Stochastic differential equations

Stefan Geiss

March 18, 2019

Contents

0. Some notation

We use standard notation from probability:

- $(\Omega, \mathcal{F}, \mathbb{P})$ stands for a probability space.
- $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{R}$, i.e. $\mathcal{B}(\mathbb{R})$ is the smallest σ-algebra containing all open intervals (a, b) with $-\infty < a < b < \infty$ (or equivalently, all open subsets of $\mathbb{R}$). Similarly, $\mathcal{B}(\mathbb{R}^n)$ is the smallest σ-algebra containing all open subsets of $\mathbb{R}^n$.
- A map $Z : \Omega \to \mathbb{R}$ is called a random variable if Z is measurable as a map from $(\Omega, \mathcal{F})$ into $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, i.e.

  $$Z^{-1}(B) := \{\omega \in \Omega : Z(\omega) \in B\} \in \mathcal{F}$$

  for all Borel sets $B \in \mathcal{B}(\mathbb{R})$.
- The Lebesgue spaces are denoted by $L_p = L_p(\Omega, \mathcal{F}, \mathbb{P})$ with

  $$\|f\|_{L_p} := \left(\int_\Omega |f(\omega)|^p d\mathbb{P}(\omega) \right)^{\frac{1}{p}} \quad \text{for} \quad p \in (0, \infty).$$

Definition 0.1. A family of random variables $X = (X_t)_{t \geq 0}$ with $X_t : \Omega \to \mathbb{R}$ is called stochastic process with index set $I = [0, \infty)$.

When do two stochastic processes X and Y coincide? There are several notions for this:

Definition 0.2. Let $X = (X_t)_{t \geq 0}$ and $Y = (Y_t)_{t \geq 0}$ be stochastic processes on $(\Omega, \mathcal{F}, \mathbb{P})$. The processes X and Y are *indistinguishable* if and only if

$$\mathbb{P}(X_t = Y_t, \ t \geq 0) = 1.$$

The definition automatically requires that the set $\{\omega \in \Omega : X_t(\omega) = Y_t(\omega), \ t \geq 0\}$ is measurable which is not the case in general. Another form of coincidence is the following:

Definition 0.3. Let $X = (X_t)_{t \geq 0}$ and $Y = (Y_t)_{t \geq 0}$ be stochastic processes on $(\Omega, \mathcal{F}, \mathbb{P})$. The processes X and Y are *modifications* of each other provided that

$$\mathbb{P}(X_t = Y_t) = 1 \quad \text{for all} \quad t \geq 0.$$

Up to now we have to have that the processes are defined on the same probability space. This can be relaxed as follows:

Definition 0.4. Let $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$ be stochastic processes on $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\Omega', \mathcal{F}', \mathbb{P}')$, respectively. Then X and Y have the same *finite-dimensional distributions* if

$$\mathbb{P}((X_{t_1}, \ldots, X_{t_n}) \in B) = \mathbb{P}'((Y_{t_1}, \ldots, Y_{t_n}) \in B)$$

for all $0 \leq t_1 < \ldots < t_n < \infty$, where $n = 1, 2, \ldots$ and $B \in \mathcal{B}(\mathbb{R}^n)$.

Proposition 0.5. (i) *If X and Y are indistinguishable, then they are modifications of each other. The converse implication is not true in general.*

(ii) *If X and Y are modifications from each other, then they have the same finite-dimensional distributions. There are examples of stochastic processes defined on the same probability space having the same finite-dimensional distributions but which are not modifications of each other.*

There are situations in which two processes are indistinguishable when they are modifications of each other.

Proposition 0.6. *Assume that X and Y are modifications of each other and that all trajectories of X and Y are left-continuous (or right-continuous). Then the processes X and Y are indistinguishable.*

We also need different types of measurability for our stochastic processes. First let us recall the notion of a filtration and a stochastic basis.

Definition 0.7. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A family of σ-algebras $(\mathcal{F}_t)_{t\geq 0}$ is called filtration if $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for all $0 \leq s \leq t < \infty$. The quadruple $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ is called stochastic basis.

The different types of measurability are given by

Definition 0.8. Let $X = (X_t)_{t\geq 0}$, $X_t : \Omega \to \mathbb{R}$ be a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$ and let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration.

(i) The process X is called *measurable* provided that the function $(\omega, t) \to X_t(\omega)$ considered as map between $\Omega \times [0, \infty)$ and $\mathbb{R}$ is measurable with respect to $\mathcal{F} \times \mathcal{B}([0, \infty))$ and $\mathcal{B}(\mathbb{R})$.

(ii) The process X is called *progressively measurable* with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ provided that for all $T \geq 0$ the function $(\omega, t) \to X_t(\omega)$ considered as map between $\Omega \times [0,T]$ and $\mathbb{R}$ is measurable with respect to $\mathcal{F}_T \times \mathcal{B}([0,T])$ and $\mathcal{B}(\mathbb{R})$.

(iii) The process X is called *adapted* with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ provided that for all $t \geq 0$ one has that X_t is $\mathcal{F}_t$-measurable.

Proposition 0.9. *A process which is progressively measurable is measurable and adapted. All other implications between* progressively measurable*,* measurable*, and* adapted *do not hold true in general.*

Proposition 0.10. *An adapted process such that all trajectories are left-continuous (or right-continuous) is progressively measurable.*

Finally let us recall the notion of a martingale.

Definition 0.11. Let $(X_t)_{t\geq 0}$ be $(\mathcal{F}_t)_{t\geq 0}$-adapted and such that $\mathbb{E}|X_t| < \infty$ for all $t \geq 0$.

(i) X is called *martingale* provided that for all $0 \leq s \leq t < \infty$ one has

$$\mathbb{E}(X_t \mid \mathcal{F}_s) = X_s \text{ a.s.}$$

(ii) X is called *sub-martingale* provided that for all $0 \leq s \leq t < \infty$ one has

$$\mathbb{E}(X_t \mid \mathcal{F}_s) \geq X_s \text{ a.s.}$$

(iii) X is called *super-martingale* provided that for all $0 \leq s \leq t < \infty$ one has that

$$\mathbb{E}(X_t \mid \mathcal{F}_s) \leq X_s \text{ a.s.}$$

Finally we use

Definition 0.12. Let $X = (X_t)_{t\geq 0}$ be a stochastic process. The process X is *continuous* provided that $t \to X_t(\omega)$ is continuous for all $\omega \in \Omega$.

1. and 2. Lecture
Gaussian processes and Brownian motion

Gaussian processes form a class of stochastic processes used in several branches in pure mathematics and in applied mathematics. Some typical examples are the following:

- The modeling of telecommunication traffic, where the fractional Brownian motion is used.
- In Real Analysis the Laplace operator is directly connected to the Brownian motion.
- In the theory of stochastic processes many processes can be represented and investigated as transformations of the Brownian motion.

We introduce Gaussian processes in two steps. First we recall Gaussian random variables with values in $\mathbb{R}^n$, then we turn to the processes.

Definition 1.1. (i) A random variable $f : \Omega \to \mathbb{R}$ is called *Gaussian* provided that $\mathbb{P}(f = m) = 1$ for some $m \in \mathbb{R}$ or there are $m \in \mathbb{R}$ and $\sigma > 0$ such that

$$\mathbb{P}(f \in B) = \int_B e^{-\frac{(x-m)^2}{2\sigma^2}} \frac{dx}{\sqrt{2\pi}\sigma}$$

for all $B \in \mathcal{B}(\mathbb{R})$. The parameters m and σ^2 are called *expected value* and *variance*, respectively.

(ii) A random vector $f = (f_1, ..., f_n) : \Omega \to \mathbb{R}^n$ is called *Gaussian* provided that for all $a = (a_1, ..., a_n) \in \mathbb{R}^n$ one has that

$$\langle f(\omega), a \rangle := \sum_{i=1}^{n} a_i f_i(\omega)$$

is Gaussian. The parameters $m = (m_1, ..., m_n)$ with $m_i := \mathbb{E} f_i$ and $\sigma = (\sigma_{ij})_{i,j=1}^n$ with

$$\sigma_{ij} := \mathbb{E}(f_i - m_i)(f_j - m_j)$$

are called *mean (vector)* and *covariance (matrix)*, respectively.

For a Gaussian random variable we can compute the expected value and the variance by

$$m = \mathbb{E}f \quad \text{and} \quad \sigma^2 = \mathbb{E}(f-m)^2.$$

Now we introduce Gaussian processes.

Definition 1.2. A stochastic process $X = (X_t)_{t\geq 0}$, $X_t : \Omega \to \mathbb{R}$, is called *Gaussian* provided that for all $n = 1, 2, ...$ and all $0 \leq t_1 < t_2 < \cdots < t_n < \infty$ one has that

$$(X_{t_1}, ..., X_{t_n}) : \Omega \to \mathbb{R}^n$$

is a Gaussian random vector. Moreover, we let

$$m_t := \mathbb{E}X_t \quad \text{and} \quad \Gamma(s,t) := \mathbb{E}(X_s - m_s)(X_t - m_t).$$

The process $m = (m_t)_{t\geq 0}$ is called *mean (process)* and the process $(\Gamma(s,t))_{s,t\geq 0}$ *covariance (process).*

Up to now we only defined Gaussian processes, however we do not know yet whether they exist. The main result in this respect is

Proposition 1.3. *Let $(\Gamma(s,t))_{s,t\geq 0}$ be positive semi-definite and symmetric, that means*

$$\sum_{i,j=1}^{n} \Gamma(t_i, t_j) a_i a_j \geq 0 \quad \textit{and} \quad \Gamma(s,t) = \Gamma(t,s)$$

for all $s, t, t_1, ..., t_n \geq 0$ and $a_1, ..., a_n \in \mathbb{R}$. Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a Gaussian process $X = (X_t)_{t\geq 0}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with

(i) $\mathbb{E}X_t = 0$,

(ii) $\mathbb{E}X_s X_t = \Gamma(s,t)$.

Remark 1.4. Given any stochastic process $X = (X_t)_{t\geq 0} \subseteq L_2$ with $\mathbb{E}X_t = 0$ and $\Gamma(s,t) := \mathbb{E}X_s X_t$ we always have that Γ is positive semi-definite and symmetric.

Let us consider some examples.

Example 1.5 (Brownian motion)**.** We let

$$\Gamma(s,t) := \min\{s,t\} = \int_0^\infty \chi_{[0,s]}(\xi)\chi_{[0,t]}(\xi)d\xi$$

so that

$$\begin{aligned}\sum_{i,j=1}^n \Gamma(t_i,t_j)a_ia_j &= \int_0^\infty \sum_{i,j=1}^n a_i\chi_{[0,t_i]}(\xi)a_j\chi_{[0,t_j]}(\xi)d\xi \\ &= \int_0^\infty \left(\sum_{i=1}^n a_i\chi_{[0,t_i]}(\xi)\right)^2 d\xi \\ &\geq 0.\end{aligned}$$

Example 1.6 (Brownian bridge)**.** Here we take for a moment the time-interval $[0,1]$ instead of $[0,\infty)$ and believe that all things from before can be done in the same way. We let

$$\Gamma(s,t) := \begin{cases} s(1-t) & : \quad 0 \leq s \leq t \leq 1 \\ t(1-s) & : \quad 0 \leq t \leq s \leq 1 \end{cases}$$

and want to get a *Gaussian* process returning to zero at time $T = 1$. The easiest way to show that Γ is positive semi-definite is to find one realization of this process: we take the Brownian motion $W = (W_t)_{t\geq 0}$ like in Example 1.5, let

$$X_t := W_t - tW_1$$

and get that

$$\begin{aligned}\mathbb{E}X_sX_t &= \mathbb{E}(W_s - sW_1)(W_t - tW_1) \\ &= \mathbb{E}W_sW_t - t\mathbb{E}W_sW_1 - s\mathbb{E}W_1W_t + st\mathbb{E}W_1^2 \\ &= s - ts - st + st \\ &= s(1-t)\end{aligned}$$

for $0 \leq s \leq t \leq 1$.

