Models in Financial Mathematics 2
- continuous time -

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Contents

1. Some stochastic calculus

Mathematical finance in continuous time is described in the language of stochastic integrals and stochastic differential equations. That's why the course begins by introducing

- Brownian motion,
- martingales,
- Itô integral and
- Itô's formula.

1.1 Brownian motion

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, i.e.

(1) Ω is a non-empty set.

(2) $\mathcal{F}$ is a σ-algebra.

(3) $\mathbb{P}$ is a probability measure.

The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is **complete**, if $B \in \mathcal{F}$ with $\mathbb{P}(B) = 0$ and $A \subseteq B$ imply $A \in \mathcal{F}$, in other words "$\mathcal{F}$ contains all $\mathbb{P}$-null sets". Let $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ be a **filtration**, i.e.

$$\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F},\ 0 \leq s \leq t < \infty,$$

where $\mathcal{F}_s$ and $\mathcal{F}_t$ are σ-algebras. A σ-algebra $\mathcal{B}(\mathbb{R})$ is the smallest σ-algebra containing all open intervals of $\mathbb{R}$, see [3], p. 12-13.
In the future, it is assumed, that $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ satisfies the **"usual conditions"**, namely

(1) $(\Omega, \mathcal{F}, \mathbb{P})$ is complete,

(2) $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$,

(3) $\mathbb{F}$ is right-continuous, i.e. $\mathcal{F}_t = \mathcal{F}_{t+} = \bigcap\limits_{s>t} \mathcal{F}_s$.

To model random phenomena in finance, the Brownian motion will be used.

Definition 1.1.1 (Brownian motion)
Assume a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\mathbb{F} = (\mathcal{F}_t)$. A family of random variables $W = (W_t)_{t \geq 0}$ is called **(standard) Brownian motion** with respect to $(\mathcal{F}_t)$, if

(a) for all $\omega \in \Omega$, $t \mapsto W_t(\omega) : [0, \infty) \to \mathbb{R}$ is a continuous function with $W_0(\omega) = 0$.

(b) (W_t) is $(\mathcal{F}_t)$-adapted and for $0 \leq s < t$ it holds that $W_t - W_s$ is independent from $\mathcal{F}_s$, meaning that for all $A \in \mathcal{F}_s$ and $B \in \mathcal{B}(\mathbb{R})$ it holds that

$$\mathbb{P}(A \cap \{\omega : W_t(\omega) - W_s(t) \in B\}) = \mathbb{P}(A)\mathbb{P}(\{\omega : W_t(\omega) - W_s(t) \in B\}).$$

(c) W_t is normally distributed for all $t > s \geq 0$ with $\mathbb{E}W_t = 0$ and $\mathbb{E}W_t^2 = t$, i.e.

$$\mathbb{P}(\{\omega : W_t \leq x\}) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{x} e^{-\frac{z^2}{2t}} dz.$$

(d) (W_t) is homogeneous:

$$\mathbb{P}(W_{t-s} \leq x) = \mathbb{P}(W_t - W_s \leq x).$$

1.1.1 Some properties of the Brownian motion

1. Brownian motion exists. The space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ can be chosen to satisfy the "usual conditions".

2. The Brownian motion can only be sketched but not drawn: The length of the path on the interval $[0, 1]$ is ∞ almost surely:

$$\mathbb{P}\left(\left\{\omega : \lim_{N\to\infty} \sum_{k=1}^{N} \left| W_{\frac{k}{N}}(\omega) - W_{\frac{k-1}{N}}(\omega) \right| = \infty \right\}\right) = 1$$

3. The paths $t \mapsto W_t(\omega)$ of the Brownian motion are for almost all $\omega \in \Omega$ nowhere differentiable.

4. For any $0 = t_0 < t_1 < ... < t_n$ the random variables $W_{t_n} - W_{t_{n-1}}$, $W_{t_{n-1}} - W_{t_{n-2}}, ..., W_{t_1}$ are independent.

   (Remember: Random variables $X_1, ..., X_n$ are **independent** if and only if

   $$\mathbb{P}\big(\{X_1 \in B_1\} \cap ... \cap \{X_n \in B_n\}\big) = \mathbb{P}\big(\{X_1 \in B_1\}\big) \times ... \times \mathbb{P}\big(\{X_n \in B_n\}\big)$$

   for all $B_i \in \mathcal{B}(\mathbb{R})$.)

5. Because W is homogeneous,

$$\mathbb{E}W_t - W_s = \mathbb{E}W_{t-s} = 0 \text{ and } \mathbb{E}(W_t - W_s)^2 = t - s.$$

1.2 Conditional expectation and martingales

The main properties of conditional expectation are recalled for later use.

Definition 1.2.1 (Conditional expectation)
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{G} \subseteq \mathcal{F}$ a sub-σ-algebra, and X a random variable such that $\mathbb{E}|X| < \infty$. If Y is $\mathcal{G}$-measurable (i.e. $\{\omega \in \Omega : a \leq Y(\omega) \leq b\} \in \mathcal{G}$ for all $a < b$) and

$$\mathbb{E}\left(X \mathbb{1}_A\right) = \mathbb{E}\left(Y \mathbb{1}_A\right) \quad \text{for all } A \in \mathcal{G},$$

then Y is called **conditional expectation of X given $\mathcal{G}$**. The conditional expectation is denoted by $\mathbb{E}[X|\mathcal{G}] := Y$.

Remark: The conditional expectation $\mathbb{E}[X|\mathcal{G}]$ is only almost surely unique.

Example 1.2.2 If for example, $X(\omega) = \omega^2$, $\omega \in [0,1]$, then by choosing $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1])$ and $\mathbb{P} = \lambda$, where λ is the Lebesgue-measure on $[0,1]$, and a σ-algebra

$$\mathcal{G} = \left\{ \left[0, \frac{1}{4}\right], \left(\frac{1}{4}, 1\right], \emptyset, \Omega \right\},$$

$\mathbb{E}[X|\mathcal{G}]$ can be determined in the following way: Any $\mathcal{G}$-measurable random variable Y is of the form $Y = a\mathbb{1}_{[0,\frac{1}{4}]} + b\mathbb{1}_{(\frac{1}{4},1]}$, $a, b \in \mathbb{R}$. Now, a and b need to be chosen such that

$$\begin{aligned} \mathbb{E}X\mathbb{1}_{[0,\frac{1}{4}]} &= \mathbb{E}Y\mathbb{1}_{[0,\frac{1}{4}]} \quad \text{and} \\ \mathbb{E}X\mathbb{1}_{(\frac{1}{4},1]} &= \mathbb{E}Y\mathbb{1}_{(\frac{1}{4},1]}. \end{aligned}$$

Since

$$\mathbb{E}X\mathbb{1}_{[0,\frac{1}{4}]} = \int_0^1 \omega^2 \mathbb{1}_{[0,\frac{1}{4}]}(\omega) d\omega = \frac{1}{3 \cdot 4^3}$$

and

$$\mathbb{E}Y\mathbb{1}_{[0,\frac{1}{4}]} = \int_0^1 Y\mathbb{1}_{[0,\frac{1}{4}]}(\omega) d\omega = \frac{a}{4}$$

implying that $a = \frac{1}{48}$. Similarly,

$$\mathbb{E}X\mathbb{1}_{(\frac{1}{4},1]} = \int_0^1 \omega^2 \mathbb{1}_{(\frac{1}{4},1]} = \frac{1}{3}\left(1 - \frac{1}{4^3}\right)$$

and

$$\mathbb{E}Y\mathbb{1}_{(\frac{1}{4},1]} = \int_0^1 Y\mathbb{1}_{[0,\frac{1}{4}]}(\omega) d\omega = \frac{3b}{4},$$

so $b = \frac{4^3-1}{4^2 \cdot 9} = \frac{7}{16}$. Hence

$$\mathbb{E}[X|\mathcal{G}](\omega) = \frac{1}{48}\mathbb{1}_{[0,\frac{1}{4}]}(\omega) + \frac{7}{16}\mathbb{1}_{(\frac{1}{4},1]}(\omega) \text{ almost surely.}$$

