus \{\tau \leq t-1\} \in \mathcal{F}_t \quad \forall t = 1, \dots, T \\ \iff \{\tau \leq t\} &= \{\tau = 0\} \cup \{\tau = 1\} \cup \dots \cup \{\tau = t\} \in \mathcal{F}_t \quad \forall t = 0, \dots, T \end{aligned}$$

**Definition 1.3** Let  $(X_t)_{t=0}^T$  be an adapted sequence and  $\tau$  a stopping time.

We define

$$X_t^\tau(\omega) := X_{t \wedge \tau(\omega)}(\omega) \quad (a \wedge b := \min\{a, b\})$$

This means on the set  $\{\omega : \tau(\omega) = k\}$

$$X_t^\tau = \begin{cases} X_k & \text{if } t \geq k \\ X_t & \text{if } t < k. \end{cases}$$

**Theorem 1.4** Let  $\tau$  be a stopping time.

- (1)  $(X_t)$  is  $(\mathcal{F}_t)$  adapted  $\Rightarrow (X_t^\tau)$  is  $(\mathcal{F}_t)$  adapted.
- (2)  $(X_t)$  is a martingale  $\Rightarrow (X_t^\tau)$  is a martingale
- (3)  $(X_t)$  is a supermartingale  $\Rightarrow (X_t^\tau)$  is a supermartingale

*Proof*

(1)

$$X_{t \wedge \tau} = X_0 + \sum_{k=1}^t \mathbb{1}_{\{k \leq \tau\}} (X_k - X_{k-1})$$

It holds  $\{k \leq \tau\} = \{k > \tau\}^c$ . But  $\{\tau < k\} = \{\tau \leq k-1\} \in \mathcal{F}_{k-1}$ . Hence  $\varphi(k) := \mathbb{1}_{\{k \leq \tau\}}$  is a predictable sequence. Clearly,  $(X_{t \wedge \tau})_{t=0}^T$  is adapted.

(2) Let  $(X_t)$  be a martingale. By Lemma 2.1: since  $(X_t^\tau - x_0)$  is a martingale transform of  $(X_t)$  by  $(\varphi(t))$  it follows  $(X_t^\tau)$  is a martingale.

(3) Can be shown similarly. □

## 2 The Snell Envelope

We want to define the price of an American option for example for

$$Z_t = (S_t - K)^+, \quad t = 0, \dots, T.$$

We use a backward in induction:

t=T:

$$\text{option price } U_T = Z_T.$$

t=T-1:

Now the option holder has 2 possibilities:

- (1) Trading at once (t=T-1) writer must pay  $Z_{T-1}$
- (2) Trading at time t=T. Writer must be able to pay  $Z_T \Rightarrow$  needs an admissible strategy with the price

$$S_{T-1}^0 \mathbb{E}_{\mathbb{Q}} \left[ \frac{Z_T}{S_T^0} \middle| \mathcal{F}_{T-1} \right] = S_{T-1} \tilde{V}_{T-1} = V_{T-1}$$

( $\mathbb{Q}$ =EMM, assume : market is complete)

$\Rightarrow$  option price

$$U_{T-1} = \max \left\{ Z_{T-1}, S_{T-1}^0 \mathbb{E}_{\mathbb{Q}} \left[ \frac{Z_T}{S_T^0} \middle| \mathcal{F}_{T-1} \right] \right\}$$

induction:

$$U_{t-1} = \max \left\{ Z_{t-1}, S_{t-1}^0 \mathbb{E}_{\mathbb{Q}} \left[ \frac{U_t}{S_t^0} \middle| \mathcal{F}_{t-1} \right] \right\}.$$

**Theorem 2.1** Let  $(\Omega, \mathcal{F}, \mathbb{Q})$  be a finite probability space,  $(X_t)_{t=0}^T$  an  $(\mathcal{F}_t)$  adapted sequence with  $X_t \geq 0$ ,  $t = 0, \dots, T$ . Then the process  $(U_t)$  with

$$U_T := X_T$$

$$U_{t-1} := \max\{X_{t-1}, \mathbb{E}_{\mathbb{Q}}[U_t | \mathcal{F}_{t-1}]\}$$

is a supermartingale. It is the smallest supermartingale dominating  $(X_t)$ , i.e. it holds

$$U_t \geq X_t, \quad t = 0, \dots, T.$$

**Remark 2.2** The process  $(U_t)$  is called the **Snell envelope** of  $(X_t)$ .

*Proof* (of the Theorem)

Clearly:  $U_t = \max\{X_t, \mathbb{E}_{\mathbb{Q}}[U_{t+1} | \mathcal{F}_t]\} \geq X_t$ ,  $t = 0, \dots, T$ .