Example 1.7 (Fractional Brownian motion)**.** The Fractional Brownian motion was considered in 1941 by Kolmogorov in connection with turbulence and in 1968 by Mandelbrot and Van Ness as fractional Gaussian noise. Let $H \in (0,1)$ be the Hurst index of the fractional Brownian motion (Hurst was an English hydrologist) and define the covariance function Γ as

$$\Gamma(s,t) := \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

This covariance function can be obtained (exercise) by looking for a stochastic process $X = (X_t)_{t \geq 0}$ such that:

- X is a continuous Gaussian process of mean zero with $X_0 \equiv 0$.
- The increments are stationary, i.e. $X_t - X_s$ and X_{t-s} have the same distribution for $0 \leq s < t < \infty$.
- The process is self-similar with exponent $\theta \in (0,1)$, i.e. the finite dimensional distributions of $(X_{At})_{t \geq 0}$ and of $A^H (X_t)_{t \geq 0}$ coincide for $A > 0$.

For $H = 1/2$ we get $\Gamma(s,t) = \min\{s,t\}$, that means the Brownian motion from Example 1.5. The main problem consists in showing that Γ is positive semi-definite. To give the idea for this proof let $t_0 := 0$ and $a_0 := -\sum_{i=1}^n a_i$ so that $\sum_{i=0}^n a_i = 0$ and

$$\sum_{i,j=1}^n \Gamma(t_i,t_j) a_i a_j = -\frac{1}{2}\sum_{i,j=0}^n |t_i - t_j|^{2H} a_i a_j.$$

Take $\varepsilon > 0$ so that

$$\begin{aligned}
\sum_{i,j=0}^n e^{-\varepsilon |t_i - t_j|^{2H}} a_i a_j &= \sum_{i,j=0}^n \left(e^{-\varepsilon |t_i - t_j|^{2H}} - 1\right) a_i a_j \\
&= -\varepsilon \sum_{i,j=0}^n |t_i - t_j|^{2H} a_i a_j + o(\varepsilon) \\
&= 2\varepsilon \sum_{i,j=1}^n \Gamma(t_i,t_j) a_i a_j + o(\varepsilon).
\end{aligned}$$

Hence it is sufficient to show that

$$\sum_{i,j=0}^n e^{-\varepsilon |t_i - t_j|^{2H}} a_i a_j \geq 0.$$

Fact: There exists a random variable Z such that

$$\mathbb{E}e^{itZ} = e^{-\varepsilon|t|^{2H}}.$$

The random variable if $2H$-stable (a random variable Z is p-stable for some $0 < p \leq 2$ if $\alpha Z + \beta Z'$ and $(|\alpha|^p + |\beta|^p)^{1/p} Z$ have the same distribution if $\alpha, \beta \in \mathbb{R}$ and Z' is an independent copy of Z). Since characteristic functions are positive semi-definite, we are done.

Up to now we have constructed stochastic processes with certain finite-dimensional distributions. In the case of Gaussian processes this can be done through the covariance structure. Now we go the next step and provide the path-properties we would like to have. Here we use the fundamental

Proposition 1.8 (Kolmogorov). *Let $X = (X_t)_{t\in[0,1]}$, $X_t : \Omega \to \mathbb{R}$, be a family of random variables such that there are constants $c, \varepsilon > 0$ and $p \in [1, \infty)$ with*

$$\mathbb{E}|X_t - X_s|^p \leq c|t - s|^{1+\varepsilon}.$$

Then there is a modification Y of the process X such that

$$\mathbb{E} \sup_{s\neq t} \left(\frac{|Y_t - Y_s|}{|t - s|^\alpha} \right)^p < \infty$$

for all $0 < \alpha < \frac{\varepsilon}{p}$ and that all trajectories are continuous.

Remark 1.9. In particular we get from Proposition 1.8:

(i) The function $f : \Omega \to [0, \infty]$ given by

$$f(\omega) := \sup_{s\neq t} \frac{|Y_t(\omega) - Y_s(\omega)|}{|t - s|^\alpha}$$

is a measurable function.

(ii) The function f is almost surely finite (otherwise $\mathbb{E}|f|^p$ would be infinite), so that there is a set Ω_0 of measure one such that for all $\omega \in \Omega_0$ there is a $c(\omega) > 0$ such that

$$|Y_t(\omega) - Y_s(\omega)| \leq c(\omega)|t - s|^\alpha$$

for all $s, t \in [0, 1]$ and $\omega \in \Omega_0$. In particular, the trajectories $t \to Y_t(\omega)$ are continuous for $\omega \in \Omega_0$.

Let us apply the proposition above to the Brownian motion.

Proposition 1.10. *Let $W = (W_t)_{t\geq 0}$ be a Gaussian process with mean $m(t) = 0$ and covariance $\Gamma(s,t) = \mathbb{E}W_sW_t = \min\{s,t\}$. Then there is a modification $B = (B_t)_{t\geq 0}$ of $W = (W_t)_{t\geq 0}$ such that all trajectories are continuous and*

$$\mathbb{E}\left(\sup_{0\leq s<t\leq T}\frac{|B_t - B_s|}{|t-s|^\alpha}\right)^p < \infty$$

for all $0 < \alpha < \frac{1}{2}$, $0 < p < \infty$, and $T > 0$.

Proof. First we fix $T > 0$ and define

$$X_t := W_{tT}$$

for $t \in [0,1]$. Then, for $p \in (0,\infty)$,

$$\begin{aligned}
\mathbb{E}|X_t - X_s|^p &= \mathbb{E}|W_{tT} - W_{sT}|^p \\
&= \mathbb{E}|W_{(t-s)T}|^p \\
&= \frac{1}{\sqrt{2\pi(t-s)T}}\int_{\mathbb{R}}|\xi|^p e^{-\frac{\xi^2}{2(t-s)T}}d\xi \\
&= ((t-s)T)^{\frac{p}{2}}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}|\xi|^p e^{-\frac{\xi^2}{2}}d\xi \\
&= \gamma_p (t-s)^{\frac{p}{2}}T^{\frac{p}{2}}
\end{aligned}$$

where we used that the covariance structure implies $W_b - W_a \sim N(0, b-a)$ for $0 \leq a < b < \infty$. Now fix $\alpha \in (0,1/2)$ and $p \in (2,\infty)$ such that

$$\frac{1}{\frac{1}{2}-\alpha} < p < \infty \quad \text{and} \quad 0 < \alpha < \frac{\varepsilon}{p} = \frac{1}{2} - \frac{1}{p}$$

and

$$\mathbb{E}|X_t - X_s|^p \leq \gamma_p T^{\frac{p}{2}}(t-s)^{1+\varepsilon}.$$

Proposition 1.8 implies the existence of a path-wise continuous modification $Y = Y(\alpha,p)$ of X such that

$$\mathbb{E}\sup_{0\leq s<t\leq 1}\left(\frac{|Y_t(\alpha,p) - Y_s(\alpha,p)|}{|t-s|^\alpha}\right)^p < \infty. \qquad (1)$$

Replacing p by $q \in (0,p)$ the same inequality remains true since $\|\cdot\|_{L_q} \leq \|\cdot\|_{L_p}$ for $0 < q < p < \infty$. Hence for all $0 < \alpha < 1/2$ and $0 < p < \infty$ we find a modification $Y(\alpha,p)$ such that (1) is satisfied. However, since $Y(\alpha_1,p_1)$ and $Y(\alpha_2,p_2)$ are continuous and modifications of each other, they are indistinguishable. Hence we can pick one process $Y = Y(p_0,\alpha_0)$ which satisfies (1) for all $0 < \alpha < 1/2$ and all $0 < p < \infty$. Coming back to our original time-scale we have found a continuous modification $(B_t^T)_{t\in[0,T]}$ of $(W_t)_{t\in[0,T]}$ such that

$$\mathbb{E} \sup_{0\leq s<t\leq T} \left(\frac{|B_s^T - B_t^T|}{|t-s|^\alpha} \right)^p < \infty$$

for all $0 < \alpha < 1/2$ and $0 < p < \infty$. We are close to the end, we only have to remove the remaining parameter T. For this purpose we let

$$\Omega_T := \left\{ \omega \in \Omega : B_t^T(\omega) = W_t(\omega), t \in Q \cap [0,T] \right\}$$

and $\widetilde{\Omega} := \bigcap_{N=1}^\infty \Omega_N$ so that $\mathbb{P}(\widetilde{\Omega}) = 1$ and

$$B_t^{N_1}(\omega) = B_t^{T_2}(\omega) \quad \text{for} \quad t \in Q \cap [0, \min\{N_1,N_2\}]$$

and $\omega \in \widetilde{\Omega}$. Since $(B_t^{N_i})_{t\in[0,N_i]}$ are continuous processes we derive that

$$B_t^{N_1}(\omega) = B_t^{N_2}(\omega) \quad \text{for} \quad t \in [0, \min\{N_1,N_2\}]$$

whenever $\omega \in \widetilde{\Omega}$. Hence we have found one process $(B_t)_{t\geq 0}$ on $\widetilde{\Omega}$ and may set the process B zero on $\Omega \setminus \widetilde{\Omega}$. □

Now we define the notion of Brownian motion we need later.

Definition 1.11. Let $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_t)_{t\geq 0})$ be a stochastic basis. An adapted stochastic process $B = (B_t)_{t\geq 0}$, $B_t : \Omega \to \mathbb{R}$, is called *standard* $(\mathcal{F}_t)_{t\geq 0}$-*Brownian motion* provided that

(i) $B_0 \equiv 0$,

(ii) for all $0 \leq s < t < \infty$ the random variable $B_t - B_s$ is independent from $\mathcal{F}_s$ that means that

$$\mathbb{P}(C \cap \{B_t - B_s \in A\}) = \mathbb{P}(C)\mathbb{P}(B_t - B_s \in A)$$

for $C \in \mathcal{F}_s$ and $A \in \mathcal{B}(\mathbb{R})$,

(iii) for all $0 \leq s < t < \infty$ one has $B_t - B_s \sim N(0, t-s)$,

(iv) for all $\omega \in \Omega$ the trajectories $t \to B_t(\omega)$ are continuous.

The two-dimensional Brownian motion was observed in 1828 by Robert Brown as diffusion of pollen in water. Later the one-dimensional Brownian motion was used by Louis Bachelier around 1900 in modeling of financial markets and in 1905 by Albert Einstein. A first rigorous proof of its (mathematical) existence was given by Norbert Wiener in 1921. Later on, various different proofs of its existence were given.

Definition 1.12. The stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_t)_{t\geq 0})$ satisfies the *usual conditions* provided that

(i) $(\Omega, \mathcal{F}, \mathbb{P})$ is complete,

(ii) $A \in \mathcal{F}_t$ for all $A \in \mathcal{F}$ with $\mathbb{P}(A) = 0$ and $t \geq 0$,

(iii) the filtration $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous that means that

$$\mathcal{F}_t = \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}.$$

Proposition 1.13. *There is a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_t)_{t\geq 0})$ satisfying the usual conditions with a standard $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion $B = (B_t)_{t\geq 0}$.*

Proof. (a) We take the process $B = (B_t)_{t\geq 0}$ from Proposition 1.10, let $\mathcal{F}_t^B := \sigma(B_s : s \in [0,t])$, and prove that it is a $(\mathcal{F}_t^B)_{t\geq 0}$-Brownian motion.
(i) Since $\mathbb{E}B_0B_0 = 0$ so that $B_0 = 0$ a.s. we can set the whole process B on the null-set $\{B_0 \neq 0\}$ to zero and the conclusion of Proposition 1.10 is still satisfied and we can assume w.l.o.g. that $B_0 \equiv 0$.
(iv) follows directly from Proposition 1.10.
(iii) follows from $\mathbb{E}(B_t - B_s) = 0$,

$$\mathbb{E}(B_t - B_s)^2 = t - 2\min\{t, s\} + s = t - s,$$

and the fact that $B_t - B_s$ is a Gaussian random variable.
(ii) Let $0 \leq s_1 \leq s_2 \leq \cdots \leq s_n \leq s < t$. The random variables $B_t - B_s, B_s - B_{s_n}, ..., B_{s_2} - B_{s_1}, B_{s_1}$ are independent since they are Gaussian random variables and any two of them are uncorrelated. Consequently

$$\mathbb{P}(B_{s_1} \in A_1, ..., B_{s_n} \in A_n, B_t - B_s \in A)$$

$$
\begin{aligned}
&= \mathbb{P}((B_{s_1},...,B_{s_n}) \in A_1 \times \cdots \times A_n, B_t - B_s \in A) \\
&= \mathbb{P}((B_{s_1}, B_{s_2} - B_{s_1} ..., B_{s_n} - B_{s_{n-1}}) \in C, B_t - B_s \in A) \\
&= \mathbb{P}((B_{s_1}, B_{s_2} - B_{s_1} ..., B_{s_n} - B_{s_{n-1}}) \in C)\mathbb{P}(B_t - B_s \in A) \\
&= \mathbb{P}(B_{s_1} \in A_1, ..., B_{s_n} \in A_n)\mathbb{P}(B_t - B_s \in A)
\end{aligned}
$$

where

$$
C := \{(y_1, ..., y_n) \in \mathbb{R}^n : y_1 \in A_1, y_1 + y_2 \in A_2, ..., y_1 + \cdots + y_n \in A_n\}.
$$

Since the events $\{B_{s_1} \in A_1, ..., B_{s_n} \in A_n\}$ generate $\mathcal{F}_s^B$ we can apply the π-system theorem and are done.

