Probability Theory 3 (2008)

(II) Characteristic functions

1. Complex numbers

- definition
- addition, multiplication, conjugate complex numbers, the modulus of complex numbers, polar coordinates
- exponential function, Euler's formula (Proposition 2.1.3)
- random variables with values in C

2. Definition and basic properties of characteristic functions

- definition of the
  - Fourier-transform of a measure $\mu \in \mathcal{M}_1^+(\mathbb{R}^d)$,
  - characteristic function of a random variable $f : \Omega \to \mathbb{R}^d$,
  - Fourier-transform of a function $\varphi : \mathbb{R}^d \to \mathbb{R}$ (Definition 2.2.6)
- connections between these three definitions (the Fourier transform of the law is the characteristic function of the random variable, which is also the Fourier transform of the density of the law in case the density exists (Proposition 2.2.9))
- basic properties of the Fourier-transform (Proposition 2.2.5; not the proof of uniform continuity)
- examples: Dirac measure, Binomial distribution, sums of Dirac measures

3. Convolutions

- definition of the
  - convolution of measures $\mu_1, ..., \mu_n \in \mathcal{M}_1^+(\mathbb{R}^d)$,
  - convolution of functions $\varphi_1, ..., \varphi : \mathbb{R}^d \to \mathbb{R}$ (Definition 2.3.6).
- convolution of the densities is the convolution of the measures (Proposition 2.3.9)
- Dirac measure in 0 is a unit for convolutions (Example 2.3.4)
- algebraic properties of convolutions (Proposition 2.3.5, $\delta_0 * \mu = \mu$
- convolution of the laws of independent random variables equals the law of the sum of the random variables

4. **Important properties of characteristic functions**
    - basic properties (Proposition 2.4.1)
    - UNIQUENESS THEOREM (Proposition 2.4.7)
    - Fourier transforms integrable functions (RIEMANN-LEBESGUE)
    - BOCHNER-CHINCIN characterization of Fourier transforms
    - Inversion formula and sufficient condition for existence of continuous densities
    - a measure is symmetric if and only if the Fourier transform is real
    - POLYA's sufficient condition for being a Fourier transform
5. **Examples**
    - Gaussian measures on $\mathbb{R}$ (Proposition 2.5.2)
    - Gaussian measures on $\mathbb{R}^d$ (Proposition 2.5.7), non-degenerate Gaussian measures (Proposition 2.5.10)
    - CAUCHY distribution on $\mathbb{R}$ (Proposition 2.5.12)
6. **Independent random variables**
    - characterization of independent random variables by the Fourier transform
    - random variables with a joint Gaussian distribution are independent if and only they are uncorrelated
7. **Moments of measures**
    - computation of moments by the Fourier transform (Proposition 2.7.1)
    - examples: Gaussian measures, Binomial distribution, CAUCHY distribution
8. **Weak convergence**
    - Definition and equivalent properties (Proposition 2.8.1, Definition 2.8.2)
    - Convergence in probability implies weak convergence (Proposition 2.8.3)
    - CENTRAL LIMIT THEOREM (CLT) (Proposition 2.8.4)
    - THEOREM OF POISSON (Proposition 2.8.5)
    - LINDEBERG CONDITION (Proposition 2.8.7)

9. **A first ergodic theorem**
    - Ergodic theorem of Birkhoff and Chincin (Proposition 2.9.2)

Literature:

[1] H. Bauer: Probability Theory (de Gruyter)

[2] A.N. Shirjaev: Probability (Springer) [the English translation is sometimes problematic; however the book itself is excellent]

[3] L. Breiman: Probability (Addison-Wesley)

Exercises:

(2) 2, 3, 4, 5

(3) 1, 2, 3, 4, 5

(4) 1, 5

(5) 2, 3