Proposition 1.2.3 (Properties of the conditional expectation) *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $\mathcal{G} \subseteq \mathcal{F}$ be a sub-σ-algebra of $\mathcal{F}$.*

1. *If $\mathbb{E}|X| < \infty$ or $X \geq 0$ a.s. then $\mathbb{E}[X|\mathcal{G}]$* **exists**.
2. *Let $\mathbb{E}|X| < \infty$ or $X \geq 0$ a.s. then*
   (a) *If X is $\mathcal{G}$-measurable, then $\mathbb{E}[X|\mathcal{G}] = X$ almost surely.*
   (b) *If X and $\mathcal{G}$ are independent, then $\mathbb{E}[X|\mathcal{G}] = \mathbb{E}X$ almost surely.*
   (c) **Tower property**: *If $\mathcal{G} \subseteq \mathcal{H} \subseteq \mathcal{F}$ are sub-σ-algebras, then*
   $$\mathbb{E}\big[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}\big] = \mathbb{E}\big[\mathbb{E}[X|\mathcal{H}]|\mathcal{G}\big] = \mathbb{E}[X|\mathcal{G}] \text{ a.s.}$$
   (d) *Linearity: If $\mathbb{E}|X| < \infty$ and $\mathbb{E}|Z| < \infty$, then*
   $$\mathbb{E}\big[\alpha X + \beta Z|\mathcal{G}\big] = \alpha\mathbb{E}[X|\mathcal{G}] + \beta\mathbb{E}[Z|\mathcal{G}] \text{ a.s.}$$
   for all $\alpha, \beta \in \mathbb{R}$.
   (e) **"Take out what is known"**: *If $\mathbb{E}|X| < \infty$ and Y is bounded (***or*** if $\mathbb{E}|X|^p < \infty$ and $\mathbb{E}|Y|^q < \infty$ for $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p, q < \infty$) and Y is $\mathcal{G}$-measurable, then*
   $$\mathbb{E}[XY|\mathcal{G}] = Y\mathbb{E}[X|\mathcal{G}] \text{ almost surely.}$$

Definition 1.2.4 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ a filtration.

(a) A **stochastic process** $X = (X_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is a collection of random variables $(X_t)_{t\geq 0}$, (i.e. X_t is $\mathcal{F}$-measurable for all $t \geq 0$.)

(b) A stochastic process $X = (X_t)_{t\geq 0}$ is **adapted** if X_t is $\mathcal{F}_t$-measurable for all $t \geq 0$.

(c) An adapted stochastic process $(X_t)_{t\geq 0}$ is called a **martingale** with respect to $(\mathcal{F}_t)_{t\geq 0}$, if
   (1) $\mathbb{E}|X_t| < \infty$ for all $t \geq 0$, i.e. X_t is integrable.
   (2) for $0 \leq s \leq t$
   $$\mathbb{E}[X_t|\mathcal{F}_s] = X_s.$$

(d) A martingale $(X_t)_{t\geq 0}$ is called **square integrable** if
$$\mathbb{E}X_t^2 < \infty \text{ for all } t \geq 0.$$

Proposition 1.2.5 *The Brownian motion $(W_t)_{t\geq 0}$ is a martingale.*

Proof:
The proof is an exercise. It can be easily seen that $\mathbb{E}|W_t| < \infty$ by using Hölder's inequality ($\mathbb{E}|XY| \leq (\mathbb{E}|X|^p)^{\frac{1}{p}}(\mathbb{E}|Y|^q)^{\frac{1}{q}}$ for $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$) by choosing $X := W_t$, $Y := 1$ and $p = q = 2$, which imply
$$\mathbb{E}|W_t| \leq (\mathbb{E}W_t^2)^{\frac{1}{2}} = \sqrt{t}$$
by definition. □

1.3 Itô's integral for simple integrands

We assume that $W = (W_t)_{t\geq 0}$ is a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ and want to define the stochastic integral (=**Itô integral**)

$$\int_0^T L_t dW_t \quad \text{for } T > 0.$$

In this section it is assumed that the stochastic process $(L_t)_{t\geq 0}$ is a **simple process**, i.e. there exists a sequence $0 = t_0 < t_1 < ... < t_n = T$ and random variables $\xi_i,\ i = 0, 1, ..., n$ with the properties

(i) ξ_i is $\mathcal{F}_{t_i}$-measurable

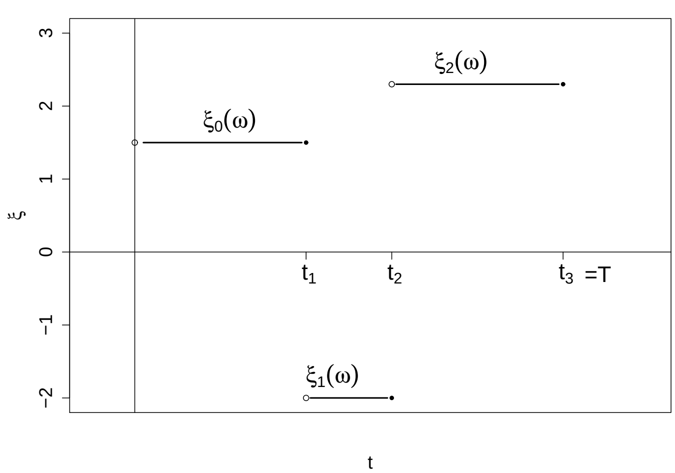
(ii) $\sup_{\omega\in\Omega} |\xi_i(\omega)| < C$ for some $C > 0$ for all $i = 1, ..., n$ such that $(L_t)_{t\geq 0}$ can be represented by

$$L_t = \sum_{i=1}^{n} \xi_{i-1} 1\!\!1_{(t_{i-1},t_i]}(t)$$

The space of simple processes is denoted by $\mathcal{L}_0$.

Remark 1.3.1

1. $(L_t)_{0\leq t\leq T}$ is a stochastic process which has piecewise constant paths for each $\omega \in \Omega$.



2. (L_t) is an adapted process:

$$L_t = \xi_{i-1} \text{ for } t \in (t_{i-1}, t_i],\ L_0 = 0.$$

Then ξ_{i-1} is $\mathcal{F}_{t_{i-1}} \subseteq \mathcal{F}_t$-measurable, hence L_t is $\mathcal{F}_t$-measurable.

If b is a continuously differentiable function on $[0,T]$, then

$$\begin{aligned}\int_0^T L_t(\omega)db(t) &= \int_0^T L_t(\omega)b'(t)dt \\ &= \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \xi_{i-1}(\omega)b'(t)dt \\ &= \sum_{i=1}^n \xi_{i-1}(\omega)(b(t_i)-b(t_{i-1})).\end{aligned}$$

This relation is the motivation for the definition of $\int_0^T L_t dW_t$ (but $(W_t(\omega)$ is **not** differentiable!).

Definition 1.3.2 (Itô integral on $\mathcal{L}_0$)
The **Itô integral** for $L=(L_t)_{t\geq 0}\in\mathcal{L}_0$ is defined by

$$I_t(L) := \sum_{i=1}^{k-1} \xi_{i-1}(W_{t_i}-W_{t_{i-1}}) + \xi_k(W_t - W_{t_k}),$$

if $t_{k-1}<t\leq t_k$ and $L_t=\sum_{i=1}^n \xi_{i-1}\mathbb{1}_{(t_{i-1},t_i]}(t)$. This can also be written as

$$I_t(L) = \sum_{i=1}^n \xi_{i-1}(W_{t_i\wedge t} - W_{t_{i-1}\wedge t},\ t\in[0,T],$$

where $a\wedge b := \min\{a,b\}$.

Notation:

$$I_t(L) = \int_0^t L_s dW_s$$

Proposition 1.3.3 (Properties of $I_t(L),\ L\in\mathcal{L}_0$)

(a) *The equality known as* **Itô isometry**

$$\mathbb{E}(I_T(L))^2 = \mathbb{E}\int_0^T L_t^2 dt$$

holds.