(b) Without less of generality we can assume that $\mathcal{F} = \sigma(B_t : t \geq 0)$. We let

$$
\mathcal{N} := \{A \subseteq \Omega : \text{ there exists a } B \in \mathcal{F} \text{ with } A \subseteq B \text{ and } \mathbb{P}(B) = 0\} \cup \{\emptyset\}.
$$

Then one has:

- If $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$, then $B \in \mathcal{G} \vee \mathcal{N}$ if and only if there is a $A \in \mathcal{G}$ such that $A \Delta B \in \mathcal{N}$.
- The measure $\mathbb{P}$ can be extended to a measure $\widetilde{\mathbb{P}}$ on $\mathcal{F} \vee \mathcal{N}$ by $\widetilde{\mathbb{P}}(B) := \mathbb{P}(A)$ for $A \in \mathcal{F}$ such that $A \Delta B \in \mathcal{N}$.

The probability space $(\Omega, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ is called *completion* of $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\mathcal{F}_t)_{t \geq 0}$ with $\mathcal{F}_t := \mathcal{F}_t^B \vee \mathcal{N}$ is called *augmentation* of $(\mathcal{F}_t^B)_{t \geq 0}$.

Now one has to show that B is an $(\mathcal{F}_t)_{t \geq 0}$-Brownian motion as well. Here we only have to check

(ii) Assume $C \in \mathcal{F}_s$ and find an $\widetilde{C} \in \mathcal{F}_s^B$ such that $\mathbb{P}(C \Delta \widetilde{C}) = 0$ where we denote the extension $\widetilde{\mathbb{P}}$ of $\mathbb{P}$ again by $\mathbb{P}$. Taking $A \in \mathcal{B}(\mathbb{R})$ we get that

$$
\begin{aligned}
\mathbb{P}(\{B_t - B_s \in A\} \cap C) &= \mathbb{P}(\{B_t - B_s \in A\} \cap \widetilde{C}) \\
&= \mathbb{P}(B_t - B_s \in A)\mathbb{P}(\widetilde{C}) = \mathbb{P}(B_t - B_s \in A)\mathbb{P}(C).
\end{aligned}
$$

The filtration $(\mathcal{F}_t)_{t \geq 0}$ is right-continuous that means that

$$
\mathcal{F}_t = \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}.
$$

The right-hand side continuity of the filtration $(\mathcal{F}_t)_{t \geq 0}$ is non-trivial and not proved here. $\square$

A. For extended reading

One can prove the existence of the Gaussian processes by analyzing the finite dimensional distributions of a stochastic process $X = (X_t)_{t\geq 0}$. What are the properties we can expect? From now we use the index set

$$\Delta := \{(t_1, ..., t_n) : n \geq 1, t_1, ..., t_n \text{ are distinct}\}.$$

Then the family $(\mu_{t_1,...,t_n})_{(t_1,...,t_n)\in\Delta}$ with

$$\mu_{t_1,...,t_n}(B) := \mathbb{P}((X_{t_1}, ..., X_{t_n}) \in B)$$

defines a family of measures so that

$$\begin{aligned} \mu_{t_1,...,t_n}(B_1 \times \cdots \times B_n) &= \mu_{(t_{\pi(1)},...,t_{\pi(n)})}(B_{\pi(1)} \times \cdots \times B_{\pi(n)}), \\ \mu_{t_1,...,t_n}(B_1 \times \cdots \times B_{n-1} \times \mathbb{R}) &= \mu_{t_1,...,t_{n-1}}(B_1 \times \cdots \times B_{n-1}) \end{aligned}$$

for all $B_1, ..., B_n \in \mathcal{B}(\mathbb{R})$ and all permutations $\pi : \{1, ..., n\} \to \{1, ..., n\}$. This is our starting point:

Definition 1.14. A family of probability measures $(\mu_{t_1,...,t_n})_{(t_1,...,t_n)\in\Delta}$, where $\mu_{t_1,...,t_n}$ is a measure on $\mathcal{B}(\mathbb{R}^n)$ is called *consistent* provided that

(i) $\mu_{t_1,...,t_n}(B_1 \times \cdots \times B_n) = \mu_{(t_{\pi(1)},...,t_{\pi(n)})}(B_{\pi(1)} \times \cdots \times B_{\pi(n)})$ for all $n = 1, 2, ...,$ $B_1, ..., B_n \in \mathcal{B}(\mathbb{R})$, and all permutations $\pi : \{1, ..., n\} \to \{1, ..., n\}$,

(ii) $\mu_{t_1,...,t_n}(B_1 \times \cdots \times B_{n-1} \times \mathbb{R}) = \mu_{t_1,...,t_{n-1}}(B_1 \times \cdots \times B_{n-1})$ for all $n \geq 2$ and $B_1, ..., B_{n-1} \in \mathcal{B}(\mathbb{R})$.

We show that a consistent family of measures can be derived from one measure. The measure will be defined on the following σ-algebra:

Definition 1.15. We let $\sigma\left(\mathbb{R}^{[0,\infty)}\right)$ be the smallest σ-algebra which contains all cylinder sets

$$B := \{(\xi_t)_{t\geq 0} : (\xi_{t_1}, ..., \xi_{t_n}) \in A\}$$

for $(t_1, ..., t_n) \in \Delta$ and $A \in \mathcal{B}(\mathbb{R}^n)$.

Proposition 1.16 (DANIELL 1918, KOLMOGOROV 1933)**.** *Assume a consistent family $(\mu_{t_1,...,t_n})_{(t_1,...,t_n)\in\Delta}$ of probability measures. Then there exists a probability measure μ on $\mathcal{B}(\mathbb{R}^{[0,\infty)})$ such that*

$$\mu((\xi_t)_{t\geq 0} : (\xi_{t_1}, ..., \xi_{t_n}) \in A) = \mu_{t_1,...,t_n}(A)$$

for all $(t_1, ..., t_n) \in \Delta$ and $A \in \mathcal{B}(\mathbb{R}^n)$.

Proof. We only give the idea of the proof. Let $\mathcal{A}$ be the algebra of cylinder sets

$$B := \{(\xi_t)_{t\geq 0} : (\xi_{t_1}, ..., \xi_{t_n}) \in A\}$$

with $(t_1, ..., t_n) \in \Delta$ and $A \in \mathcal{B}(\mathbb{R}^n)$, that means we have that

- $\mathbb{R}^{[0,\infty)} \in \mathcal{A}$,
- $B_1, ..., B_n \in \mathcal{B}$ implies that $B_1 \cup \cdots \cup B_n \in \mathcal{B}$,
- $B \in \mathcal{A}$ implies that $B^c \in \mathcal{A}$.

Now we define $\nu : \mathcal{A} \to [0, 1]$ by

$$\nu((\xi_t)_{t\geq 0} : (\xi_{t_1}, ..., \xi_{t_n}) \in A) := \mu_{t_1,...,t_n}(A).$$

In fact, the definition is correct. Assume that

$$\{(\xi_t)_{t\geq 0} : (\xi_{s_1}, ..., \xi_{s_m}) \in A\} = \{(\xi_t)_{t\geq 0} : (\xi_{t_1}, ..., \xi_{t_n}) \in B\}.$$

Let $(r_1, ..., r_N) \in \Delta$ such that $\{r_1, ..., r_N\} = \{s_1, ..., s_m, t_1, ..., t_n\}$. By adding coordinates we find an $C \in \mathcal{B}(\mathbb{R}^N)$ such that the sets above are equal to

$$\{(\xi_t)_{t\geq 0} : (\xi_{r_1}, ..., \xi_{r_N}) \in C\}.$$

By the consistency we have that

$$\mu_{s_1,...,s_m}(A) = \mu_{r_1,...,r_N}(C) = \mu_{t_1,...,t_n}(B).$$

More difficult would be to check that ν is σ-additive on $\mathcal{A}$, which means that

$$\nu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \nu(B_n)$$

for $B_1, B_2, ... \in \mathcal{A}$, $B_i \cap B_j = \emptyset$ for $i \neq j$, **and** $\bigcup_{n=1}^{\infty} B_n \in \mathcal{A}$. This we leave out for the moment. Having this we might finish with CARATHÉODORY's extension theorem. $\square$

As an application we get the

Proof of Proposition 1.3. We will construct a consistent family of probability measures. Given $(t_1, ..., t_n) \in \Delta$, we let $\mu_{t_1,...,t_n}$ be the Gaussian measure on $\mathbb{R}^n$ with mean zero and covariance

$$\int_{\mathbb{R}^n} \xi_i \xi_j d\mu_{t_1,...,t_n}(\xi_1, ..., \xi_n) = \Gamma(t_i, t_j).$$

If the measure exists, then it is unique. To obtain the measure we let $C := (\Gamma(t_i, t_j))_{i,j=1}^n$, so that C is symmetric and positive semi-definite. We know from algebra that there is a matrix A such that $C = AA^T$. Let γ_n be the standard Gaussian measure on $\mathbb{R}^n$ and μ be the image with respect to $A : \mathbb{R}^n \to \mathbb{R}^n$. Then

$$\int_{\mathbb{R}^d} \langle x, e_i \rangle d\mu(x) = \int_{\mathbb{R}^d} \langle Ax, e_i \rangle d\gamma_n(x) = 0$$

and

$$\begin{aligned} \int_{\mathbb{R}_n} \langle x, e_i \rangle \langle x, e_j \rangle d\mu(x) &= \int_{\mathbb{R}_n} \langle Ax, e_i \rangle \langle Ax, e_j \rangle d\gamma_n(x) \\ &= \int_{\mathbb{R}_n} \langle x, A^T e_i \rangle \langle x, A^T e_j \rangle d\gamma_n(x) \\ &= \langle A^T e_i, A^T e_j \rangle \\ &= \langle e_i, AA^T e_j \rangle \\ &= \langle e_i, C e_j \rangle \\ &= \Gamma(t_i, t_j). \end{aligned}$$

The defined family of measures is easily seen to be consistent: given a permutation $\pi : \{1, ..., n\} \to \{1, ..., n\}$ we have that the covariance of $\mu_{t_{\pi(1)},...,t_{\pi(n)}}$ is $\Gamma(t_{\pi(i)}, t_{\pi(j)})$ which proves property (i). Hence $\mu_{t_{\pi(1)},...,t_{\pi(n)}}$ can be obtained from μ by permutation of the coordinates. To prove that

$$\mu_{t_1,...,t_{n-1},t_n}(B_1 \times \cdots \times B_{n-1} \times \mathbb{R}) = \mu_{t_1,...,t_{n-1}}(B_1 \times \cdots \times B_{n-1})$$

we consider the linear map $A : \mathbb{R}^n \to \mathbb{R}^{n-1}$ with $A(\xi_1, ..., \xi_n) := (\xi_1, ..., \xi_{n-1})$ so that

$$A^{-1}(B_1 \times \cdots \times B_{n-1}) = B_1 \times \cdots \times B_{n-1} \times \mathbb{R}$$

and

$$\mu_{t_1,\ldots,t_{n-1},t_n}(B_1 \times \cdots \times B_{n-1} \times \mathbb{R}) = \mu_{t_1,\ldots,t_{n-1},t_n}(A^{-1}(B_1 \times \cdots \times B_{n-1}))$$

and we need to show that

$$\nu := \mu_{t_1,\ldots,t_{n-1},t_n}(A^{-1}(\cdot)) = \mu_{t_1,\ldots,t_{n-1}}.$$