So  $(U_t)$  is dominating  $(X_t)$ .

$(U_t)$  is adapted.

$$\begin{aligned} \mathbb{E}[U_t | \mathcal{F}_{t-1}] &\leq \max\{X_{t-1}, \mathbb{E}_{\mathbb{Q}}[U_t | \mathcal{F}_{t-1}]\} \\ &= U_{t-1} \Rightarrow \text{supermartingale} \end{aligned}$$

We have to show that  $(U_t)$  is the smallest one:

Suppose  $(Y_t)$  is a supermartingale dominating  $(X_t)$ . Then  $Y_T \geq X_T = U_T$ .

**Backward induction:** Assume for some  $t \leq T$  that  $Y_t \geq U_t$ . Then it follows

$$Y_{t-1} \geq \mathbb{E}_{\mathbb{Q}}[Y_t | \mathcal{F}_{t-1}] \geq \mathbb{E}_{\mathbb{Q}}[U_t | \mathcal{F}_{t-1}]$$

↑

Supermartingale

↑

assumption  $Y_t \geq U_t$

But  $Y_{t-1} \geq X_{t-1}$  holds also.

$$\Rightarrow Y_{t-1} \geq \max\{X_{t-1}, \mathbb{E}_{\mathbb{Q}}[U_t | \mathcal{F}_{t-1}]\} = U_{t-1}$$

□

**Theorem 2.3** 1.  $\tau^* = \min\{t \geq 0 : U_t = X_t\}$  is a stopping time.

2. The stopped process  $(U_t^{\tau^*})$  is a martingale.

*Proof*

1.

$\{\tau^* = 0\} = \{U_0 = X_0\} \in \mathcal{F}_0$  since  $U_0, X_0$  are  $\mathcal{F}_0$ -measurable.

$$\{\tau^* = t\} = \bigcap_{s=0}^{t-1} \{U_s > X_s\} \cap \{U_t = X_t\}$$

Since  $\{U_s > X_s\} \in \mathcal{F}_s \subseteq \mathcal{F}_t$  and  $\{U_t = X_t\} \in \mathcal{F}_t$  we have  $\{\tau^* = t\} \in \mathcal{F}_t$ .

2. Define  $\varphi(t) := \mathbb{1}_{\{\tau^* \geq t\}}$  ( $\Rightarrow (\varphi(t))$  is predictable)

It holds

$$\begin{aligned} U_t^{\tau^*} &= U_0 + \sum_{s=1}^t \varphi(s)(U_s - U_{s-1}) \\ \Rightarrow U_t^{\tau^*} - U_{t-1}^{\tau^*} &= \varphi(t)(U_t - U_{t-1}) = \mathbb{1}_{\{\tau^* \geq t\}}(U_t - U_{t-1}) \\ &= \mathbb{1}_{\{\tau^* \geq t\}}(U_t - \mathbb{E}_{\mathbb{Q}}[U_t | \mathcal{F}_{t-1}]) \end{aligned}$$

since on the set  $\{\tau^* \geq t\} = \{\tau^* > t-1\}$  it holds  $U_{t-1} > X_{t-1}$  and hence

$$\begin{aligned} U_{t-1} &= \max\{X_{t-1}, \mathbb{E}_{\mathbb{Q}}[U_t | \mathcal{F}_{t-1}]\} \\ &= \mathbb{E}_{\mathbb{Q}}[U_t | \mathcal{F}_{t-1}]. \end{aligned}$$