(b) $(I_t(L))_{t\geq 0}$ *is a square integrable, continuous martingale.*

(c) $I_t(\alpha L+\beta K) = \alpha I_t(L)+\beta I_t(K)$ *for all* $L,K\in\mathcal{L}_0$ *and* $\alpha,\beta\in\mathbb{R}$.

Proof:

(a) By a direct computation,

$$\begin{aligned}\mathbb{E}(I_T(L))^2 &= \mathbb{E}\left(\sum_{i=1}^n \xi_{i-1}(W_{t_i}-W_{t_{i-1}})\right)^2\\ &= \sum_{i=1}^n\sum_{k=1}^n \mathbb{E}\Big(\xi_{i-1}\xi_{k-1}(W_{t_i}-W_{t_{i-1}})(W_{t_k}-W_{t_{k-1}})\Big)\\ &= \sum_{i=1}^n \mathbb{E}\xi_{i-1}^2(t_i-t_{i-1})+0,\end{aligned}$$

because if $i \neq k$, for example $i < k$, then by using the tower property and taking out what is known

$$\begin{aligned}&\mathbb{E}\xi_{i-1}\xi_{k-1}(W_{t_i}-W_{t_{i-1}})(W_{t_k}-W_{t_{k-1}})\\ =&\ \mathbb{E}\mathbb{E}\left[\xi_{i-1}\xi_{k-1}(W_{t_i}-W_{t_{i-1}})(W_{t_n}-W_{t_{n-1}})|\mathcal{F}_{t_{k-1}}\right]\\ =&\ \mathbb{E}\xi_{i-1}\xi_{k-1}(W_{t_i}-W_{t_{i-1}})\mathbb{E}\left[W_{t_k}-W_{t_{k-1}}|\mathcal{F}_{t_{k-1}}\right]=0,\end{aligned}$$

since $W_{t_k}-W_{t_{k-1}}$ is independent from $\mathcal{F}_{t_{k-1}}$ and $\mathbb{E}(W_{t_k}-W_{t_{k-1}})=0$. If $i=k$, then

$$\begin{aligned}\mathbb{E}\xi_{i-1}^2(W_{t_i}-W_{t_{i-1}})^2 &= \mathbb{E}\mathbb{E}\left[\xi_{i-1}^2(W_{t_i}-W_{t_{i-1}})^2|\mathcal{F}_{t_{i-1}}\right]\\ &= \mathbb{E}\xi_{i-1}^2\mathbb{E}\left[(W_{t_i}-W_{t_{i-1}})^2|\mathcal{F}_{t_{i-1}}\right]\\ &= \mathbb{E}\xi_{i-1}^2(t_i-t_{i-1}). \qquad \text{(P1)}\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathbb{E}\int_0^T L_t^2dt &= \mathbb{E}\int_0^T\left(\sum_{i=1}^n \xi_{i-1}\mathbb{1}_{(t_{i-1},t_i]}(t)\right)^2dt\\ &= \mathbb{E}\int_0^T\left(\sum_{i=1}^n \xi_{i-1}^2\mathbb{1}_{(t_{i-1},t_i]}(t)\right)dt\\ &= \mathbb{E}\sum_{i=1}^n \xi_{i-1}^2\int_0^T \mathbb{1}_{(t_{i-1},t_i]}(t)dt\\ &= \mathbb{E}\sum_{i=1}^n \xi_{i-1}^2(t_i-t_{i-1}). \qquad \text{(P2)}\end{aligned}$$

Comparing (P1) and (P2) implies the claim (a).

(b) From (a) the square integrability of $(I_t(L))_{t\geq 0}$ follows, because

$$\mathbb{E}(I_t(L))^2=\mathbb{E}\int_0^t L_s^2ds=\mathbb{E}\sum_{i=1}^n\xi_{i-1}^2(t_i\wedge t-t_{i-1}\wedge t)\leq c^2t, \qquad \text{(P3)}$$

because $\xi_{i-1}^2 \leq c^2$ by the definition of simple processes. A martingale is said to be **continuous** if it has almost surely continuous paths. Hence, it needs to be verified that

$$\mathbb{P}(\{\omega \in \Omega : (I_t(L))_{t \leq 0}(\omega) \text{ is continuous in } t\}) = 1.$$

By the definition of the Brownian motion, $t \mapsto W_t(\omega)$ is a continuous function for all $\omega \in \Omega$. This implies

$$\begin{aligned} I_t(L)(\omega) &= \sum_{i=1}^{n} \xi_{i-1}(\omega)(W_{t_i \wedge t}(\omega) - W_{t_{i-1} \wedge t}(\omega)) \\ &\to \sum_{i=1}^{n} \xi_{i-1}(\omega)(W_{t_i \wedge s}(\omega) - W_{t_{i-1} \wedge s}(\omega)) = I_s(L)(\omega), \end{aligned}$$

as $t \to s$ and thus $I_t(L)(\omega)$ is continuous in t for all $\omega \in \Omega$.

Yet it needs to be shown that $(I_t(L))_{t \geq 0}$ is a martingale:

(1) If $t \in (t_{k-1}, t_k]$, then

$$I_t(L) = \sum_{i=1}^{k-1} \xi_{i-1}(W_{t_i} - W_{t_i - 1}) + \xi_{k-1}(W_t - W_{t_{k-1}}).$$

The random variables ξ_{i-1} are $\mathcal{F}_{t_{i-1}}$-measurable, ξ_{k-1} is $\mathcal{F}_{t_{k-1}}$-measurable, the terms $(W_{t_i} - W_{t_{i-1}})$ are $\mathcal{F}_{t_i}$-measurable and the term $(W_t - W_{t_{k-1}})$ is $\mathcal{F}_t$-measurable. Since $\mathcal{F}_{t_{i-1}} \subseteq \mathcal{F}_t$, $I_t(L)$ is $\mathcal{F}_t$-measurable, and $(I_t(L))_{t \geq 0}$ is adapted.

(2) $\mathbb{E}|I_t(L)| \leq (\mathbb{E}|I_t(L)|^2)^{\frac{1}{2}} < \infty$, because inequality (P3).

(3) For $0 \leq s < t$, assume $t_{k-1} < s < t \leq t_k$. Then

$$\begin{aligned} \mathbb{E}[I_t(L)|\mathcal{F}_s] &= \sum_{i=1}^{k-1} \mathbb{E}\left[\xi_{i-1}(W_{t_i} - W_{t_{i-1}})|\mathcal{F}_s\right] \\ &+ \mathbb{E}\left[\xi_{k-1}(W_t - W_{t_{k-1}})|\mathcal{F}_s\right] \\ &= \sum_{i=1}^{k-1} \xi_{i-1}(W_{t_i} - W_{t_{i-1}}) + \xi_{k-1}(W_s - W_{t_{k-1}}) \\ &= I_s(L) \text{ almost surely.} \end{aligned}$$

(c) Clear from the definition.

□

1.4 Itô's integral for general integrands

Is it possible to define

$$\int_0^T W_t dW_t \ ?$$

The process (W_t) is not piecewise constant, so $(W_t) \notin \mathcal{L}_0$. In this section, the definition of $I_t(L)$ will be extended to a larger class of integrands L. The results are not proven here, but the proofs can be found in [2], [3] and [4]. Let $\mathcal{L}_2$ be the space of the processes $L = (L_t)_{t\in[0,T]}$ which are

- $\mathcal{B}([0,T]) \times \mathcal{F}$-measurable,
- $(\mathcal{F}_t)$-adapted,
- $\mathbb{E}\int_0^T L_t^2 dt < \infty$.

