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Probability 3
Exercises 5 (4th of Dezember 2008)

(1) Does there exist a measure $\mu \in \mathcal{M}_1^+(\mathbb{R}^n)$ such that $\hat{\mu}(x) = 1$ for all $x \in \mathbb{R}$? If yes, how can one find out μ?

(2) Assume probability measures $\mu_1, ..., \mu_n$ on $\mathbb{R}$ with densities $\varphi_1, ..., \varphi_n$, i.e.

$$\mu_k(B) = \int_B \varphi_k(x) d\lambda(x)$$

where λ is the one-dimensional Lebesgue measure. Show that the function $\varphi : \mathbb{R}^n \to [0, \infty)$ defined by

$$\varphi(x_1, ..., x_n) := \varphi_1(x_1) \cdots \varphi_n(x_n)$$

is the density of the product measure $\mu_1 \times \cdots \times \mu_n$ which is a Borel measure on $\mathbb{R}^n$. Does this apply to the standard Gaussian measures $\gamma^{(1)}$ and $\gamma^{(n)}$?

Hint: Define a measure $\nu \in \mathcal{M}_1^+(\mathbb{R}^n)$ through the density φ and show that ν and $\mu_1 \times \cdots \times \mu_n$ coincide by testing on an appropriate π-system.

(3) Let $g_1, g_2, g_3, g_4 : \Omega \to \mathbb{R}$ be independent standard Gaussian random variables, i.e. the laws of the g_k's are the standard Gaussian measure on $\mathbb{R}$. Compute the covariance matrix of the Gaussian measure $\mu = law(f) \in \mathcal{M}_1^+(\mathbb{R}^4)$, where $f(\omega) = (f_1(\omega), f_2(\omega), f_3(\omega), f_4(\omega))$ and

$$\begin{aligned} f_1(\omega) &:= 1 + g_1(\omega) + g_2(\omega) + g_3(\omega) + g_4(\omega) \\ f_2(\omega) &:= 2 + g_1(\omega) - g_2(\omega) + g_3(\omega) - g_4(\omega) \\ f_3(\omega) &:= 3 + g_1(\omega) + g_2(\omega) - g_3(\omega) - g_4(\omega) \\ f_4(\omega) &:= 4 + g_1(\omega) - g_2(\omega) - g_3(\omega) + g_4(\omega). \end{aligned}$$

(4*) Do there exist random variables $f_1, f_2 : \Omega \to \mathbb{R}$ such that $law(f_1)$ and $law(f_2)$ are Gaussian measures, but $law(F)$, $F(\omega) := (f_1(\omega), f_2(\omega)) \in \mathbb{R}^2$ is not a Gaussian measure?