So it follows

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[U_t^{\tau^*} - U_{t-1}^{\tau^*} | \mathcal{F}_{t-1}] &= \mathbb{E}_{\mathbb{Q}}[\mathbb{1}_{\{\tau^* \geq t\}}(U_t - \mathbb{E}_{\mathbb{Q}}[U_t | \mathcal{F}_{t-1}]) | \mathcal{F}_{t-1}] \\ &= \mathbb{1}_{\{\tau^* \geq t\}}(\mathbb{E}_{\mathbb{Q}}[U_t | \mathcal{F}_{t-1}] - \mathbb{E}_{\mathbb{Q}}[U_t | \mathcal{F}_{t-1}]) \\ &= 0 \\ &\Rightarrow (U_t^{\tau^*}) \text{ martingale} \end{aligned}$$

□

**Definition 2.4** A stopping time  $\sigma : \Omega \rightarrow \mathbb{T}$  is **optimal** for  $(X_t)$  if

$$\mathbb{E}_{\mathbb{Q}}X_{\sigma} = \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}}X_{\tau}$$

where  $\mathcal{T}$  denotes the set of stopping times  $\tau : \Omega \rightarrow \mathbb{T}$ .

**Interpretation:** If we think of  $X_n$  as the total winnings at time  $n$ , then stopping at time  $\sigma$  would maximize the expected gain.

**Corollary 2.1**  $\tau^* = \min\{t \geq 0 : U_t = X_t\}$  is an optimal stopping time for  $(X_t)$  and

$$U_0 = \mathbb{E}_{\mathbb{Q}}X_{\tau^*} = \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}}X_{\tau}.$$

*Proof*  $(U_t^{\tau^*})$  is a martingale. It holds

$$U_0 = U_0^{\tau^*} = \mathbb{E}_{\mathbb{Q}}U_T^{\tau^*} = \mathbb{E}_{\mathbb{Q}}U_{T \wedge \tau^*} = \mathbb{E}_{\mathbb{Q}}X_{\tau^*}$$

because  $T \wedge \tau^* = \tau^*$  and  $U_{\tau^*} = X_{\tau^*}$  by definition.

On the other hand  $(U_t^{\tau})$  is a supermartingale for any  $\tau \in \mathcal{T}$  by Theorem 1.4. So it follows because  $U_t^{\tau}$  is a supermartingale and  $(U_t)$  dominates  $(X_t)$  that

$$U_0 = U_0^{\tau} \geq \mathbb{E}_{\mathbb{Q}}U_{\tau} \geq \mathbb{E}_{\mathbb{Q}}X_{\tau}.$$

Which implies  $U_0 \geq \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}}X_{\tau}$ , and since  $\tau^* \in \mathcal{T}$  and  $\mathbb{E}_{\mathbb{Q}}X_{\tau^*} = U_0$  we get  $U_0 = \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}}X_{\tau}$  □

There is the following characterization for optional stopping times:

**Theorem 2.5** *A stopping time  $\sigma$  is optional for  $(X_t)$*

$$\Leftrightarrow (1) U_\sigma = X_\sigma,$$

and

$$(2) U^\sigma \text{ in an } ((\mathcal{F}_t), \mathbb{Q}) \text{ martingale .}$$

( $U$  denotes the Snell envelope of  $(X_t)$ ).

*Proof* "  $\Leftarrow$  "

If  $U^\sigma$  is a martingale, it holds

$$U_0 = \mathbb{E}_{\mathbb{Q}} U_T^\sigma = \mathbb{E}_{\mathbb{Q}} U_\sigma = \mathbb{E}_{\mathbb{Q}} X_\sigma.$$

On the other hand  $(U_t^\tau)$  is a supermartingale for any  $\tau \in \mathcal{T}$  (since  $(U_t)$  is a supermartingale, see Theorem 1.4.) Hence

$$U_0 = U_0^\tau \geq \mathbb{E}_{\mathbb{Q}} U_T^\tau = \mathbb{E}_{\mathbb{Q}} U_\tau \geq \mathbb{E}_{\mathbb{Q}} X_\tau$$

because  $(U_t)$  dominates  $(X_t)$ .

$$\Rightarrow_{\sigma \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}} X_\sigma = \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}} X_\tau, \text{ i.e. } \sigma \text{ is optimal}$$

"  $\Rightarrow$  " Assume  $\sigma$  is optimal; i.e.  $\mathbb{E}_{\mathbb{Q}} X_\sigma = \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}} X_\tau$

By Collary 2.1:  $U_0 = \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}} X_\tau$ .

$$\text{Hence } U_0 = \mathbb{E}_{\mathbb{Q}} X_\sigma \leq \mathbb{E}_{\mathbb{Q}} U_\sigma$$

because  $(U_t)$  dominates  $(X_t)$ .