Lemma 1.4.1 *Let $L \in \mathcal{L}_2$. Then there exists a sequence $(L^n)_{n\geq 0}$ of simple processes such that*

$$\lim_{n\to\infty} \mathbb{E} \int_0^T |L_t - L_t^n|^2 dt = 0.$$

Definition 1.4.2 Let $L \in \mathcal{L}_2$. Then define

$$I_t(L) := \lim_{n\to\infty} \int_0^t L_s^n dW_s,$$

where the limit is in L_2-sense, i.e. $I_t(L)$ is the random variable such that

$$\lim_{n\to\infty} \mathbb{E}(I_t(L) - I_t(L^n))^2 = 0.$$

Notation:

$$I_t(L) := \int_0^t L_s dW_s$$

Proposition 1.4.3 (Properties of $\int_0^t L_s dW_s, L \in \mathcal{L}_2$)

(a) **Itô isometry:** $\mathbb{E}(I_t(L))^2 = \mathbb{E}\int_0^t L_s^2 ds$

(b) $(I_t(L))_{t\geq 0}$ *is a square integrable martingale.*

(c) $I_t(\alpha L + \beta K) = \alpha I_t(L) + \beta I_t(K)$ *almost surely for all* $\alpha, \beta \in \mathbb{R}$, $L, K \in \mathcal{L}_2$.

(d) $\mathbb{E} I_t(L) = 0$.

Remark 1.4.4 By (b), $(I_t(L))_{t\geq 0}$ is a square integrable martingale. What about the continuity of $t \mapsto I_t(L)(\omega)$? Here, for each $t \in [0,T]$ the random variable $I_t(L)$ has been defined as a limit in L_2-sense, i.e. $I_t(L)$ is $\mathbb{P}$-a.s. unique.
The stochastic processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are called **modifications** of each other, if $X_t = Y_t$ almost surely for all $t \geq 0$. It can be shown, that $(I_t(L))_{t\geq 0}$ has a modification which has almost surely continuous paths $t \mapsto I_t(L)(\omega)$. In the future, it is assumed, that $(I_t(L))$ refers to the modification, which has almost surely continuous paths. It can be shown, that

$$\lim_{n,m\to\infty} \mathbb{E} \sup_{0\leq t\leq T} \left| \int_0^t L_s^n - L_s^m dW_s \right|^2 = 0.$$

1.5 Itô's formula

Assume a given stochastic process

$$X_t = X_0 + \int_0^t b(s)ds + \int_0^t \delta(s)dW_s,$$

where the second term is a Riemann-integral with

- $b(s) = b(s,\omega)$ is jointly measurable, i.e. b is $\mathcal{B}([0,T]) \otimes \mathcal{F}$-measurable,
- $b(s)$ is $\mathcal{F}_s$-measurable, $\mathbb{E}\int_0^T |b(s)|ds < \infty$, and
- $\delta \in \mathcal{L}_2$.

Then the Itô integral is defined.

Proposition 1.5.1 (Itô's formula) *Let* $f \in C^{1,2}([0,\infty) \times \mathbb{R})$. *Then*

$$\begin{aligned} f(t,X_t) &= f(0,X_0) + \int_0^t \frac{\partial f}{\partial t}(s,X_s)ds \\ &+ \int_0^t \frac{\partial f}{\partial x}(s,X_s)(b(s)ds + \sigma(s)dW_s) + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s,X_s)\delta^2(s)ds. \end{aligned}$$

Remark 1.5.2 If

$$\left(\frac{\partial f}{\partial x}(s,X_s)\delta(s)\right)_{s\in[0,T]} \notin \mathcal{L}_2,$$

then a more general definition is needed for

$$\int_0^t \frac{\partial f}{\partial x}(s,X_s)\delta(s)dW_s.$$

(see, for example [2], section 3.1.)

Example 1.5.3 Let $f(t,x) = e^{x-\frac{t}{2}}$, $X_t = W_t$. Then

$$\begin{aligned} f(t, W_t) &= e^{W_t - \frac{t}{2}} = 1 + \int_0^t -\frac{1}{2} e^{W_s - \frac{s}{2}} ds + \int_0^t e^{W_s - \frac{s}{2}} dW_s + \frac{1}{2} \int_0^t e^{W_s - \frac{s}{2}} ds \\ &= 1 + \int_0^t e^{W_s - \frac{s}{2}} dW_s. \end{aligned}$$

2. Continuous time market models

2.1 The stock price process

A **continuous time market model** consists of

(1) a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$,

(2) a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$, that

- satisfies the "usual conditions"
- $\mathcal{F}_0$ is **trivial**, i.e. for $A \in \mathcal{F}_0$, $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$.
- $\mathcal{F}_T = \mathcal{F}$.

(3) $d + 1$ traded assets:

- d stocks: $S_1(t), ..., S_d(t)$
- one bank account $S_0(t)$

Now assume, that $r : [0, T] \to [0, \infty)$ with $r(0) = 0$, for example $r(t) = r_0 t$, $r_0 \ge 0$, and $S_0(t) = e^{r(t)}$. Stocks model as "generalized geometric Brownian motions". For $d = 1$,

$$S_1(t) = S_1(0) \exp\left(\int_0^t \delta(s) dW_s + \int_0^t \left(\alpha(s) - \frac{1}{2}\delta^2(s) \right) ds \right), \qquad (*)$$

where $\alpha, \delta \in \mathcal{L}_2$. Now Itô's formula implies that

$$\begin{aligned} & (S_1(t)) \text{ is given by } (*) \\ \iff \quad & S_1(t) = S_1(0) + \int_0^t \alpha(s) S_1(s) ds + \int_0^t \delta(s) S_1(s) dW_s, \end{aligned}$$

for all $t \in [0, T]$.
Special case: geometric Brownian motion (with drift)

$$S_1(t) = S_1(0) e^{\delta W_t + (\alpha - \frac{\delta^2}{2})t}, \; t \in [0, T]$$

$$\Longleftrightarrow \quad S_1(t) = S_1(0) + \alpha \int_0^t S_1(s)ds + \delta \int_0^t S_1(s)dW_s, \; t \in [0,T]$$

For $d > 1$, the random influence is assumed to come from a d**-dimensional Brownian motion**

$$W = (W_t^1, ...W_t^d)_{t\in[0,T]},$$

where $(W_t^1)_{t\in[0,T]}, (W_t^2)_{t\in[0,T]}, ..., (W_t^d)_{t\in[0,T]}$ are **independent** Brownian motions. Then for $i = 1, ..., d$, $S_i(t)$ is defined as

$$S_i(t) := S_i(0) + \int_0^t \alpha_i(s)S_i(s)ds + \sum_{i=1}^{d} \int_0^t S_i(s)\delta_{ij}(s)dW_s^j$$

for all $t \in [0,T]$, $\alpha_i, \delta_{ij} \in \mathcal{L}_2$.

2.2 Trading strategies

Assume, that there are shares/stocks $S_1(t), ..., S_d(t), \; t \in [0,T]$, and a non-random bank account $S_0(t), \; t \in [0,T]$.

Definition 2.2.1 The stochastic processes

$$\varphi(t) := (\varphi_0(t), ..., \varphi_d(t)), \; t \in [0,T],$$

form a **trading strategy**, if

(a) the $\varphi_i : [0,T] \times \Omega \to \mathbb{R}$ are $\mathcal{B}([0,T]) \otimes \mathcal{F}$-measurable and adapted ($\varphi_i(t)$ is $\mathcal{F}_t$-measurable for all t), $i = 0, ..., d$.

(b) $\sum\limits_{i=0}^{d} \int\limits_0^T \mathbb{E}(\varphi_i(t)^2)dt < \infty$.

In the definition above, $\varphi_i(t)$ denotes the amount of shares of asset i ($1 \le i \le d$) held in the portfolio at time t. $\varphi_i(t) < 0$ means **short sales**: selling a stock which is not owned, only borrowed.

3. Risk neutral pricing

We want to have a method to compute the fair price of an option (so that riskless profit = **arbitrage** is not possible). The method will be "risk neutral pricing" using an equivalent martingale measure.

Definition 3.1.1 A probability measure $\mathbb{Q}$ defined on $(\Omega, \mathcal{F})$ is a **(strong) equivalent martingale measure** if

(i) $\mathbb{Q}$ is equivalent to $\mathbb{P}$, i.e. $\mathbb{Q}(A) = 0 \iff \mathbb{P}(A) = 0$ for all $A \in \mathcal{F}$

(ii) The **discount price process**

$$\tilde{S} := \left(\frac{S(t)}{S_0(t)} \right)_{t \geq 0}$$

is a $\mathbb{Q}$-martingale.