The measure ν is the image measure of $\mu_{t_1,\ldots,t_{n-1},t_n}$ with respect to A so that it is a Gaussian measure. Moreover,

$$\begin{aligned}
&\int_{\mathbb{R}^{n-1}} \eta_i \eta_j d\nu(\eta_1, ..., \eta_{n-1}) \\
&= \int_{\mathbb{R}^n} \langle A\xi, e_i\rangle \langle A\xi, e_j\rangle d\mu_{t_1,\ldots,t_n}(\xi) \\
&= \int_{\mathbb{R}^n} \langle \xi, A^T e_i\rangle \langle \xi, A^T e_j\rangle d\mu_{t_1,\ldots,t_n}(\xi) \\
&= \sum_{k,l=1}^{n} \langle e_k, A^T e_i\rangle \langle e_l, A^T e_j\rangle \int_{\mathbb{R}^n} \langle \xi, e_k\rangle \langle \xi, e_l\rangle d\mu_{t_1,\ldots,t_n}(\xi) \\
&= \sum_{k,l=1}^{n} \langle e_k, A^T e_i\rangle \langle e_l, A^T e_j\rangle \sigma_{kl} \\
&= \sum_{k,l=1}^{n} \langle A e_k, e_i\rangle \langle A e_l, e_j\rangle \sigma_{kl} \\
&= \sum_{k,l=1}^{n-1} \langle e_k, e_i\rangle \langle e_l, e_j\rangle \sigma_{kl} \\
&= \sigma_{ij}.
\end{aligned}$$

Now the process $X = (X_t)_{t \geq 0}$ is obtained by $X_t : \mathbb{R}^{[0,\infty)} \to \mathbb{R}$ with $X_t((\xi_s)_{s\geq 0}) := \xi_t$. □

Proof of Proposition 1.8. (a) For $m = 1, 2, ...$ we let

$$D_m := \left\{0, \frac{1}{2^m}, ..., \frac{2^m}{2^m}\right\} \quad \text{and} \quad D := \bigcup_{m=1}^{\infty} D_m.$$

Moreover, we set

$$\Delta_m := \left\{(s,t) \in D_m \times D_m : |s-t| = 2^{-m}\right\} \quad \text{and} \quad K_m := \sup_{(s,t)\in\Delta_m} |X_t - X_s|.$$

Then $\operatorname{card}(\Delta_m) \le 22^m$ and

$$\begin{aligned}
\mathbb{E}K_m^p &= \mathbb{E} \sup_{(s,t)\in\Delta_m} |X_t - X_s|^p \\
&\le \sum_{(s,t)\in\Delta_m} \mathbb{E}|X_t - X_s|^p \\
&\le \operatorname{card}(\Delta_m) c \left(\frac{1}{2^m}\right)^{1+\varepsilon} \\
&\le 2c 2^m 2^{-m} 2^{-m\varepsilon} \\
&= 2c 2^{-m\varepsilon}.
\end{aligned}$$

(b) Let $s, t \in D$ and

$$\begin{aligned}
S_k &:= \max\{s_k \in D_k : s_k \le s\} \in D_k \\
T_k &:= \max\{t_k \in D_k : t_k \le t\} \in D_k
\end{aligned}$$

so that $S_k \uparrow s$, $T_k \uparrow t$, and $S_k = s$ and $T_k = t$ for $k \ge k_0$. For $|t - s| \le 2^{-m}$ we get that

$$X_s - X_t = \sum_{i=m}^{\infty} \left(X_{S_{i+1}} - X_{S_i}\right) + X_{S_m} + \sum_{i=m}^{\infty} \left(X_{T_i} - X_{T_{i+1}}\right) - X_{T_m}$$

where we note that the sums are finite sums, that $|T_m - S_m| \in \{0, 2^{-m}\}$, $S_{i+1} - S_i \in \left\{0, 2^{-(i+1)}\right\}$, and that $T_{i+1} - T_i \in \left\{0, 2^{-(i+1)}\right\}$. Hence

$$|X_t - X_s| \le K_m + 2\sum_{i=1}^{m} K_{i+1} \le 2\sum_{i=m}^{\infty} K_i.$$

(c) Let

$$M_\alpha := \sup\left\{\frac{|X_t - X_s|}{|t - s|^\alpha} : s, t \in D, s \ne t\right\}.$$

Now we estimate M_α from above by

$$\begin{aligned}
M_\alpha &= \sup_{m=0,1,\dots} \sup\left\{\frac{|X_t - X_s|}{|t - s|^\alpha} : s, t \in D, s \ne t, 2^{-m-1} \le |t - s| \le 2^{-m}\right\} \\
&\le \sup_{m=0,1,\dots} 2^{(m+1)\alpha} \sup\left\{|X_t - X_s| : s, t \in D, s \ne t, |t - s| \le 2^{-m}\right\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \quad 2 \sup_{m=0,1,\ldots} 2^{(m+1)\alpha} \sum_{i=m}^{\infty} K_i \\
&\leq \quad 2^{1+\alpha} \sum_{i=0}^{\infty} 2^{\alpha i} K_i,
\end{aligned}
$$

where we used step (b), and

$$
\begin{aligned}
\|M_\alpha\|_{L_p} &\leq 2^{1+\alpha} \sum_{i=0}^{\infty} 2^{\alpha i} \|K_i\|_{L_p} \\
&\leq 2^{1+\alpha} \sum_{i=0}^{\infty} 2^{\alpha i} (2c)^{\frac{1}{p}} 2^{-\frac{i\varepsilon}{p}} \\
&= 2^{1+\alpha} (2c)^{\frac{1}{p}} \sum_{i=0}^{\infty} 2^{i\left(\alpha - \frac{\varepsilon}{p}\right)} \\
&< \infty
\end{aligned}
$$

where we used step (a).
(d) Hence there is a set $\Omega_0 \subseteq \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that $t \to X_t(\omega)$ is uniformly continuous on D for $\omega \in \Omega_0$. We define

$$
Y_t(\omega) := \begin{cases} X_t(\omega) & : \quad \omega \in \Omega_0, t \in D \\ \lim_{s \uparrow t, s \in D} X_s(\omega) & : \quad \omega \in \Omega_0, t \notin D \\ 0 & : \quad \omega \notin \Omega_0 \end{cases} .
$$

It remains to show that $\mathbb{P}(X_t = Y_t) = 1$. Because of our assumption we have that

$$
\|X_{t_n} - X_t\|_{L_p} \to 0 \quad \text{as} \quad t_n \uparrow t.
$$

Take $t_n \in D$ and find a subsequence $(n_k)_{k=1}^{\infty}$ such that

$$
\mathbb{P}(\lim_k X_{t_{n_k}} = X_t) = 1.
$$

Since $\mathbb{P}(\lim_k X_{t_{n_k}} = Y_t) = 1$ by construction, we are done. $\square$

3. Lecture

Properties of the Brownian motion

Proposition 3.1. *The trajectories of the standard Brownian motion are Hölder continuous with exponent $\alpha \in (0, 1/2)$, i.e. the set*

$$A_{\alpha,T} := \left\{ \omega \in \Omega : \sup_{0 \le s < t \le T} \frac{|B_t(\omega) - B_s(\omega)|}{|t-s|^\alpha} < \infty \right\}$$

is measurable and of measure one for all $\alpha \in (0, 1/2)$ and $T > 0$.

Proof. Since the Brownian motion is continuous we get that $A_{\alpha,T} \in \mathcal{F}$. Moreover, $\mathbb{E} B_s B_t = \min\{s, t\}$ implies by Proposition 1.10 that there is a continuous modification $\widetilde{B} = (\widetilde{B}_t)_{t \ge 0}$ of B such that the corresponding set $\widetilde{A}_{\alpha,T}$ has measure one. However, B and $\widetilde{B}$ are indistinguishable, so that $\mathbb{P}(A_{\alpha,T}) = 1$ as well. $\square$

Proposition 3.2 (Law of iterated logarithm, Hinčin 1933)**.** *Almost surely one has that*

(i) $\limsup_{t \downarrow 0} \frac{B_t(\omega)}{\psi(1/t)} = \limsup_{t \uparrow \infty} \frac{B_t(\omega)}{\psi(t)} = 1,$

(ii) $\liminf_{t \downarrow 0} \frac{B_t(\omega)}{\psi(1/t)} = \liminf_{t \uparrow \infty} \frac{B_t(\omega)}{\psi(t)} = -1,$

where $\psi(t) := \sqrt{2t \log \log t}$.

Proposition 3.3 (Paley, Wiener-Zygmund)**.** *Given a standard Brownian motion on a stochastic basis satisfying the usual conditions, one has*

$$\mathbb{P}\left(\omega \in \Omega : t \to B_t(\omega) \textit{ is nowhere differentiable }\right) = 1.$$

Proof. It is sufficient to prove that

$$\mathbb{P}\left(\exists s \in [0, 1) : B_t(\omega) \text{ differentiable in } s\right) = 0.$$

Fix $s \in [0, 1)$, $\omega \in \Omega$, and assume that $B_t(\omega)$ is differentiable in s. Then there exists an integer $M = M(\omega) \ge 1$ such that

$$|B_t(\omega) - B_s(\omega)| \le \frac{M}{2}|t - s|$$

for all $t \in [0,1]$. For

$$t_1 = \frac{[ns]+j-1}{n} \quad \text{and} \quad t_2 = \frac{[ns]+j}{n},$$

where $[x]$ is the largest integer $N \le x$, with $j = 1, ..., n-[ns]$ one gets that

$$|B_{t_1}(\omega) - B_{t_2}(\omega)| \le |B_{t_1}(\omega) - B_s(\omega)| + |B_{t_2}(\omega) - B_s(\omega)| \le M\frac{j}{n}.$$

Hence there is an $s \in [0,1)$ and an $M \ge 1$ such that for all $n = 1, 2, ...$ and $j = 1, ..., n-[ns]$ one has that

$$\left| B_{\frac{[ns]+j-1}{n}}(\omega) - B_{\frac{[ns]+j}{n}}(\omega) \right| \le M\frac{j}{n}.$$

If n is large enough there are at least three possible j. Hence there are $M \ge 1$ and $m \ge 1$ such that for all $n \ge m$ there are three subsequent $k \in \{1, ..., n\}$ such that

$$\left| B_{\frac{k-1}{n}}(\omega) - B_{\frac{k}{n}}(\omega) \right| \le M\frac{3}{n}.$$

Estimating the probability, where we use FATOU's lemma, gives that

$$\begin{aligned}
& \sum_{M=1}^{\infty} \mathbb{P}\left(\bigcup_{m=1}^{\infty} \bigcap_{n \ge m} \bigcup_{i=1}^{n-2} \bigcap_{k=i,i+1,i+2} \left| B_{\frac{k-1}{n}}(\omega) - B_{\frac{k}{n}}(\omega) \right| \le M\frac{3}{n} \right) \\
= & \sum_{M=1}^{\infty} \mathbb{P}\left(\liminf_n \bigcup_{i=1}^{n-2} \bigcap_{k=i,i+1,i+2} \left| B_{\frac{k-1}{n}}(\omega) - B_{\frac{k}{n}}(\omega) \right| \le M\frac{3}{n} \right) \\
\le & \sum_{M=1}^{\infty} \liminf_n \mathbb{P}\left(\bigcup_{i=1}^{n-2} \bigcap_{k=i,i+1,i+2} \left| B_{\frac{k-1}{n}}(\omega) - B_{\frac{k}{n}}(\omega) \right| \le M\frac{3}{n} \right) \\
= & \sum_{M=1}^{\infty} \liminf_n n\mathbb{P}\left(\bigcap_{k=1,2,3} \left| B_{\frac{k-1}{n}}(\omega) - B_{\frac{k}{n}}(\omega) \right| \le M\frac{3}{n} \right) \\
= & \sum_{M=1}^{\infty} \liminf_n n\mathbb{P}\left(\left| B_{\frac{1}{n}}(\omega) \right| \le M\frac{3}{n} \right)^3 \\
= & \sum_{M=1}^{\infty} \liminf_n n\mathbb{P}\left(|B_1(\omega)| \le M\frac{3}{\sqrt{n}} \right)^3
\end{aligned}$$

$$\leq \quad c \sum_{M=1}^{\infty} \liminf_n n \left(M \frac{3}{\sqrt{n}} \right)^3 = 0.$$

