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Probability 3
Exercises 2 (13th of November 2008)

(1) Assume that $f, g : \Omega \to \mathbb{R}$ are independent random variables with the same distribution. Show that $\mathbb{E}|f| < \infty$ if and only if $\mathbb{E}|f+g| < \infty$.

(2) Show that

(a) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$ for all $z_1, z_2 \in C$,

(b) $e^{z_1+z_2} = e^{z_1} e^{z_2}$ for all $z_1, z_2 \in C$,

(c) $\left|e^{i\alpha} - e^{i\beta}\right| \le |\alpha - \beta|$ for all $\alpha, \beta \in R$.

(3) Let $n \in \{1, 2, 3, ...\}$, $\Omega = \mathbb{R}$, and $\mathcal{F} = \mathcal{B}(\mathbb{R})$. Define

$$\mu(B) := \frac{\operatorname{card}(B \cap \{1, 2, ..., n\})}{n}.$$

(a) Show that

$$\widehat{\mu}(x) = \frac{e^{ix}}{n} \frac{e^{inx} - 1}{e^{ix} - 1},$$

where $x \in \mathbb{R}$ and $x \neq 2\pi k$ for all $k \in \{0, -1, 1, -2, 2, ...\}$.

(b) Compute $\widehat{\mu}(2\pi k)$ for all $k \in \{0, -1, 1, -2, 2, ...\}$.

(4) For $\lambda > 0$ and $\mu := \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} \delta_k$, where δ_k is the Dirac-measure at k, prove that $\widehat{\mu}(x) = e^{\lambda(e^{ix}-1)}$.

(5) For $a > 0$ and

$$\mu(B) := \frac{1}{2a} \int_{-a}^{a} \chi_B(x)dx, \quad B \in \mathcal{B}(\mathbb{R})$$

compute $\widehat{\mu}(x)$.