It will be shown that (like in the discrete time case) for certain models an equivalent martingale measure (EMM) exists

- uniquely
- not uniquely (there are more than one EMM) or
- not at all.

Definition 3.1.2

(a) The **value of the portfolio** $\varphi_t = (\varphi_0(t), \varphi_1(t), ..., \varphi_d(t))$ at time t is given by

$$V_\varphi(t) := \sum_{i=0}^{d} \varphi_i(t) S_i(t), \; t \in [0, T].$$

The process $(V_\varphi(t))_{t \in [0,T]}$ is called the **value process** (or wealth process) of the trading strategy φ.

(b) The **gains process** is

$$G_\varphi(t) := \sum_{i=0}^{d} \int_0^t \varphi_i(u) dS_i(u),$$

if the stochastic integral is well-defined.

(c) A trading strategy is called **self-financing** if

- $\int_0^t \varphi_i(u)dS_i(u)$ is well-defined
- $V_\varphi(t) = V_\varphi(0) + G_\varphi(t)$, $t \in [0,T]$.

Remark 3.1.3 If

$$S_i(t) = S_i(0) + \int_0^t \alpha_i(s)S_i(s)ds + \sum_{j=1}^d \int_0^t S_i(s)\delta_{ij}(s)dW_s^j,$$

then

$$\int_0^t \varphi_i(u)dS_i(u) = \int_0^t \varphi_i(u)\alpha_i(u)S_i(u)du + \sum_{j=1}^d \int_0^t \varphi_i(u)S_i(u)\delta_{ij}dW_u^j.$$

By denoting the first term by A and the second term by B, then by Hölder's inequality

$$\mathbb{E}\Big(\int_0^t \varphi_i(u)dS_i(u)\Big)^2 = \mathbb{E}(A+B)^2 \le 2\mathbb{E}A^2 + 2\mathbb{E}B^2.$$

For

$$\mathbb{E}A^2 = \mathbb{E}\Big(\int_0^t \varphi_i(u)\alpha_i(u)S_i(u)du\Big)^2 \le t\mathbb{E}\int_0^t \Big(\varphi_i(u)S_i(u)\alpha_i(u)\Big)^2 du < \infty$$

and for $\mathbb{E}B^2$,

$$\mathbb{E}\Big(\int_0^t \varphi_i(u)S_i(u)\delta_{ij}(u)dW_u^j\Big)^2 = \sum_{j=1}^d \mathbb{E}\int_0^t \Big(\varphi_i(u)S_i(u)\delta_{ij}(u)\Big)^2 du < \infty.$$

The above is true, because

$$\mathbb{E}\int_0^t a(u)dW_u^1 \int_0^t b(u)dW_u^2 = 0$$

if W^1 and W^2 are independent Brownian motions, and $a, b \in \mathcal{L}_2$. See the exercises for $(a(u))$ and $(b(u))$ simple processes.

Definition 3.1.4 Assume $\mathbb{Q}$ is an equivalent martingale measure. A strategy φ is **admissible**, if

(i) it is self-financing,

(ii) $V_\varphi(t) \ge 0,\ t \in [0,T]$

(iii) $\mathbb{E}_{\mathbb{Q}} \sup_{0\le t\le T} \tilde{V}_\varphi(t)^2 < \infty$, where

$$\tilde{V}_\varphi(t) := \frac{V_\varphi(t)}{S_0(t)}.$$

Definition 3.1.5 A trading strategy φ is called an **arbitrage opportunity** if

(i) φ is admissible,

(ii) $V_\varphi(0) = 0$ and $\mathbb{P}(V_\varphi(T) > 0) > 0$.

Any non-negative random variable H we will call **an option**. Often H is a function of $S_i(T)$, the terminal value of the i-th share price process. For example,

$$H = f(S_i(T)) = (S_i(T) - K)^+,$$

the European call option. But there also exists **"basket options"** like

$$H = \big(S_1(T) + ... + S_d(T) - K\big)^+$$

or **"Asian options"**, depending not only on the last time T,

$$H = \left(\frac{1}{T}\int_0^T S_1(t)dt - K\right)^+.$$

Proposition 3.1.6 *If the equivalent martingale measure $\mathbb{Q}$ exists uniquely and $\mathbb{E}_{\mathbb{Q}}H^2 < \infty$, then there exists an admissible trading strategy*

$$\varphi = \big(\varphi_0(s), ..., \varphi_d(s)\big)_{s\in[0,T]}$$

such that

$$\tilde{H} = V_0(\varphi) + \sum_{i=1}^d \int_0^T \varphi_i(s)d\tilde{S}_i(s).$$

Proof:
Follows from the "Martingale representation theorem". The theorem says that if there is exactly one $\mathbb{Q} \sim \mathbb{P}$ such that $(\tilde{S}_i(t))_{t\in[0,T]}$, $i = 1, ..., d$ are $\mathbb{Q}$-martingales, then any option $H \ge 0$ with $\mathbb{E}_{\mathbb{Q}}H^2 < \infty$ can be hedged. It can be shown, that $\mathbb{E}_{\mathbb{Q}} \int_0^T \varphi_i(t)d\tilde{S}_i(t) = 0$. Hence

$$\mathbb{E}_{\mathbb{Q}}\tilde{H} = V_0(\varphi),$$

which is the **fair price** of the option H.

Since arbitrage opportunities appear in reality only temporarily, the aim is to construct market models which do not admit arbitrage opportunities. We continue with a formulation of the **"Fundamental Theorem of Asset pricing"**:

Proposition 3.1.7 *If a market model has an equivalent martingale measure* $\mathbb{Q}$*, then it does not admit arbitrage.*

Remark 3.1.8 The other implication is not always true in this general setting. But there exists an "if and only if"-relation, if "no arbitrage opportunities" is replaced by "no free lunch with vanishing risk" (see [1], page 235).

4. The Black-Scholes model

Assume a probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ and a Brownian motion W with respect to $\mathbb{F}$. We will consider the model suggested by Black and Scholes to do some computations related to the theory introduced before.
The model consists of

- one riskless asset $S_0(t) = e^{rt},\ t \geq 0$, where $r > 0$ is the (instantaneous) interest rate, and
- one risky asset $S(t) = s_0 e^{\delta W_t - \frac{\delta^2}{2}t + \mu t}$.

It can be easily checked, that $S_0(t)$ solves the differential equation

$$\begin{cases} dS_0(t) &= rS_0(t)dt \\ S_0(0) &= 1, \end{cases} \tag{*}$$

and that $S_0(t)$ is the only solution of (*). It is also true, that

$$\begin{cases} dS(t) &= \delta S(t)dW_t + \mu S(t)dt \\ S(0) &= s_0 \end{cases} \tag{**}$$

is solved by

$$S(t) = s_0 e^{\delta W_t - \frac{\delta^2}{2}t + \mu t}.$$

4.1 The equivalent martingale measure $\mathbb{Q}$

We want to determine the equivalent martingale measure $\mathbb{Q}$. It has the properties:

1. $\mathbb{Q} \sim \mathbb{P}$
2. $\tilde{S}(t) = \frac{S(t)}{S_0(t)} = s_0 e^{\delta W_t - \frac{\delta^2}{2}t + (\mu - r)t}$, $t \in [0, T]$ is a $\mathbb{Q}$-martingale.

The measure $\mathbb{Q}$ can be found using Girsanov's theorem:

Proposition 4.1.1 (Girsanov's theorem)
Let $(\theta_t)_{t \in [0,T]}$ be an adapted process satisfying $\int_0^T \theta_s^2 ds < \infty$ almost surely and such that the process

$$H_t = \exp\left(-\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right),\ t \in [0, T]$$

is a martingale. Then with respect to $\mathbb{Q}$,

$$\mathbb{Q}(A) := \int_A H_T d\mathbb{P}(\omega),$$

the process

$$B_t := W_t + \int_0^t \theta_s ds, \ 0 \le t \le T$$

is a **standard Brownian motion**.