$(U_t)$  is a supermartingale, therefore  $(U_t^\sigma)$  is a supermartingale

$$U_0 = U_0^\sigma \geq \mathbb{E}_{\mathbb{Q}} U_\sigma$$

$$\Rightarrow \mathbb{E}_{\mathbb{Q}} X_\sigma = U_0 = \mathbb{E}_{\mathbb{Q}} U_\sigma. \quad (6.1)$$

$$\text{By } U_t \geq X_t \forall t, \omega \Rightarrow X_\sigma = U_\sigma.$$

Since  $(U_t^\sigma)$  is a supermartingale:

$$\mathbb{E}_{\mathbb{Q}}[U_T^\sigma | \mathcal{F}_t] \leq U_t^\sigma \quad (6.2)$$

$$\begin{aligned} U_0 = U_0^\sigma &\geq \mathbb{E}_{\mathbb{Q}} U_t^\sigma \geq \mathbb{E}_{\mathbb{Q}} \mathbb{E}_{\mathbb{Q}}[U_T^\sigma | \mathcal{F}_t] \\ &= \mathbb{E}_{\mathbb{Q}} U_\sigma = U_0 \end{aligned}$$

since  $(U_t^\sigma)$  is a supermartingale and because of the relations (6.1) and (6.2).

$$\text{Hence } \mathbb{E}_{\mathbb{Q}}[U_T^\sigma | \mathcal{F}_t] = U_t^\sigma, \quad (U_t^\sigma) \text{ martingale.}$$

**Remark 2.6**  $\tau^*$  from the Corollary is the *smallest* optimal stopping time for  $(X_t)$ .

### 3 Decomposition of Supermartingales

We will consider the so called "Doob decomposition" which we will use to find trading strategies for American options. It is also used to model trading strategies with consumption.

**Theorem 3.1** *Every supermartingale  $(U_t)_{t=0}^T$  has the following unique decomposition*

$$U_t = M_t - A_t,$$

where  $(M_t)$  is a martingale and  $(A_t)$  is a non-decreasing predictable process with  $A_0 = 0$ .

*Proof* Induction **t=0**: if  $A_0 = 0 \Rightarrow M_0 = U_0$  is uniquely determined.

**t**  $\rightarrow$  **t+1**:

$$U_{t+1} - U_t = M_{t+1} - M_t - (A_{t+1} - A_t) \quad (6.3)$$

We take the conditional expectation on both sides and assume  $(M_t)$  is a martingale and that  $(A_t)$  is predictable:

$$\begin{aligned} \mathbb{E}[U_{t+1}|\mathcal{F}_t] - U_t &= \mathbb{E}[M_{t+1}|\mathcal{F}_t] - M_t - (A_{t+1} - A_t) \\ \Rightarrow -(A_{t+1} - A_t) &= \mathbb{E}[U_{t+1}|\mathcal{F}_t] - U_t \leq 0 \end{aligned} \quad (6.4)$$

$$\Rightarrow A_t \leq A_{t+1} \text{ (non-decreasing)}$$

□

**Remark 3.2**

$$M_{t+1} - M_t = U_{t+1} - \mathbb{E}[U_{t+1}|\mathcal{F}_t] \text{ (use (6.3) and (6.4))}$$

$$A_{t+1} - A_t = U_t - \mathbb{E}[U_{t+1}|\mathcal{F}_t]$$

One can find also the **largest** optimal stopping time for  $(X_t)$ :

**Definition 3.3** Define  $\sigma : \Omega \rightarrow \{0, 1, \dots, T\}$  by setting

$$\sigma(\omega) := \begin{cases} T & \text{if } A_T(\omega) = 0, \\ \min\{t \geq 0; A_{t+1} > 0\} & \text{if } A_T(\omega) > 0. \end{cases}$$

if  $(U_t)$  is the Snell envelope of  $(X_t)$  and  $U_t = M_t - A_t$  (Doob decomposition).