$\square$

Now we turn to the Markov property. A basic example to motivate the strong Markov property is the *Reflection Principle* (André, Lévy 1948) for a standard Brownian motion $B = (B_t)_{t \geq 0}$. Given $b > 0$, we are interested in the distribution of

$$\tau_b := \inf\{t \geq 0 \,:\, B_t = b\}.$$

First we write

$$\mathbb{P}(\tau_b < t) = \mathbb{P}(\tau_b < t,\, B_t > b) + \mathbb{P}(\tau_b < t,\, B_t < b).$$

Then our heuristic *Reflection Principle* says that

$$\mathbb{P}(\tau_b < t,\, B_t < b) = \mathbb{P}(\tau_b < t,\, B_t > b). \tag{2}$$

On the other hand

$$\mathbb{P}(\tau_b < t, B_t > b) = \mathbb{P}(B_t > b)$$

which implies

$$\begin{aligned} \mathbb{P}(\tau_b < t) &= \mathbb{P}(\tau_b < t,\, B_t > b) + \mathbb{P}(\tau_b < t,\, B_t < b) \\ &= 2\mathbb{P}(\tau_b < t,\, B_t > b) \\ &= 2\mathbb{P}(B_t > b). \end{aligned}$$

From this one can deduce at least two things:

- The Brownian motion reaches with probability one any level because

$$\begin{aligned} \mathbb{P}(\tau_b < \infty) = \lim_{t\to\infty} \mathbb{P}(\tau_b < t) = 2 \lim_{t\to\infty} \mathbb{P}(B_t > b) \\ = 2 \lim_{t\to\infty} \mathbb{P}\left(B_1 > \frac{b}{\sqrt{t}} \right) = 1. \end{aligned}$$

- One can deduce the distribution of the running maximum of the Brownian motion $M_t(\omega) := \sup_{s\in[0,t]} B_s(\omega)$ because

$$\{M_t \geq b\} = \{\tau_b \leq t\} \quad \text{so that} \quad \mathbb{P}(M_t \geq b) = 2\mathbb{P}(B_t > b).$$

To justify (2) would require a considerable amount of work. Here we only indicate some concepts around the random time $\tau_b : \Omega \to [0,\infty]$.

Definition 3.4. Assume a measurable space $(\Omega, \mathcal{F})$ equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$. The map $\tau : \Omega \to [0,\infty]$ is called *stopping time* with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ provided that

$$\{\tau \leq t\} \in \mathcal{F}_t$$

for all $t \geq 0$. Moreover,

$$\mathcal{F}_\tau := \{A \in \mathcal{F} \ : \ A \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}.$$

In a sense, $\mathcal{F}_\tau$ contains those events that can be decided until time τ.

Proposition 3.5. *Let $\tau : \Omega \to [0,\infty]$ be a stopping time. Then*

(i) *the system of sets $\mathcal{F}_\tau$ is a σ-algebra,*

(ii) *one has $\{\tau \leq s\} \in \mathcal{F}_\tau$ for $s \geq 0$ so that τ is an extended $\mathcal{F}_\tau$-measurable random variable.*

Proof. (i) Since $\emptyset \cap \{\tau \leq t\} = \emptyset \in \mathcal{F}_t$ we have $\emptyset \in \mathcal{F}_\tau$. Assume that $B_1, B_2, ... \in \mathcal{F}_\tau$. Then

$$\left(\bigcup_{n=1}^{\infty} B_n\right) \cap \{\tau \leq t\} = \bigcup_{n=1}^{\infty}(B_n \cap \{\tau \leq t\}) \in \mathcal{F}_t.$$

Finally, for $B \in \mathcal{F}_\tau$ we get that

$$B^c \cap \{\tau \leq t\} = \{\tau \leq t\} \setminus (B \cap \{\tau \leq t\}) \in \mathcal{F}_t.$$

(ii) For $s, t \in [0,\infty)$ we get that

$$\{\tau \leq s\} \cap \{\tau \leq t\} = \{\tau \leq \min\{s,t\}\} \in \mathcal{F}_{\min\{s,t\}} \subseteq \mathcal{F}_t.$$

We conclude the proof by remarking that the system $\{x \in \mathbb{R} : x \leq t\}$, $t \in \mathbb{R}$, generates the Borel σ-algebra and that, trivially, $\{\tau \leq t\} = \emptyset \in \mathcal{F}_\tau$ for $t < 0$. $\square$

Example 3.6. Let $X = (X_t)_{t\geq 0}$ be continuous and adapted, $\Gamma \subseteq \mathbb{R}$ be non-empty, and define the hitting time

$$\tau_\Gamma := \inf\{t \geq 0 \,:\, X_t \in B\}$$

with the convention that $\inf \emptyset := \infty$. If Γ is open or closed, then τ_Γ is an stopping time.

The proof will be an exercise.

As an application we prove

Proposition 3.7. *Assume a standard Brownian motion $B = (B_t)_{t\geq 0}$ with $B_0 \equiv 0$. Then, a.s., the Brownian motion changes infinitely often its sign on $[0,\varepsilon]$ for all $\varepsilon > 0$.*

Proof. Let $(\mathcal{F}_t)_{t\geq 0}$ be the augmentation of the natural filtration (used in Proposition 1.13). We define the stopping times

$$\tau_- := \inf\{t \geq 0 : B_t < 0\} \quad \text{and} \quad \tau_+ := \inf\{t \geq 0 : B_t > 0\}.$$

Then $\{\tau_- = 0\} \in \mathcal{F}_0$ and $\{\tau_+ = 0\} \in \mathcal{F}_0$, so that

$$\mathbb{P}(\tau_- = 0) \in \{0,1\} \quad \text{and} \quad \mathbb{P}(\tau_+ = 0) \in \{0,1\}.$$

By symmetry $\mathbb{P}(\tau_- = 0) = \mathbb{P}(\tau_+ = 0)$. Assuming them to be zero would imply (exercise) that there is an $\varepsilon > 0$ such that

$$\mathbb{P}(B_t = 0, t \in [0,\varepsilon]) > 0$$

which is impossible (exercise). Hence

$$\mathbb{P}(\tau_- = 0) = \mathbb{P}(\tau_+ = 0) = 1$$

which implies the claim. $\square$

To give a rigorous justification of the reflection principle one would need to introduce the strong Markov property. In this course we restrict ourselves to the introduction of the basic concepts.

Definition 3.8. Let $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_t)_{t\geq 0})$ be a stochastic basis satisfying the usual assumptions and $X = (X_t)_{t\geq 0}$, $X_t : \Omega \to \mathbb{R}$, be a stochastic process.

(i) The process X is a *Markov process* provided that X is adapted and for all $s, t \geq 0$ and $B \in \mathcal{B}(\mathbb{R})$ one has that

$$\mathbb{P}(X_{s+t} \in B | \mathcal{F}_s) = \mathbb{P}(X_{s+t} \in B | \sigma(X_s)) \; a.s.$$

(ii) The process X is a *strong Markov process* provided that X is progressively measurable and for all $t \geq 0$, stopping times $\tau : \Omega \to [0, \infty]$, and $B \in \mathcal{B}(\mathbb{R})$ one has that, a.s.,

$$\mathbb{P}(\{X_{\tau+t} \in B\} \cap \{\tau < \infty\} \,|\mathcal{F}_\tau) = \mathbb{P}(\{X_{\tau+t} \in B\} \cap \{\tau < \infty\} \,|\sigma(X_\tau)),$$

where $\sigma(X_\tau) := \sigma(\tau^{-1}(\infty), \{X_\tau^{-1}(B)\} \cap \{\tau < \infty\})$.

Proposition 3.9. *Assume a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_t)_{t\geq 0})$ satisfying the usual conditions and an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion $B = (B_t)_{t\geq 0}$ like in Definition 1.11. Then B is a strong Markov process.*

4. Lecture

Stochastic integration

We will define stochastic integrals for local martingales and assume in this chapter that the usual conditions on the stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_t)_{t\geq 0})$ are satisfied.

Definition 4.1. Let $(M_t)_{t\geq 0}$ be $(\mathcal{F}_t)_{t\geq 0}$-adapted.

(i) M is called *martingale* provided that $\mathbb{E}|M_t| < \infty$ for all $t \geq 0$ and

$$\mathbb{E}(M_t \mid \mathcal{F}_s) = M_s \text{ a.s.}$$

for all $0 \leq s \leq t < \infty$.

(ii) We denote by $\mathcal{M}_0^{2,c}$ the space of all martingales M such that all paths $t \to M_t(\omega)$ are continuous, $M_0 \equiv 0$, and $\mathbb{E}M_t^2 < \infty$ for all $t \geq 0$.

(iii) $M = (M_t)_{t\geq 0}$ is called a continuous local martingale provided that $M_0 \equiv 0$ and there exists a sequence of stopping times $0 \leq \tau_1 \leq \tau_2 \leq \tau_3 \leq ... < \infty$ with $\lim_n \tau_n(\omega) = \infty$ for all $\omega \in \Omega$ such that the processes $(M_{t\wedge\tau_n})_{t\geq 0} \in \mathcal{M}_0^{2,c}$ for all $n = 1, 2, ...$ In this case we write $M \in \mathcal{M}_0^{loc,c}$.

Example 4.2. (i) The Brownian motion is a martingale.

(ii) The geometric Brownian motion $S = (S_t)_{t\geq 0}$ defined by $S_t := e^{B_t - \frac{t}{2}}$ is a martingale.

Proof. (i) For $0 \leq s < t < \infty$ one has that, a.s.,

$$\mathbb{E}(B_t|\mathcal{F}_s) = \mathbb{E}(B_t - B_s|\mathcal{F}_s) + \mathbb{E}(B_s|\mathcal{F}_s) = \mathbb{E}(B_t - B_s) + B_s = B_s.$$

The proof of (ii) is an exercise. □

Proposition 4.3. *Let $M \in \mathcal{M}_0^{loc,c}$.*

(i) *Then there exists a unique adapted path-wise non-decreasing continuous process $\langle M \rangle$ with $\langle M \rangle_0 \equiv 0$ such that, for all $T > 0$,*

$$\sup_{t\in[0,T]} \left| \sum_{i=1}^{n} |M_{t\wedge t_i^n} - M_{t\wedge t_{i-1}^n}|^2 - \langle M \rangle_t \right| \to_{\mathbb{P}} 0$$

for any sequence of time nets $0 = t_0^n < \cdots < t_n^n < \infty$ with $\sup_{i=1,\dots,n} |t_i^n - t_{i-1}^n| \to_n 0$ and $t_n^n \to_n \infty$.

(ii) *The process $\langle M\rangle$ is the unique adapted increasing continuous process starting in zero such that $M^2 - \langle M\rangle$ is a local martingale.*

(iii) *If $M \in \mathcal{M}_0^{2,c}$, then the process $\langle M\rangle$ is the unique adapted increasing continuous process starting in zero such that $M^2 - \langle M\rangle$ is a martingale.*

The process $\langle M\rangle = (\langle M\rangle_t)_{t\geq 0}$ is called quadratic variation of M.

Example 4.4. *For the Brownian motion $B = (B_t)_{t\geq 0}$ one has that $\langle B\rangle_t = t$, $t \geq 0$, a.s.*

Proof. We get that

$$\begin{aligned}
\mathbb{E}\left|t - \sum_{i=1}^N \left[B_{t\frac{i}{N}} - B_{t\frac{i-1}{N}}\right]^2\right|^2 &= t^2\mathbb{E}\left|1 - \sum_{i=1}^N \left[B_{\frac{i}{N}} - B_{\frac{i-1}{N}}\right]^2\right|^2 \\
&= t^2\mathbb{E}\left|\sum_{i=1}^N \left[\left[B_{\frac{i}{N}} - B_{\frac{i-1}{N}}\right]^2 - \frac{1}{N}\right]\right|^2 \\
&= t^2\sum_{i=1}^N \mathbb{E}\left|\left[\left[B_{\frac{i}{N}} - B_{\frac{i-1}{N}}\right]^2 - \frac{1}{N}\right]\right|^2 \\
&= t^2\sum_{i=1}^N [\mathbb{E}B_1^4 - 1]\frac{1}{N^2} \to_N 0.
\end{aligned}$$

$\square$

Proposition 4.5 (Burkholder-Davis-Gundy)**.** *For all $p \in (1,\infty)$ there is a constant $c_p > 0$ such that*

$$\frac{1}{c_p}\|\sqrt{\langle M\rangle_t}\|_{L_p} \leq \|M_t\|_{L_p} \leq c_p\|\sqrt{\langle M\rangle_t}\|_{L_p}$$

for all $t \geq 0$ and $M \in \mathcal{M}_0^{loc,c}$.