Remark 4.1.2 It is often difficult in applications to check whether $(H_t)_t$ is a martingale. The **Novikov condition**

$$\mathbb{E} e^{\frac{1}{2}\int_0^T \theta_t^2 dt} < \infty$$

is a sufficient condition for (H_t) being a martingale.
Now by Itô's formula,

$$\begin{aligned}
\tilde{S}(t) &= s_0 + \delta \int_0^t \tilde{S}(s) dW_s + \int_0^t \tilde{S}(s)(-\frac{\delta^2}{2} + (\mu - r))ds \\
&+ \frac{\delta^2}{2} \int_0^t \tilde{S}(s) ds \\
&= s_0 + \delta \int_0^t \tilde{S}(s) dW_s + \int_0^t \tilde{S}(s)(\mu - r) ds \\
&= s_0 + \delta \int_0^t \tilde{S}(s) dB_s,
\end{aligned}$$

with $B_s = W_s + \frac{\mu - r}{\delta} s$. Hence $\mathbb{Q}$ is given by $\mathbb{Q}(A) := \int_A H_t d\mathbb{P}$, where

$$H_t := \exp\left(-\frac{\mu - r}{\delta} W_t + \frac{(\mu - r)^2}{2\delta^2} t\right).$$

4.2 Pricing: The Black-Scholes formula

Assume we have an European call-option with strike price $K > 0$ at time $T > 0$,

$$f(x) = (x - K)^+$$

What is the fair price of the option $f(S_T) = (S_T - K)^+$? This question will be answered using the so-called **martingale representation** (see [4]).

Definition 4.2.1 Let $(X_t)_{0 \le t \le T}$ be a stochastic process on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

(i) Set

$$\begin{aligned}\mathcal{F}_t &:= \sigma(X_s, 0 \le s \le t)\\ &= \text{the smallest } \sigma\text{-algebra, such that } X_s \text{ is}\\ &\quad\ \mathcal{F}_t\text{-measurable for all } s \in [0,t].\end{aligned}$$

Then $(\mathcal{F}_t)_{t\in[0,T]}$ is the **filtration generated by** (X_t).

(ii) Now all $\mathbb{P}$-null sets of $\mathcal{F}$ are added to each $\mathcal{F}_t$:

$$\mathcal{F}_t^X := \mathcal{F}_t \cup \{A \in \mathcal{F} : \mathbb{P}(A) = 0\}.$$

Then $(\mathcal{F}_t^X)_{t\in[0,T]}$ is again a filtration, and it is called the **natural filtration of** $(X_t)_{t\in[0,T]}$.

Proposition 4.2.2 (Brownian martingale representation)
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $(W_t)_{t\in[0,T]}$ a Brownian motion, $(\mathcal{F}_t^W)_{t\in[0,T]}$ its natural filtration. Let $(M_t)_{t\in[0,T]}$ be a square integrable martingale with respect to $(\mathcal{F}_t^W)_{t\in[0,T]}$. Then there exists an $(\mathcal{F}_t^W)_{t\in[0,T]}$-adapted process $(L_t)_{t\in[0,T]}$, such that $\mathbb{E}\int_0^T L_t^2 dt < \infty$, and

$$M_t = M_0 + \int_0^t L_s dW_s \text{ almost surely for all } t \in [0,T].$$

The above Proposition 4.2.2 is applied in the following way:

- $(\widetilde{S(T) - K})^+$ is square integrable:

$$\mathbb{E}_{\mathbb{Q}}((\widetilde{S(T) - K})^+)^2 \le \mathbb{E}_{\mathbb{Q}}(\widetilde{S(T)})^2 = 1 < \infty$$

- $M_t := \mathbb{E}_{\mathbb{Q}}\left[(\widetilde{S(T) - K})^+ | \mathcal{F}_t^B\right]$ is a square integrable martingale with respect to $(\mathcal{F}_t^B)_{t\in[0,T]}$, where $(B_s)_{s\in[0,T]}$ is the Brownian motion with respect to $\mathbb{Q}$, see page 24.

- Proposition 4.2.2 implies that there exists $(L_s)_{s\in[0,T]}$ which is $(\mathcal{F}_t^B)$-adapted and $\mathbb{E}_{\mathbb{Q}}\int_0^T L_s^2 ds < \infty$, such that

$$\left(\widetilde{S(T) - K}\right)^+ = M_T = M_0 + \int_0^T L_s dB_s. \qquad (*)$$

On the other hand, if the equation

$$\left(\widetilde{S(T) - K}\right)^+ = \widetilde{V_\varphi(T)} = V_\varphi(0) + \int_0^T \varphi(u) d\widetilde{S(u)} \qquad (**)$$

would be true with $(\varphi(u))$ being adapted and $\mathbb{E}_{\mathbb{Q}}(\int_0^T \varphi(u)d\widetilde{S(u)})^2 < \infty$, then $(\varphi(u))_{u\in[0,T]}$ would be a self-financing trading strategy. Because of

$$\widetilde{S(t)} = s_0 + \delta \int_0^t \widetilde{S(s)}dB_s,$$

where $(B_s)_{s\in[0,T]}$ is the Brownian motion with respect to $\mathbb{Q}$, (**) implies

$$(\widetilde{S(T)-K})^+ = V_\varphi(0) + \int_0^T \varphi(u)\tilde{S}(u)\delta dB_u. \qquad (***)$$

By comparing (*) and (***),

$$\begin{aligned}
0 &= \mathbb{E}_{\mathbb{Q}}\Big|(\widetilde{S(T)-K})^+ - (\widetilde{S(T)-K})^+\Big|^2 \\
&= \mathbb{E}_{\mathbb{Q}}\Big(M_0 + \int_0^T L_s dB_s - V_\varphi(0) - \int_0^T \varphi(u)\widetilde{S(u)}\delta dB_u\Big)^2 \\
&= \mathbb{E}_{\mathbb{Q}}\Big(M_0 + V_\varphi(0)\Big)^2 \\
&+ 2\mathbb{E}_{\mathbb{Q}}\Big(M_0 - V_\varphi(0)\Big)\Big(\int_0^T L_s dB_s - \int_0^T \varphi(u)\widetilde{S(u)}\delta dB_u\Big) \\
&+ \mathbb{E}_{\mathbb{Q}}\Big(\int_0^T L_s dB_s - \int_0^T \varphi(u)\widetilde{S(u)}\delta dB_u\Big)^2 \\
&= \mathbb{E}_{\mathbb{Q}}\Big(M_0 - V_\varphi(0)\Big)^2 + \mathbb{E}_{\mathbb{Q}}\int_0^T \Big|L_u - \varphi(u)\widetilde{S(u)}\delta\Big|^2 du,
\end{aligned}$$

where the last equality follows the fact that $\mathbb{E}_{\mathbb{Q}}\int_0^T L_u - \varphi(u)\widetilde{S(u)}\delta dB_u = 0$ and Itô isometry. This implies, that

$$\begin{aligned}
V_\varphi(0) &= M_0 \\
\varphi(u) &= \frac{L_u}{\widetilde{S(u)}\delta} \text{ and} \\
\widetilde{V_\varphi(t)} &= M_t
\end{aligned}$$

$\mathbb{Q}\otimes dt$-almost everywhere. Since (B_u) is a Brownian motion with respect to $\mathbb{Q}$, by Proposition 1.4.3

$$\tilde{V}_\varphi(t) = V_0(0) + \int_0^t \varphi(u)\tilde{S}(u)\delta dB_u, \ t\in[0,T]$$

is a square integrable martingale.
Now, if

$$V_\varphi(0) \;=\; \text{the price of the option at time 0.}$$

$$\tilde{V}_\varphi(t) \quad = \quad \text{discounted price of the option at time } t.$$

