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Probability 2
Exercises 1 (14. September 2006)

Solve 4 of the 5 problems:

(1) Let $\mathcal{B}(\mathbb{R})$ be the Borel σ-algebra on $\mathbb{R}$. Using $\mathcal{B}(\mathbb{R}) = \sigma((a,b) : -\infty < a < b < \infty)$ show that $[a,b) \in \mathcal{B}(\mathbb{R})$, $[a,b] \in \mathcal{B}(\mathbb{R})$, and $\{a\} \in \mathcal{B}(\mathbb{R})$ for all $-\infty < a < b < \infty$.

(2) As in the course let $\Omega = \{(k,l) : k,l = 1,...,6\}$ and let $\mathcal{F}$ be the smallest σ-algebra containing the sets $\Omega_1 := \{(k,l) : k+l = 2\}$,..., $\Omega_{11} := \{(k,l) : k+l = 12\}$. How many elements does $\mathcal{F}$ have?

(3) Let $\Omega = [0,1]$ and define $\mathcal{F}$ to be the system of all subsets $A \subseteq [0,1]$ such that A or A^c have countable many or finitely many elements.

  (a) Prove that $\mathcal{F}$ is a σ-algebra.
  (b) Prove that $\mathcal{F} = \sigma(\{x\} : x \in [0,1])$.

(4) Assume that $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$ are probability spaces. Using Carathéodory's theorem show that there exists at most one measure $\mathbb{P}$ on $\mathcal{F}_1 \otimes \mathcal{F}_2$ such that

$$\mathbb{P}(A_1 \otimes A_2) = \mathbb{P}_1(A_1)\mathbb{P}_2(A_2).$$

You do not need to prove the existence, just the uniqueness.

(5) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $A \in \mathcal{F}$ with $\mathbb{P}(A) > 0$. Then

$$\mathbb{P}(B|A) := \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)}, \quad \text{for } B \in \mathcal{F},$$

is called *conditional probability of B given A.*

For $\lambda > 0$ define the exponential distribution with parameter λ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ by

$$\mu_\lambda(A) := \int_{A \cap [0,\infty)} \lambda e^{-\lambda x} dx$$

where A is (for example) a finite union of intervals.

  (a) Prove that $\mu_\lambda([a+b,\infty)|[a,\infty)) = \mu_\lambda([b,\infty))$ for $a,b \geq 0$, i.e. the exponential distribution does not have a memory.
  (b) Suppose that the amount of time one spends in a post office is exponential distributed with $\lambda = \frac{1}{10}$.
    (a) What is the probability, that a customer will spend more than 15 minutes?
    (b) What is the probability, that a customer will spend more than 15 minutes from the beginning in the post office, given that the customer already spent at least 10 minutes?