Given, for example, a Brownian motion $B = (B_t)_{t\geq 0}$, we would like to define

$$\int_0^T L_t dB_t$$

for a large class of stochastic processes $L = (L_t)_{t\geq 0}$. A first approach would be to write

$$\int_0^T L_t dB_t = \int_0^T L_t \frac{dB_t}{dt} dt.$$

However this is not possible (at least in this naive form) because the Brownian motion is not differentiable as we learned earlier. So we have to proceed slightly differently. We will first define the stochastic integral for simple processes and extend then the definition to an appropriate class of processes.

Definition 4.6. (i) Let $M \in \mathcal{M}_0^{2,c}$. We define $\mathcal{L}_2(M)$ to be the set of all progressively measurable $L = (L_t)_{t\geq 0}$ such that

$$\int_0^T \mathbb{E} L_t^2 d\langle M\rangle_t < \infty$$

for all $T \geq 0$. Moreover, for $L \in \mathcal{L}_2(M)$ we let

$$|L|_{L_2(M)} := \sum_{n=1}^{\infty} 2^{-n} \min\left\{1, \sqrt{\int_0^n \mathbb{E} L_t^2 d\langle M\rangle_t}\right\}.$$

(ii) We let $\mathcal{L}_0$ be the space of all simple integrands, i.e. there is a sequence $t_0 < t_1 < t_2 < \cdots$ with $\lim_n t_n = \infty$ and uniformly bounded (in i and ω) random variables $v_i : \Omega \to \mathbb{R}$ such that v_i is $\mathcal{F}_{t_i}$-measurable and

$$L_t = \sum_{i=1}^{\infty} v_{i-1} \chi_{(t_{i-1},t_i]}(t).$$

Lemma 4.7. *For any $L \in \mathcal{L}_2(M)$ there are $L^n \in \mathcal{L}_0$ such that*

$$\lim_n |L - L^n|_{L_2(M)} = 0.$$

For $L \in \mathcal{L}_0$ we can easily define a stochastic integral by

$$I_t^M(L)(\omega) := \sum_{i=1}^{\infty} v_{i-1}(\omega)(M_{t_i\wedge t}(\omega) - M_{t_{i-1}\wedge t}(\omega)).$$

The key properties of this construction are the following:

Proposition 4.8. *For $M \in \mathcal{M}_0^{2,c}$ and $L \in \mathcal{L}_0$ one has:*

(i) $(I_t^M(L))_{t\geq 0} \in \mathcal{M}_0^{2,c}$.

(ii) *For* $0 \leq s \leq t$ *one has that*

$$\mathbb{E}\left([I_t^M(L) - I_s^M(L)]^2 | \mathcal{F}_s\right) = \mathbb{E}\left(\int_s^t L_u^2 d\langle M\rangle_u | \mathcal{F}_s\right) a.s.$$

Proof. (i) By definition we have that $I_0^M(L) \equiv 0$ and that the process $t \to I_t^M(L)(\omega)$ is continuous for all $\omega \in \Omega$. Since

$$v_{i-1}(M_{t_i\wedge t} - M_{t_{i-1}\wedge t}) = \begin{cases} 0 & : \ t \leq t_{i-1} \\ v_{i-1}(M_{t_i\wedge t} - M_{t_{i-1}}) & : \ t > t_{i-1} \end{cases}$$

we get that $I_t^M(L)$ is $\mathcal{F}_t$-measurable. Now we observe that

$$\mathbb{E}|v_{i-1}(M_b - M_{t_{i-1}})|^2 \leq c^2 \mathbb{E}|M_b - M_{t_{i-1}}|^2 < \infty$$

for $t_{i-1} \leq b < \infty$ so that

$$\begin{aligned} \left(\mathbb{E}|I_t(L)|^2\right)^{\frac{1}{2}} &\leq \left(\mathbb{E}\left|\sum_{i=1}^{n_0} v_{i-1}(M_{t_i\wedge t} - M_{t_{i-1}\wedge t})\right|^2\right)^{\frac{1}{2}} \\ &\leq \sum_{i=1}^{n_0} \left(\mathbb{E}|v_{i-1}(M_{t_i\wedge t} - M_{t_{i-1}\wedge t})|^2\right)^{\frac{1}{2}} < \infty \end{aligned}$$

whenever $t_{n_0-1} < t \leq t_{n_0}$. It remains to show the martingale property

$$\mathbb{E}(I_t^M(L)|\mathcal{F}_s) = I_s^M(L) \ a.s.$$

for $0 \leq s \leq t < \infty$. We only check $0 < s \leq t < \infty$ and find n_0 and m_0 such that $t_{n_0-1} < t \leq t_{n_0}$ and $t_{m_0-1} < s \leq t_{m_0}$. Then, a.s.,

$$\begin{aligned} \mathbb{E}(I_t^M(L)|\mathcal{F}_s) &= \mathbb{E}\left(\sum_{i=1}^{n_0} v_{i-1}(M_{t_i\wedge t} - M_{t_{i-1}\wedge t})|\mathcal{F}_s\right) \\ &= \sum_{i=1}^{n_0} \mathbb{E}\left(v_{i-1}(M_{t_i\wedge t} - M_{t_{i-1}\wedge t})|\mathcal{F}_s\right). \end{aligned}$$

For $1 \leq i \leq m_0 - 1$ we get

$$v_{i-1}(M_{t_i\wedge t} - M_{t_{i-1}\wedge t}) = v_{i-1}(M_{t_i} - M_{t_{i-1}}) = v_{i-1}(M_{t_i\wedge s} - M_{t_{i-1}\wedge s})$$

which is $\mathcal{F}_s$-measurable. In the case $n_0 \geq i \geq m_0 + 1$ we may deduce, a.s., that

$$\begin{aligned}
\mathbb{E}(v_{i-1}(M_{t_i \wedge t} - M_{t_{i-1}\wedge t})|\mathcal{F}_s) &= \mathbb{E}(v_{i-1}(M_{t_i \wedge t} - M_{t_{i-1}})|\mathcal{F}_s) \\
&= \mathbb{E}\left(\mathbb{E}(v_{i-1}(M_{t_i \wedge t} - M_{t_{i-1}})|\mathcal{F}_{t_{i-1}})|\mathcal{F}_s\right) \\
&= \mathbb{E}\left(v_{i-1}\mathbb{E}(M_{t_i \wedge t} - M_{t_{i-1}}|\mathcal{F}_{t_{i-1}})|\mathcal{F}_s\right) \\
&= \mathbb{E}\left(v_{i-1}(M_{t_i \wedge t \wedge t_{i-1}} - M_{t_{i-1}})|\mathcal{F}_s\right) \\
&= 0.
\end{aligned}$$

Finally, for $i = m_0$ one obtains, a.s., that

$$\begin{aligned}
\mathbb{E}(v_{i-1}(M_{t_i \wedge t} - M_{t_{i-1}\wedge t})|\mathcal{F}_s) &= v_{i-1}\mathbb{E}(M_{t_i \wedge t} - M_{t_{i-1}\wedge t}|\mathcal{F}_s) \\
&= v_{i-1}(M_{t_i \wedge t \wedge s} - M_{t_{i-1}}) \\
&= v_{i-1}(M_{t_i \wedge s} - M_{t_{i-1}\wedge s}).
\end{aligned}$$

(ii) By introducing new time knots we can assume without loss of generality that $s = t_n$ and $t = t_N$. Let

$$X_k := I^M_{t_k}(L) \quad \text{and} \quad \mathcal{G}_k := \mathcal{F}_{t_k}.$$

Hence, a.s.,

$$\begin{aligned}
\mathbb{E}\left([I^M_t(L) - I^M_s(L)]^2|\mathcal{F}_s\right) &= \mathbb{E}\left([X_N - X_n]^2|\mathcal{G}_n\right) \\
&= \mathbb{E}\left(\sum_{l=n+1}^{N} (X_l - X_{l-1})^2|\mathcal{G}_n\right) \\
&= \mathbb{E}\left(\sum_{l=n+1}^{N} v^2_{l-1}(M_{t_l} - M_{t_{l-1}})^2|\mathcal{F}_{t_n}\right) \\
&= \sum_{l=n+1}^{N} \mathbb{E}\left(\mathbb{E}(v^2_{l-1}(M_{t_l} - M_{t_{l-1}})^2|\mathcal{F}_{t_{l-1}})|\mathcal{F}_{t_n}\right) \\
&= \sum_{l=n+1}^{N} \mathbb{E}\left(v^2_{l-1}\mathbb{E}((M_{t_l} - M_{t_{l-1}})^2|\mathcal{F}_{t_{l-1}})|\mathcal{F}_{t_n}\right) \\
&= \sum_{l=n+1}^{N} \mathbb{E}\left(v^2_{l-1}\mathbb{E}\left(\int_{t_{l-1}}^{t_l} d\langle M\rangle_u|\mathcal{F}_{t_{l-1}}\right)|\mathcal{F}_{t_n}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{l=n+1}^{N} \mathbb{E}\left(v_{l-1}^2 \int_{t_{l-1}}^{t_l} d\langle M\rangle_u | \mathcal{F}_{t_n} \right) \\
&= \mathbb{E}\left(\int_s^t L_u^2 d\langle M\rangle_u | \mathcal{F}_s \right).
\end{aligned}
$$

□

Proposition 4.9. *For $M \in \mathcal{M}_0^{2,c}$ and $L \in \mathcal{L}_2(M)$ there exists a unique martingale $X = (X_t)_{t\geq 0} \in \mathcal{M}_0^{2,c}$ such that for all $L^n \in \mathcal{L}_0$ with*

$$\lim_n |L - L^n|_{L_2(M)} = 0$$

one has that $\|X_t - I_t^M(L^n)\|_{L_2} \to 0$ as $n \to \infty$.

A first impression that stochastic integration and usual integration differ gives

Example 4.10. *One has that*

$$\int_0^t B_u dB_u = \frac{1}{2}(B_t^2 - t) \quad \textit{for} \quad t \geq 0 \; a.s.$$

Now we extend our stochastic integral simultaneously into two directions. First we enlarge the class of integrands we can use:

Definition 4.11. Let $\mathcal{L}_2^{\text{loc}}(M)$ be the set of all progressively measurable processes $L = (L_t)_{t\geq 0}$ such that

$$\mathbb{P}\left(\omega \in \Omega : \int_0^t L_u^2(\omega) d\langle M\rangle_u(\omega) < \infty \right) = 1 \quad \text{for all} \quad t \geq 0.$$

Given $M \in \mathcal{M}_0^{loc,c}$ and $L \in \mathcal{L}_2^{\text{loc}}(M)$ there exists a sequence of stopping times $\tau_1 \leq \tau_2 \leq \cdots < \infty$ with $\lim_n \tau_n(\omega) = \infty$ such that $M^n = (M_{t\wedge\tau_n})_{t\geq 0} \in \mathcal{M}_0^{2,c}$ and $(L_t^n)_{t\geq 0} := (L_t \chi_{\{t\leq\tau_n\}})_{t\geq 0} \in \mathcal{L}_2(M^n)$ for all $n = 1, 2, \ldots$

Proposition 4.12. *There exists a unique $X \in \mathcal{M}_0^{loc,c}$ such that*

$$I_t^{M^n}(L^n)(\omega)\chi_{\{t\leq\tau_n(\omega)\}} = X_t(\omega)\chi_{\{t\leq\tau_n(\omega)\}}$$

for $t \geq 0$ $\mathbb{P}$-a.s.

Definition 4.13. We let

$$\int_0^t L_u dM_u := X_t \quad \text{and} \quad \int_s^t L_u dM_u := X_t - X_s.$$

Now we summarize some of the properties of our stochastic integral:

Proposition 4.14. (i) *For $K, L \in \mathcal{L}_2^{\mathrm{loc}}(M)$ and $\alpha, \beta \in \mathbb{R}$ one has*

$$\int_0^t (\alpha K_u + \beta L_u) dM_u = \alpha \int_0^t K_u dM_u + \beta \int_0^t L_u dM_u, \ t \geq 0, \ a.s.$$

(ii) *Itô-Isometry: for $M \in \mathcal{M}_0^{2,c}$, $K, L \in \mathcal{L}_2(M)$, and $0 \leq s < t < \infty$ one has*

$$\mathbb{E}\left(\int_s^t K_u dM_u \int_s^t L_u dM_u | \mathcal{F}_s \right) = \mathbb{E}\left(\int_s^t K_u L_u d\langle M \rangle_u | \mathcal{F}_s \right) \ a.s.$$

(iii) *For $L \in \mathcal{L}_2^{\mathrm{loc}}(M)$ and a stopping time $\tau : \Omega \to [0, \infty)$ one has that*

$$\left(\int_0^{t \wedge \tau(\omega)} L_u dM_u \right)(\omega) = \left(\int_0^t L_u \chi_{\{u \leq \tau\}} dM_u \right)(\omega)$$

for $t \geq 0$ a.s.