Then the price is

$$\begin{aligned}
V_\varphi(t) &= e^{rt}\mathbb{E}_{\mathbb{Q}}\left[\frac{(S(T)-K)^+}{e^{rT}}\middle|\mathcal{F}_t^B\right] \\
&= \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}(S(T)-K)^+|\mathcal{F}_t^B] \\
&= \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}\left(S(t)\frac{S(T)}{S(t)}-K\right)^+\middle|\mathcal{F}_t^B\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}\left(x\frac{S(T)}{S(t)}-K\right)^+\middle|\mathcal{F}_t^B\right]\Bigg|_{x=S(t)},
\end{aligned}$$

where $S(t)$ is independent from $\mathcal{F}_t^B$ and $\mathcal{F}_t^B$-measurable. For the last equality, see [5], proposition A.25.
Now, set

$$F(t,x) := \mathbb{E}_{\mathbb{Q}}e^{-r(T-t)}\left(x\frac{S(T)}{S(t)}-K\right)^+,$$

then $V_\varphi(t) = F(t,S(t))$. By $\widetilde{S(u)} = s_0 e^{\delta B_u - \frac{\delta^2 u}{2}}$ (see p.24)

$$\begin{aligned}
F(t,x) &= \mathbb{E}_{\mathbb{Q}}e^{-r(T-t)}\left(x\frac{e^{rT}\tilde{S}(T)}{e^{rt}\tilde{S}(t)}-K\right)^+ \\
&= e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\left(xe^{r(T-t)}e^{\delta(B_T-B_t)-\frac{\delta^2(T-t)}{2}}-K\right)^+ \\
&= e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\left(xe^{r(T-t)}e^{\delta(B_{T-t})-\frac{\delta^2(T-t)}{2}}-K\right)^+,
\end{aligned}$$

since $B_T - B_t \overset{d}{=} B_{T-t}$. By substituting $T-t =: a$ and denoting $X = \frac{B_{T-t}}{\sqrt{T-t}}$, making X standard Gaussian, i.e. $\mathbb{P}(X \leq z) = \mathcal{N}_{0,1}(z)$, $F(t,x)$ can be expressed as

$$\begin{aligned}
F(t,x) &= e^{-ra}\mathbb{E}_{\mathbb{Q}}\left(xe^{ra}e^{\delta\sqrt{a}X-\frac{\delta^2 a}{2}}-K\right)^+ \\
&= e^{-ra}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(xe^{ra}e^{\delta\sqrt{a}z-\frac{\delta^2 a}{2}}-K\right)^+ e^{-\frac{z^2}{2}}dz \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(xe^{\delta\sqrt{a}z-\frac{\delta^2 a}{2}}-Ke^{-ra}\right)\mathbb{1}_{\{z+d_2\geq 0\}}e^{-\frac{z^2}{2}}dz,
\end{aligned}$$

where

$$d_1 = \frac{\log\frac{x}{K}+(r+\frac{\delta^2}{2})a}{\delta\sqrt{a}} \quad \text{and } d_2 = d_1 - \delta\sqrt{a},$$

with $a = T-t$. Then $\{z+d_2 \geq 0\} = \{-z \leq d_2\}$, and

$$F(t,x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_2}\left(xe^{-\delta\sqrt{a}z-\frac{\delta^2 a}{2}}-Ke^{-ra}\right)e^{-\frac{z^2}{2}}dz$$

$$= \quad ... \quad = x\mathcal{N}_{0,1}(d_1) - Ke^{-ra}\mathcal{N}_{0,1}(d_2).$$

This is the Black-Scholes formula

$$F(t,x) = x\mathcal{N}_{0,1}(d_1) - Ke^{-r(T-t)}\mathcal{N}_{0,1}(d_2).$$

4.3 Example of infinitely many EMM's

Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space and (W_t^1) and (W_t^2) are independent $\mathbb{F}-$Brownian motions. Let $(S_t)_{t\in[0,T]}$, $S_t := e^{\delta W_t - \frac{\delta^2 t}{2}}$ be the share price process. Then (S_t) is a $\mathbb{P}$-martingale with respect to $(\mathcal{F}_t^{W^1})_{t\in[0,T]}$, the natural filtration of (W_t^1). Set $H_t := e^{\theta W_t^2 - \frac{T\theta^2}{2}}$, $\theta > 0$. Then

$$\mathbb{Q}^\theta(A) := \mathbb{E}_\mathbb{P}\left(1\!\!1_A H_T\right)$$

is an equivalent martingale measure for any $\theta > 0$:

- $0 < H_T < \infty$ $\mathbb{P}$-a.s.
- Since $H_T \frac{S_t}{S_s}$ and $\mathcal{F}_s^{W_1}$ are independent, as are H_T and $\frac{S_t}{S_s}$, and H_T and S_T are square integrable, the martingale property is satisfied as

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}^\theta}[S_t|\mathcal{F}_s^{W_1}] &= \mathbb{E}\left[H_T S_t|\mathcal{F}_s^{W_1}\right] \\
&= S_s\mathbb{E}\left[H_T\frac{S_t}{S_s}\middle|\mathcal{F}_s^{W_1}\right] \\
&= S_s\mathbb{E}\left(H_T\frac{S_t}{S_s}\right) \\
&= S_s\mathbb{E}H_T\mathbb{E}\frac{S_t}{S_s} = S_s,
\end{aligned}$$

  because $\mathbb{E}H_T = 1 = \mathbb{E}\frac{S_t}{S_s}$.

4.4 Example of no EMM's

Assume $r = 0$ and price processes are chosen as

$$S_i(t) = S_i(0)e^{W_t + (\mu_i - \frac{1}{2})t}, \ i = 1, 2 \ \text{ and } \mu_1 \neq \mu_2.$$

Then $H_i^1 = e^{-\mu_i W_T + \frac{\mu_i^2}{2}T}$ defines according to Proposition 4.1.1 the equivalent martingale measure $\mathbb{Q}_i$ for S_i. For $\mathbb{Q}_1 = \mathbb{Q}_2$ the condition $\mu_1 = \mu_2$ would be needed.

5. Bonds

A **bond** is a debt security. The issuer (government, credit institutes, companies) has to pay interest (coupon) once or twice a year and to repay the amount (principal) and the interest at the maturity time. Zero-coupon bonds do not pay interest until maturity time. Modeling of a zero-coupon bond:

- $p(t,T)$ = price of a zero-coupon bond at time $t \leq T$ that pays 1 € at time T.
- $p(t,t) = 1$.

If the "instantaneous" interest rate is a constant r, then

$$p(t,T) = e^{-r(T-t)}$$

because for 1 € one would get $e^{r(T-t)}$ interest for the time amount $T-t$. (The relation between annual interest rate r_a and instantaneous interest rate r is

$$r_a = \lim_{n\to\infty} \left(1 + \frac{rT}{n}\right)^n - 1 = e^{rT} - 1 \quad \text{for } T = 1 \text{ year.)}$$

Now we assume that the interest rate is not fixed, but changes randomly.

Similar to Proposition 3.1.7, it holds that if the market model $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}, S)$ has an equivalent martingale measure, then it does not admit arbitrage.
If $\mathbb{Q}$ is an equivalent martingale measure, then $\mathbb{Q} \sim \mathbb{P}$ and the discounted **price processes of the basic securities** have to be $\mathbb{Q}$-martingales. If we consider the bond market, then the discounted zero-coupon bonds have to be $\mathbb{Q}$-martingales.

Definition 5.1.1 The **fair price for a zero-coupon bond** with maturity T at time t is

$$\frac{p(t,T)}{S_0(t)} := \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{S_0(T)}\middle|\mathcal{F}_t\right] \text{ almost surely,}$$

under the assumption $S_0(t) = \exp(\int_0^t r(s)ds)$ with a random adapted process $(r(s))_{s\in[0,T]}$. So $p(t,T)$, the price of a bond at time t which matures at T is

$$p(t,T) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds}|\mathcal{F}_t\right].$$

Remark 5.1.2 There are other approaches to describe bonds and bond prices. See [1] for the Heath-Jarrow-Morton-model.
The "short rate" r(s) can be modeled with several proposals:
Vasicek model, see [2]:

$$r(s) = r_0 + \int_0^s \alpha - \beta r(u) du + \int_0^s \gamma dW_u,$$

Cox-Ingersoll-Ross, see [2]:

$$r(s) = r_0 + \int_0^s \alpha - \beta r(u) du + \int_0^s \delta(u) \sqrt{r(u)} dW_u,$$

and other more complicated ones.