**Theorem 3.4**  $\sigma$  is the largest optimal stopping time for  $(X_t)$

*Proof*(1)  $\sigma$  is a stopping time:

$$\{\sigma = t\} = \cap_{s \leq t} \{A_s = 0\} \cap \{A_{t+1} > 0\} \text{ is } \mathcal{F}_t \text{ - measurable}$$

because  $\{A_s = 0\}$  is  $\mathcal{F}_{s-1} \subseteq \mathcal{F}_{t-1}$ -measurable and  $\{A_{t+1} > 0\}$  is  $\mathcal{F}_t$ -measurable since  $A$  is predictable.

(2) it is **optimal**

We conclude from

$$U_t = M_t - A_t \quad \text{and} \quad U_t^\sigma = M_t^\sigma \text{ for } t \leq \sigma(\omega)$$

that  $(U_t^\sigma)$  is a martingale. This gives us property (2) of Theorem 2.5 and hence that  $\sigma$  is optimal. We still have to show  $U_\sigma = X_\sigma$ .

$$\begin{aligned} U_\sigma &= \sum_{s=0}^{T-1} \mathbb{1}_{\{\sigma=s\}} U_s + \mathbb{1}_{\{\sigma=T\}} U_T \\ &= \sum_{s=0}^{T-1} \mathbb{1}_{\{\sigma=s\}} \max\{X_s, \mathbb{E}[U_{s+1} | \mathcal{F}_s]\} + \mathbb{1}_{\{\sigma=s\}} U_T \end{aligned}$$

We have  $\mathbb{E}[U_{s+1} | \mathcal{F}_s] = \mathbb{E}[M_{s+1} - A_{s+1} | \mathcal{F}_s] = M_s - A_{s+1}$  and  $A_{s+1} > 0$  on  $\{\sigma = s\}$

$$U_s = M_s - A_s \text{ and } A_s = 0 \text{ on } \{\sigma = s\}$$

$$\mathbb{E}[U_{s+1} | \mathcal{F}_s] < U_s \Rightarrow$$

$$U_s = \max\{X_s, \mathbb{E}[U_{s+1} | \mathcal{F}_s]\} = X_s$$

So we get

$$\begin{aligned} U_\sigma &= \sum_{s=0}^{T-1} \mathbb{1}_{\{\sigma=s\}} X_s + \mathbb{1}_{\{\sigma=T\}} U_T \\ &= X_\sigma \end{aligned}$$

because  $U_T = X_T$  by construction

$\Rightarrow$  **optimal.**

(3)  $\sigma$  is the **largest**:

Assume  $\tau \geq \sigma$  and  $\mathbb{Q}(\tau > \sigma) > 0$ .

$$\text{then } \mathbb{E}U_\tau = \mathbb{E}M_\tau - \mathbb{E}A_\tau = \mathbb{E}M_0 - \mathbb{E}A_\tau < \mathbb{E}U_0 = U_0$$

which means  $\tau$  is not optimal. □

## 4 Pricing and hedging of American options

We assume the market  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), (S_t))$  is viable and complete  $((\Omega, \mathcal{F}, \mathbb{P})$  finite probability space) and  $\mathbb{Q}$  is the EMM. An **American option** is an adapted sequence  $(Z_t)_{t=0}^T$  with  $Z_t \geq 0$ . In Section 2 of this chapter we saw: Given an American option  $(Z_t)_{t=0}^T$  its value process  $(U_t)_{t=0}^T$  can be described by

$$\begin{cases} U_T = Z_T \\ U_t = \max\{Z_t, S_t^0 \mathbb{E}_{\mathbb{Q}}[\frac{U_{t+1}}{S_{t+1}} | \mathcal{F}_t]\} \quad 0 \leq t \leq T-1 \end{cases}$$