A. For extended reading

It is not easy to find local martingales which are not martingales. We indicate a construction, but do not go into any details. The example is intended as motivation for Itô's formula presented in the next section.

Example 4.15. Given $d = 1, 2, ...$ we let $(W_t)_{t\geq 0}$ be the d-dimensional standard Brownian motion where $W_t^{(d)} := (B_{t,1}, ..., B_{t,d})$, $W_0 \equiv 0$, and $(B_{t,i})_{t\geq 0}$ are independent Brownian motions. The filtration is obtained as in the one-dimensional case as the augmentation of the natural filtration. Let $d = 3$ and

$$M_t := \frac{1}{|x + W_t|}$$

with $|x| = 2$ where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^d$. Then $M = (M_t)_{t\geq 0}$ is a local martingale, but not a martingale.

Proof. To justify the construction one would need the following:
(a) For a d-dimensional standard Brownian motion W with $d \geq 2$ the sets $\{y\}$ with $y \neq 0$ are *polar sets*, that means

$$\mathbb{P}(\tau_y < \infty) = 0 \quad \text{with} \quad \tau_y := \inf\{t \geq 0 : W_t = y\}.$$

(b) For $d \geq 3$ one has that $\mathbb{P}(\lim_{t\to\infty} |W_t| = \infty) = 1$.
(c) Assuming that M is a martingale property we would get $\mathbb{E}M_t = \mathbb{E}M_0 = \frac{1}{2}$. But a direct computation yields to

$$\mathbb{E}\frac{1}{|x + W_t|} = \mathbb{E}\frac{1}{|x + \sqrt{t}(g_1, g_2, g_3)|} \to_{t\to\infty} 0$$

where $g_1, g_2, g_3 \sim N(0, 1)$ are independent.
(d) How to show that M is a *local* martingale? This gives us a first impression of Itô-formula which will read for $f(\xi_1, \xi_2, \xi_3) := \frac{1}{\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}}$ and $X_t := x + (B_{t,1}, B_{t,2}, B_{t,3})$ as

$$\begin{aligned} f(X_t) &= f(x) + \sum_{i=1}^{3} \int_0^t \frac{\partial f}{\partial x_i}(X_u) dB_{u,i} + \frac{1}{2}\int_0^t (\Delta f)(X_u) du \ a.s. \\ &= f(x) + \sum_{i=1}^{3} \int_0^t \frac{\partial f}{\partial x_i}(X_u) dB_{u,i} \end{aligned}$$

where the latter term turns out to be a local martingale. □

5. and 6. Lecture

Itô's formula

In calculus there is the fundamental formula

$$f(y) = f(x) + \int_x^y f'(u)du$$

for, say, $f \in C^1(\mathbb{R})$ and $-\infty < x < y < \infty$. Is there a similar formula for stochastic integrals?

Definition 5.1. *Let $f : [0,\infty) \to \mathbb{R}$ be a function. Then*

$$\operatorname{var}(f,t) := \sup_{0=t_0\leq\cdots\leq t_n=t} \sum_{k=1}^n |f(t_k) - f(t_{k-1})| \in [0,\infty].$$

The 1-variation $\operatorname{var}(f,t)$ is always lower bounded by the quadratic variation because

$$\left(\sum_{k=1}^n |f(t_k) - f(t_{k-1})|^2\right)^{\frac{1}{2}} \leq \sum_{k=1}^n |f(t_k) - f(t_{k-1})|.$$

More precisely, to require that a process has a bounded variation is strictly stronger than to require that a process has path-wise a bounded 2-variation.

Lemma 5.2. (i) *The function* $\operatorname{var}(f,\cdot)$ *is increasing.*

(ii) *If $f : [0,\infty) \to \mathbb{R}$ is continuous, then* $\operatorname{var}(f,\cdot)$ *is left-continuous.*

Definition 5.3. A stochastic process $A = (A_t)_{t\geq 0}$, $A_t : \Omega \to \mathbb{R}$, is called of *bounded variation* provided that

$$\operatorname{var}(A_\cdot(\omega),t) = \sup_{0=t_0\leq\cdots\leq t_n=t} \sum_{k=1}^n |A_{t_k}(\omega) - A_{t_{k-1}}(\omega)| < \infty \quad \text{for all} \quad t\geq 0 \quad a.s.$$

Lemma 5.4. *If $M = (M_t)_{t\geq 0} \in \mathcal{M}_0^{loc,c}$ is of bounded variation, then*

$$\mathbb{P}\left(\omega \in \Omega : M_t(\omega) = 0, t \geq 0\right) = 1.$$

Proof. Since M has continuous paths it is sufficient to show that

$$\mathbb{P}(M_t = 0) = 1 \quad \text{for all} \quad t \geq 0.$$

(a) First we consider $M \in \mathcal{M}_0^{2,c}$. Assume that

$$\operatorname{var}(M_\cdot(\omega), t) \leq c < \infty \; a.s.$$

and let $t_i^n := \frac{it}{n}$. Then

$$\begin{aligned}
\mathbb{E}M_t^2 &= \mathbb{E}\left[\sum_{i=1}^n \left[M_{t_i^n} - M_{t_{i-1}}^n\right]\right]^2 \\
&= \sum_{i=1}^n \mathbb{E}\left[M_{t_i^n} - M_{t_{i-1}}^n\right]^2 \\
&\leq \mathbb{E}\operatorname{var}(M_\cdot, t) \sup_{i=1,\ldots,n} \left|M_{t_i^n} - M_{t_{i-1}^n}\right| \\
&\leq c\,\mathbb{E} \sup_{i=1,\ldots,n} \left|M_{t_i^n} - M_{t_{i-1}^n}\right|.
\end{aligned}$$

Since

$$\sup_{i=1,\ldots,n} \left|M_{t_i^n}(\omega) - M_{t_{i-1}^n}(\omega)\right| \to_n 0$$

for all $\omega \in \Omega$ by the uniform continuity of the paths of M on compact intervals and

$$\sup_{i=1,\ldots,n} |M_{t_i^n} - M_{t_{i-1}^n}| \leq 2 \sup_{u\in[0,t]} |M_u| \in L_2$$

by DOOB's maximal inequality, majorized convergence implies that

$$\lim_n \mathbb{E}\left|M_{t_i^n} - M_{t_{i-1}^n}\right| = 0 \quad \text{so that} \quad \mathbb{E}M_t^2 = 0.$$

(b) Now let $N \in \{1, 2, ...\}$, $T > 0$, and

$$\tau_N(\omega) := \inf\{t \geq 0 : \operatorname{var}(M_\cdot(\omega), t) > N\} \wedge T.$$

Because of Lemma 5.2 the random time τ_N is a stopping time. To check this it is sufficient to show that

$$\sigma_N(\omega) := \inf\{t \geq 0 : \operatorname{var}(M_\cdot(\omega), t) > N\}$$

is a stopping time. Indeed

$$\{t \leq \sigma_N(\omega)\} = \{\mathrm{var}(M_\cdot(\omega), t) \leq N\} \in \mathcal{F}_t$$

yields that σ_N is an optional time, so that we conclude that σ_N is a stopping time by the usual conditions. Moreover,

$$(M_{t\wedge\tau_N})_{t\geq 0} \in \mathcal{M}_2^{c,0}$$

by the optional stopping theorem and

$$\mathrm{var}(M_\cdot^{\tau_N}(\omega), t) \leq N.$$

Applying (a) gives

$$\mathbb{E}M^2_{\tau_N\wedge T} = 0.$$

Consequently,

$$\mathbb{E}M_T^2 = \mathbb{E}\lim_N M^2_{T\wedge\tau_N} = \lim_N \mathbb{E}M^2_{T\wedge\tau_N} = 0$$

since $\tau_N \uparrow T$ a.s. and

$$M^2_{T\wedge\tau_N} \leq \sup_{t\in[0,T]} M_t \in L_1.$$

(c) Now we assume a local martingale with a localizing sequence $(\sigma_n)_{n=0}^\infty$ for M. In addition, we let

$$\rho_n := \inf\{t \geq 0 : |M_t| \geq n\}$$

so that $\tau_n := \sigma_n \wedge \rho_n$ is a localizing sequence with $|M_t^{\tau_n}| \leq n$. The variation of M^{τ_n} is bounded by the variation of M, so that

$$\mathbb{P}(M_{t\wedge\sigma_n} = 0) = 1$$

for all $t \geq 0$ and $n = 0, 1, 2, ...$ by (a) and (b). Consequently,

$$\mathbb{P}(M_t = 0) = \mathbb{E}\left(\lim_{n\to\infty} \chi_{\{M_{t\wedge\tau_n}=0\}}\right) = \lim_{n\to\infty} \mathbb{E}\left(\chi_{\{M_{t\wedge\tau_n}=0\}}\right) = 0$$

where we have used dominated convergence and $\lim_n \tau_n(\omega) = \infty$ for all $\omega \in \Omega$. $\square$

Definition 5.5. A continuous adapted stochastic process $X = (X_t)_{t\geq 0}$ is called *continuous semi-martingale* provided that

$$X_t = x_0 + M_t + A_t$$

where $x_0 \in \mathbb{R}$, $M \in \mathcal{M}_0^{loc,c}$, and A is of bounded variation with $A_0 \equiv 0$.

Because of Lemma 5.4 the decomposition is unique.

Proposition 5.6 (Itô's formula for continuous semimartingales)**.** *Let $f \in C^2(\mathbb{R}^d)$ and $X_t = (X_t^1, ..., X_t^d)$ be a vector of continuous semi-martingales. Then one has that, a.s.,*

$$f(X_t) = f(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(X_u) dX_u^i + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(X_u) d\langle M^i, M^j\rangle_u$$

where $dX_u^i = dM_u^i + dA_u^i$ and

$$\langle M^i, M^j\rangle_u := \frac{1}{4}\left[\langle M^i + M^j\rangle_u - \langle M^i - M^j\rangle_u\right].$$

Proposition 5.7 (Partial integration)**.** *For continuous semi-martingales X and Y one has that*

$$X_t Y_t = X_0 Y_0 + \int_0^t Y_u dX_u + \int_0^t X_u dY_u + \langle X, Y\rangle_t \quad a.s.$$

or, in differential form,

$$d(X_t Y_t) = Y_t dX_t + X_t dY_t + d\langle X, Y\rangle_t.$$

Proof. We take $d = 2$ and $f(x, y) := xy$ so that, a.s.,

$$\begin{aligned}
X_t Y_t &= f(X_t, Y_t) \\
&= f(X_0, Y_0) + \int_0^t \frac{\partial f}{\partial x}(X_u, Y_u) dX_u + \int_0^t \frac{\partial f}{\partial y}(X_u, Y_u) dY_u \\
&\quad + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x \partial y}(X_u, Y_u) d\langle X, Y\rangle_u \\
&\quad + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial y \partial x}(X_u, Y_u) d\langle Y, X\rangle_u \\
&= X_0 Y_0 + \int_0^t Y_u dX_u + \int_0^t X_u dY_u + \int_0^t d\langle X, Y\rangle_u
\end{aligned}$$

because $(\partial f^2/\partial x^2) = (\partial f^2/\partial y^2) = 0$. $\square$

Proposition 5.8 (Compensator)**.** *For $M \in \mathcal{M}_0^{loc,c}$ one has that*

$$M_t^2 - \langle M \rangle_t = 2 \int_0^t M_u dM_u$$

is a local martingale.

Proof. One takes $d = 1$ and $f(x) = x^2$. □

The proposition above says that $\langle M \rangle_t$ is the compensation for M_t^2 to get a local martingale.