6. Currency markets

Sometimes the assets are in several countries simultaneously. We consider two countries: a domestic country with interest rate r_d and a foreign country with interest rate r_f, and bank accounts in both countries, denoted by $B_d(t) = e^{r_d t}$ and $B_f(t) = e^{r_f t}$. We introduce an exchange rate process $(Q(t))_{t\geq 0}$ to pass denomination in foreign to domestic currency. $(Q(t))_{t\geq 0}$ depends on the two economies, the policies of the governments, etc, so $(Q(t))$ is influenced by a "multidimensional noise", modelled with independent Brownian motions $(W_t^1)_t, ..., (W_t^d)_t$. Then

$$\begin{cases} dQ(t) &= Q(t)\mu dt + Q(t)(\delta_1 dW_t^1 + ... + \delta_d dW_t^d) \\ Q(0) &= q_0 > 0 \end{cases}$$

if and only if

$$Q(t) = q_0 \exp\left(\sum_{i=1}^{d} \delta_i W_t^i + \left(\mu - \frac{1}{2}\sum_{i=1}^{d}\delta_i^2\right) t\right) \qquad (\dagger)$$

We compute the discounted value of the foreign savings account (discounted by the domestic interest rate) in domestic currency,

$$\tilde{Q}(t) := \frac{B_f(t)Q(t)}{B_d(t)}, \qquad (\ddagger)$$

and get from (†) and (‡) that

$$\tilde{Q}(t) = q_0 \exp\left(\sum_{i=1}^{d} \delta_i W_t^i + \left(\mu - \frac{1}{2}\sum_{i=1}^{d}\delta_i^2\right) t + (r_f - r_d)t\right).$$

To avoid arbitrage between domestic and foreign bond markets, such an equivalent martingale measure $\tilde{Q}(t)$ is needed that

$$\tilde{Q}(t) = q_0 + \int_0^t \tilde{Q}(s)(\mu + r_f - r_d)ds + \sum_{i=1}^{d} \delta_i \int_0^t \tilde{Q}(s) dW_s^i$$

does not have the drift term $\int_0^t \tilde{Q}(s)(\mu + r_f - r_d)ds$. In case there exist d independent trading assets the equivalent martingale measure $\mathbb{Q}$ is unique.

It is called the **domestic martingale measure**. By considering **currency options**, for example the currency European call-option:

$$f(Q(t)) := (Q(T) - K)^+$$

Then the fair price at time t is given by

$$C(t) = \mathbb{E}_{\mathbb{Q}} e^{-r_d(T-t)}(Q(T) - K)^+.$$

7. Credit risk

Credit risk is the risk of loss if a loan or its interest is not paid back. There exists two methods to model credit risk:

- **reduced form models**
  A point process is used to describe the default event.

- **structural models**
  The traded assets itself are used to describe the default event.

Default event is the event where the debtor does not make a scheduled payment - "the debtor defaults".

7.1 A simple model for credit risk

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ be a filtered probability space, and $T^* > 0$ a time horizon. We assume a firm's value process

$$V(t) = V(0) + r\int_0^t V(s)ds - \delta\int_0^t V(s)ds + \delta\int_0^t V(s)dW_s,$$

where r is the interest rate and δ is either the constant dividend rate if $\delta > 0$ or the constant "pay-in" rate, if $\delta < 0$. The process can be equivalently written (see page 24) as

$$V(t) = V(0)\exp\left(\delta W_t - \frac{\delta^2 t}{2} + (r-\delta)t\right).$$

We assume that there exists an equivalent martingale measure $\mathbb{Q}$ to have an arbitrage-free market. Then the discounted tradable securities which pay no dividends follow $\mathbb{Q}$-martingales. By assuming that the firm's value process **is** a tradable security, we get for $\delta \geq 0$

$$\widetilde{V}(t) = V(0)e^{\delta W_t - \frac{\delta^2 t}{2}}.$$

If (W_t) is a $\mathbb{Q}$-Brownian motion, then $(\widetilde{V}(t))$ is a $\mathbb{Q}$-martingale. Now we consider a zero-coupon bond with notational amount F (=face value) and maturity $T \leq T^*$.

So the cash received by the "owner of the defaultable claim" (= the one who gives the credit to the firm in form of buying the bond) is

$$D_T = \begin{cases} F, & \text{if } V_T \geq F \\ V_T, & \text{if } V_T < F. \end{cases}$$

Now, define a "default time"

$$\tau := T\mathbb{1}_{\{V_T<F\}} + \infty\mathbb{1}_{\{V_T \geq F\}}.$$

Then the payoff for the "equity owners" (=owners of the firm) is

$$C_T = (V_T - F)^+,$$

which can be interpreted as a **call option on the value of the firm**. By Black-Scholes formula (see page 28),

$$C_t = \mathbb{E}_{\mathbb{Q}}(\widetilde{V_T - F})^+ = V_t e^{-\delta(T-t)}\mathcal{N}_{0,1}(d_1) - Fe^{-r(T-t)}\mathcal{N}_{0,1}(d_2) \qquad (\maltese)$$

with

$$d_1 = \frac{\log\frac{V_t}{F} + (r - \delta + \frac{\delta^2}{2})(T-t)}{\delta\sqrt{T-t}} = d_2 + \delta\sqrt{T-t}.$$

The bond owners (= the one who gave the credit) payoff is

$$D_T = F - (F - V_T)^+.$$

What is the **fair price of this bond**, or in other words, how much **credit** can this firm get?

Proposition 7.1.1 *Assume the above assumptions. Then*

Bonds = a risk-free payment − put-option on the value of the firm,

so we have

$$\begin{aligned} p(t,T) &= Fe^{-r(T-t)} - \mathbb{E}_{\mathbb{Q}}\big[e^{-r(T-t)}(F - V_T)^+\big|\mathcal{F}_T\big] \\ &= V_t e^{-\delta(T-t)}\mathcal{N}_{0,1}(-d_1) + Fe^{-r(T-t)}\mathcal{N}_{0,1}(d_2). \end{aligned}$$

Proof:
The proof is done by using the call-put-parity

$$C_T - P_T = (V_T - F)^+ - (F - V_T)^+ = V_T - F$$

and knowing that

$$e^{-(r-\delta)t}V_t = M_t$$

is a martingale. We have

$$C_t = e^{rt}\mathbb{E}_{\mathbb{Q}}\big[\widetilde{C_T}\big|\mathcal{F}_t\big]$$

and

$$P_t = e^{rt}\mathbb{E}_{\mathbb{Q}}\big[\widetilde{P_T}\big|\mathcal{F}_t\big],$$

which imply that

$$\begin{aligned} C_t - P_t &= e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\big[V_T\big|\mathcal{F}_t\big] - e^{-r(T-t)}F \\ &= e^{-r(T-t)+(r-\delta)T-(r-\delta)t}V_t - e^{-r(T-t)}F \\ &= V_t e^{-\delta(T-t)} - e^{-r(T-t)}F. \end{aligned}$$

This yields

$$Fe^{-r(T-t)} - P_t = V_t e^{-\delta(T-t)} - C_t$$

But by (✠),

$$\begin{aligned} p(t,T) &= Fe^{-r(T-t)} - P_t \\ &= V_t e^{-\delta(T-t)} - V_t e^{-\delta(T-t)}\mathcal{N}_{0,1}(d_1) + Fe^{-r(T-t)}\mathcal{N}_{0,1}(d_2) \\ &= V_t e^{-\delta(T-t)}\mathcal{N}_{0,1}(-d_1) + Fe^{-r(T-t)}\mathcal{N}_{0,1}(d_2) \end{aligned}$$

□

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