That means, **the discounted price of the option**  $\tilde{U}_t := \frac{U_t}{S_t^0}$ ,  $t=0, \dots, T$  is **the Snell envelope under  $\mathbb{Q}$**  of  $(\tilde{Z}_t)_{t=0}^T$ . Like in Section 2 one can show

$$\tilde{U}_t = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_{\mathbb{Q}}[\frac{Z_{\tau}}{S_{\tau}^0} | \mathcal{F}_t],$$

where  $\mathcal{T}_{t,T}$  denotes the set of all stopping time  $\tau : \Omega \rightarrow \{t, \dots, T\}$ . Consequently, the price  $U_t$  of the option  $(Z_t)_{t=0}^T$  is

$$U_t = S_t^0 \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_{\mathbb{Q}}[\frac{Z_{\tau}}{S_{\tau}^0} | \mathcal{F}_t].$$

Now we use the Doob decomposition:

$$\tilde{U}_t = \tilde{M}_t - \tilde{A}_t$$

where  $\tilde{M}_t$  is a  $\mathbb{Q}$ -martingale and  $\tilde{A}_t$  is non-decreasing, predictable,  $\tilde{A}_0=0$ . By assumption, the market is complete.

$\Rightarrow \exists \varphi$ , self-financing such that for  $H = S_T^0 \tilde{M}_T$  it holds

$$V_T(\varphi) = S_T^0 \tilde{M}_T$$

$$i.e. \tilde{V}_T(\varphi) = \tilde{M}_T \Rightarrow \tilde{V}_t(\varphi) = \tilde{M}_t, t = 0, \dots, T.$$

because  $(\tilde{V}_t(\varphi))$  is a  $\mathbb{Q}$ -martingale.

$$\Rightarrow \tilde{U}_t = \tilde{V}_t(\varphi) - \tilde{A}_t, \quad U_t = V_t(\varphi) - A_t$$

Optionprice  $V_0(\varphi) = U_0 = \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}} \frac{Z_{\tau}}{S_{\tau}^0}$   
 = investment needed for a hedging strategy  
 = rational ( or "fair" ) price or "no-arbitrage price".

$V_0(\varphi)$  is the minimal investment capital to hedge (using a self-financing strategy) the option

$$Z_t, t = 1, \dots, T$$

**Theorem 4.1** *A stopping time  $\sigma \in \mathcal{T}$  is an optimal exercise time for the American Option  $(Z_t)_{t=0}^T$*

$$\Leftrightarrow \mathbb{E}_{\mathbb{Q}} \frac{Z_{\sigma}}{S_{\sigma}^0} = \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}} \frac{Z_{\tau}}{S_{\tau}^0} \quad (6.5)$$

*Proof*

An optimal date to exercise the option has to be a stopping time (traders do not have information about the future). An option holder would not exercise (=use) the option in case

$$U_t > Z_t$$

because he would trade an asset worth  $U_t$  (=the price of the option at time  $t$ ) for an amount  $Z_t$  (= profit he gets from exercising the option at time  $t$ ). We know  $(\tilde{U}_t)$  dominates  $(\tilde{Z}_t) \Rightarrow U_t \geq Z_t \forall t$ . So the option holder waits till  $U_{\tau^*} = Z_{\tau^*} (\Leftrightarrow \tilde{U}_{\tau^*} = \tilde{Z}_{\tau^*})$ . But this is the property (1) of an optimal stopping time (compare Theorem 2.5). So also

$$\sigma_{\max} = \begin{cases} \inf\{0 \leq t \leq T-1, A_{t+1} \neq 0\} \\ T, \end{cases} \quad A_T = 0$$

is an optimal exercise time. **After**  $\sigma_{\max}$  one should not exercise: From

$$U_t = \vee_t(\varphi) - A_t$$

It follows  $U_t < \vee_t(\varphi)$  on  $\{t > \sigma_{\max}\}$ . If the holder would sell the option at the time  $\sigma_{\max}$  he would get  $U_{\sigma_{\max}}$ . By using  $U_{\sigma_{\max}} = \vee_{\sigma_{\max}}(\varphi)$  and trading with the trading strategy  $\varphi$  he creates a parfolio  $\varphi_t$  such that

$$\vee_{\sigma_{\max}+1}(\varphi) > U_{\sigma_{\max}+1}, \dots, \vee_T(\varphi) > U_T$$

