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Stochastic Partial Differential Equations

Exercises 1 for the 22-th January 2002

1. For $L > 0$, $t \geq 0$, $x \in [0, L]$, and $n \geq 1$ let
$$u_n(t,x) := \sin\left(\frac{nx\pi}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 t}.$$
Show that $u = \sum_{n=1}^{N} \alpha_n u_n$ with $\alpha_n \in \mathbb{R}$ satisfies the HEAT EQUATION
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

2. For $t > 0$ and $x, y \in \mathbb{R}$ let
$$\Gamma(t,x,y) := \frac{2}{L} \sum_{n=1}^{\infty} e^{-\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{nx\pi}{L}\right) \sin\left(\frac{ny\pi}{L}\right).$$
   (a) Show that the sum in the definition of $\Gamma(t,x,y)$ exists for $t > 0$.

   (b) Letting $\varphi(y) = \sum_{n=1}^{N} \alpha_n \sin\left(\frac{ny\pi}{L}\right)$ and
$$u(t,x) := \int_{\mathbb{R}} \Gamma(t,x,y)\varphi(y)dy,$$
one shows that $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ for $t > 0$ and $\lim_{t \downarrow 0} u(t,x) = \varphi(x)$.

   (c^*) One shows that there is a constant $c > 0$ such that for all $t > 0$ and all $x, y \in \mathbb{R}$ one has that
$$|\Gamma(t,x,y)| \leq \frac{c}{\sqrt{t}}.$$