Hence, for an optimal exercise time  $\tau$  it holds  $(U_t^\tau)$  is a martingale, so also property (2) of an optimal stopping time must hold (compare Theorem 2.5)  
Remark:

Or, from the writer's point of view: If he uses  $\varphi$  as defined above and the buyer exercises at  $\tau \neq$  'optimal' then

$$U_\tau > Z_\tau \text{ or } A_\tau > 0.$$

$\Rightarrow$  Writer makes riskless profit:

$$\begin{aligned} U_t &= \vee_t(\varphi) - A_t \\ \Rightarrow \vee_\tau(\varphi) - Z_\tau &= U_\tau + A_\tau - Z_\tau > 0 \end{aligned}$$

if  $U_\tau > Z_\tau, A_\tau \geq 0, A_\tau > 0$  and  $U_\tau \geq Z_\tau$ .  $\vee_\tau(\varphi)$  is the amount the writer gets by hedging, and  $Z_\tau$  is the amount the writer has to pay to the holder.

## 5 American options and European options

**Theorem 5.1** *Let  $C_t^A$  be the value of an American option at time  $t$  described by an adapted sequence  $(Z_t)_{t=0}^T$  and let  $C_t^E$  be the value of an European option at time  $t$  defined by the  $\mathcal{F}_T$  measurable random variable  $H = Z_T$ . Then it holds*

$$C_t^A \geq C_t^E.$$

Moreover, if  $C_t^E \geq Z_t$  for all  $t$ , then

$$C_t^A = C_t^E \quad \forall t = 1, \dots, T.$$

**Remark 5.2** *One can imagine that  $C_t^A \geq C_t^E$  should be true because the American holder has more choices to exercise than the European.*

*Proof* Put  $C_t^E := S_t^0 \mathbb{E}_{\mathbb{Q}}[\frac{H}{S_t^0} | \mathcal{F}_t]$ .  $\tilde{C}_t^A = \frac{C_t^A}{S_t^0}$  is a  $\mathbb{Q}$ - supermartingale and because of the assumption  $C_T^A = C_T^E$  we have

$$\tilde{C}_t^A \geq \mathbb{E}_{\mathbb{Q}}[\tilde{C}_T^A | \mathcal{F}_t] = \mathbb{E}_{\mathbb{Q}}[\tilde{C}_T^E | \mathcal{F}_t] = \tilde{C}_t^E.$$

If  $C_t^E \geq Z_t$  for all  $t$  then  $\tilde{C}_t^E$  is a  $\mathbb{Q}$ - martingale (and as any martingale is a supermartingale) it is a  $\mathbb{Q}$ - supermartingale which dominates  $\tilde{Z}_t$ . Hence

$$\tilde{C}_t^A \leq \tilde{C}_t^E \quad t = 0, \dots, T.$$

□

**Remark 5.3** *The price of the European call and the American call are equal:*

$$\begin{aligned} \tilde{C}_t^E &= (1+r)^{-T} \mathbb{E}_{\mathbb{Q}}[(S_T - K)^+ | \mathcal{F}_t] \\ &\geq \mathbb{E}_{\mathbb{Q}}[\tilde{S}_T - K(1+r)^{-T} | \mathcal{F}_t] \\ &= \tilde{S}_t - K(1+r)^{-T}. \end{aligned}$$

*Thus*

$$C_t^E \geq S_t - K(1+r)^{-(T-t)} \geq S_t - K, \quad r \geq 0.$$

*Because  $C_t^E \geq 0$  we have  $C_t^E \geq (S_t - K)^+ Z_t$*

# Bibliography

- [1] R. Elliott, P. Kopp Mathematics of Financial Markets. Springer, 2000
- [2] D. Lamberton and B Lapeyre Stochastic Calculus Applied to Finance. Chapman & Hall, 1996.