Definition 5.9. A continuous and adapted process $X = (X_t)_{t \geq 0}$, $X_t : \Omega \to \mathbb{R}$, is called Itô-process provided there exist $L \in \mathcal{L}_2^{\mathrm{loc}}(B)$ and a progressively measurable process $a = (a_t)_{t \geq 0}$ with

$$\int_0^t |a_u(\omega)| du < \infty$$

for all $t \geq 0$ and $\omega \in \Omega$ and $x_0 \in \mathbb{R}$ such that

$$X_t(\omega) = x_0 + \left(\int_0^t L_u dB_u \right)(\omega) + \int_0^t a_u(\omega) du \quad \text{for} \quad t \geq 0, \ a.s.$$

To formulate Itô's formula in this case we need

Definition 5.10. A continuous function $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ belongs to $C^{1,2}([0, \infty) \times \mathbb{R})$ provided that all partial derivatives $\partial f / \partial t$, $\partial f / \partial x$, and $\partial^2 f / \partial x^2$ exist on $(0, \infty) \times \mathbb{R}$, are continuous, and can be continuously extended to $[0, \infty) \times \mathbb{R}$.

Before we state Itô's formula for Itô-processes we need

Lemma 5.11. *Let $M_t = \int_0^t L_u dB_u$ for some $L \in \mathcal{L}_2(B)$. Then*

$$\langle M \rangle_t = \int_0^t L_u^2 du \quad a.s.$$

Proof. By Proposition 4.3 it is sufficient to check that

$$\left(M_t^2 - \int_0^t L_u^2 du\right)_{t\geq 0}$$

is a martingale. For $0 \leq s < t < \infty$ we have to show that

$$\mathbb{E}\left(M_t^2 - \int_0^t L_u^2 du | \mathcal{F}_s\right) = M_s^2 - \int_0^s L_u^2 du \quad a.s.$$

or

$$\mathbb{E}\left(M_t^2 - M_s^2 | \mathcal{F}_s\right) = \mathbb{E}\left(\int_s^t L_u^2 du | \mathcal{F}_s\right) \quad a.s.$$

Using $\mathbb{E}\left(M_t M_s | \mathcal{F}_s\right) = M_s \mathbb{E}\left(M_t | \mathcal{F}_s\right) = M_s^2$ a.s., the left-hand side computes to

$$\mathbb{E}\left(M_t^2 - M_s^2 | \mathcal{F}_s\right) = \mathbb{E}\left(M_t^2 - 2M_t M_s + M_s^2 | \mathcal{F}_s\right) = \mathbb{E}\left((M_t - M_s)^2 | \mathcal{F}_s\right)$$

and the assertion follows from Proposition 4.14. □

Proposition 5.12 (ITÔ's formula for ITÔ-processes)**.** *Let $X = (X_t)_{t\geq 0}$ be an* ITÔ*-process with representation*

$$X_t = x_0 + \int_0^t L_u dB_u + \int_0^t a_u du, \ t \geq 0, a.s.$$

and let $f \in C^{1,2}$. Then one has that

$$f(t, X_t) = f(0, X_0) + \int_0^t \frac{\partial f}{\partial u}(u, X_u) du + \int_0^t \frac{\partial f}{\partial x}(u, X_u) L_u dB_u + \int_0^t \frac{\partial f}{\partial x}(u, X_u) a_u du + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(u, X_u) L_u^2 du$$

for $t \geq 0$ a.s.

Proof. We only indicate the case $f \in C^2(\mathbb{R}^2)$. We let $d = 2$ and consider the processes $(Y_t, X_t) := (t, X_t)$ and apply the general ITÔ rule. Since the martingale part of Y is zero, the only cross variation which is left is $\langle M \rangle$ where M is the (local) martingale part of X. Here we use Lemma 5.11 to deduce

$$\int_0^t \frac{\partial^2 f}{\partial x^2}(u, X_u) d\langle M \rangle_u = \int_0^t \frac{\partial^2 f}{\partial x^2}(u, X_u) L_u^2 du.$$

□

Example 5.13 (Exponential martingales). *Let* $L \in C[0,\infty)$ *and* $X_t := \int_0^t L_u dB_u$. *Then*

$$\mathcal{E}(X)_t := e^{X_t - \frac{1}{2}\int_0^t L_u^2 du} = e^{X_t - \frac{1}{2}\langle X\rangle_t}$$

is a martingale and called exponential martingale*.*

The proof will be an exercise.

A. For extendend reading: Three applications of Itô's formula

A.1 Behavior of the three dimensional Brownian motion

Let $B = (B_t^1, ..., B_t^d)$ a d-dimensional standard Brownian motion where the filtration is taken to be the augmentation of the natural filtration and the usual conditions are satisfied. The process

$$R_t := |x_0 + B|$$

where $|\cdot|$ is the d-dimensional euclidean norm is called d-dimensional Bessel process starting in $x_0 \in \mathbb{R}^d$. We want to prove the following

Proposition 5.14. *Let $d = 3$ and $0 < c < r = |x_0|$. Then one has that*

$$\mathbb{P}\left(\inf_{t \geq 0} R_t \leq c\right) = \frac{c}{r}.$$

Proof. Let

$$\tau := \inf\{t \geq 0 : R_t = c\} \quad \text{and} \quad \sigma_k := \inf\{t \geq 0 : R_t = k\}$$

for an integer $k > r$. Let

$$\rho_{k,n} := \tau \wedge \sigma_k \wedge n.$$

By Itô's formula we get

$$\frac{1}{R_{\rho_{k,n}}} = \frac{1}{r} - \int_0^{\rho_{k,n}} \langle \nabla f, dB \rangle_u$$

with $f(x) := 1/|x|$. Taking the expected value gives

$$\frac{1}{r} = \mathbb{E}\frac{1}{R_{\rho_{k,n}}} = \frac{1}{c}\mathbb{P}(\tau \leq \sigma_k \wedge n) + \frac{1}{k}\mathbb{P}(\sigma_k \leq \tau \wedge n) + \mathbb{E}\frac{1}{R_n}\chi_{\{n < \sigma_k \wedge \tau\}}.$$

By $n \to \infty$ we get that

$$\frac{1}{r} = \frac{1}{c}\mathbb{P}(\tau \leq \sigma_k) + \frac{1}{k}\mathbb{P}(\sigma_k \leq \tau).$$

By $k \to \infty$ we end up with

$$\frac{1}{r} = \frac{1}{c}\mathbb{P}(\tau < \infty).$$

□

A.2 Lévy's characterization of the Brownian motion

Proposition 5.15. *Let $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_t)_{t\geq 0})$ satisfy the usual conditions and let M be a path-wise continuous martingale with $M_0 \equiv 0$. Then the following conditions are equivalent:*

(i) *M is an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion.*

(ii) *$\langle M\rangle_t = t$ for $t \geq 0$ a.s.*

Proof. We only have to show that (ii) implies (i). Let

$$f(x) := e^{i\lambda x} \quad \text{for some} \quad \lambda \in \mathbb{R}.$$

By Itô's formula (here we ignore, that we have complex numbers),

$$e^{i\lambda(M_t - M_s)}\chi_A = \chi_A + \chi_A \int_s^t i\lambda e^{i\lambda(M_u - M_s)} dM_u - \frac{\lambda^2}{2}\chi_A \int_s^t e^{i\lambda(M_u - M_s)} du$$

for all $A \in \mathcal{F}_s$. Taking the expected value implies that

$$\mathbb{E}e^{i\lambda(M_t - M_s)}\chi_A = \mathbb{P}(A) - \frac{\lambda^2}{2}\int_s^t \mathbb{E}e^{i\lambda M_u - M_s}\chi_A du.$$

Letting

$$H(u) := \mathbb{E}e^{i\lambda M_u - M_s}\chi_A$$

gives that

$$H(t) = \mathbb{P}(A) - \frac{\lambda^2}{2}\int_s^t H(u)du,$$

so that $H(t) = \mathbb{P}(A)e^{-\frac{\lambda^2}{2}(t-s)}$. This implies that $M_t - M_s$ is independent from $\mathcal{F}_s$ and that $M_t - M_s \sim N(0, t-s)$. □

A.3 Local time

Given a Borel set $A \subseteq \mathbb{R}$ and a Brownian motion $B = (B_t)_{t\geq 0}$ we want to compute the occupation time of B in A until time t, i.e.

$$\Gamma_t(A, \omega) := \int_0^t \chi_A(B_s(\omega))ds = \lambda(s \in [0,t] : B_s(\omega) \in A).$$

It is not difficult to show that $\Gamma_t(A,\omega) = 0$ $\mathbb{P}$-a.s. if $\lambda(A) = 0$ so that one can ask for a density

$$\Gamma_t(A,\omega) = \int_A 2L_t(x,\omega)dx$$

where the factor 2 is for cosmetics reason.

Definition 5.16. A stochastic process $L = (L_t(x,\cdot))_{t\geq 0, x\in\mathbb{R}}$ is called Brownian local time provided that

(i) $L_t(x,\cdot) : \Omega \to \mathbb{R}$ is $\mathcal{F}_t$-measurable,

(ii) there exists $\Omega_0 \in \mathcal{F}$ of measure one such that for all $\omega \in \Omega_0$ one has

  (a) $(t,x) \to L_t(x,\omega)$ is continuous,

  (b) $\Gamma_t(A,\omega) = \int_A 2L_t(x,\omega)dx$ for all Borel sets $A \subseteq \mathbb{R}$.

To get a candidate for $L_t(x,\cdot)$ we use Itô's formula: Let $\varphi_\varepsilon \in C_0^\infty$ be such that $\operatorname{supp}(\varphi_\varepsilon) \subseteq [-\varepsilon,\varepsilon]$, $\varphi_\varepsilon \geq 0$, and $\int_{\mathbb{R}} \varphi_\varepsilon(x)dx = 1$. Let

$$f_\varepsilon(x) := \int_{-\infty}^x \int_{-\infty}^y \varphi_\varepsilon(u)dudy$$

so that

$$\begin{aligned} f_\varepsilon'(x) &= \int_{-\infty}^x \varphi_\varepsilon(u)du, \\ f_\varepsilon''(x) &= \varphi_\varepsilon(x). \end{aligned}$$

By Itô's formula, a.s.

$$\begin{aligned} f_\varepsilon(B_t - a) &= f_\varepsilon(-a) + \int_0^t f_\varepsilon'(B_s - a)dB_s + \frac{1}{2}\int_0^t f_\varepsilon''(B_s - a)ds \\ &= f_\varepsilon(-a) + \int_0^t f_\varepsilon'(B_s - a)dB_s + \frac{1}{2}\int_0^t \varphi_\varepsilon(B_s - a)ds. \end{aligned}$$

Now

$$\mathbb{E}\int_0^t |f_\varepsilon'(B_s - a) - \chi_{(0,\infty)}(B_s - a)|^2 ds \to 0$$

and

$$\sup_x |f_\varepsilon(x) - x^+| \to 0$$

as $\varepsilon \downarrow 0$, so that, a.s.,

$$\lim_{\varepsilon_n \downarrow 0} \frac{1}{2} \int_0^t \varphi_{\varepsilon_n}(B_s - a) ds = (B_t - a)^+ - (-a)^+ - \int_0^t \chi_{(0,\infty)}(B_s) dB_s$$

for some sequence $\varepsilon_n \downarrow 0$. But the left-hand side is - formally -

$$\frac{1}{2} \int_0^t \delta(B_s - a) ds = L_t(a, \cdot).$$

Proposition 5.17 (Trotter). *The Brownian local time exists.*

Proof. (Idea) (a) Let

$$M_t(x, \omega) := (B_t(\omega) - a)^+ - (-a)^+ - \left(\int_0^t \chi_{(a,\infty)}(B_s) dB_s \right)(\omega).$$

By a version of Kolmogorov's Proposition 1.8 one can show that there exists a continuous (in (t, x)) version $L = (L_t(x, \cdot))_{t \geq 0, x \in \mathbb{R}}$ of M.
(b) Let $-\infty < a_1 < a_2 < b_2 < b_1$ and define the continuous function $h : \mathbb{R} \to \mathbb{R}$ as $h(x) = 1$ on $[a_2, b_2]$, zero outside $[a_1, b_1]$, and linear otherwise. Let

$$H(x) := \int_{-\infty}^x \int_{-\infty}^y h(u) du dy = \int_{\mathbb{R}} (x - u)^+ du$$

so that

$$\begin{aligned} H'(x) &= \int_{-\infty}^x h(u) du = \int_{\mathbb{R}} h(u) \chi_{(u,\infty)}(x) du \\ H''(x) &= h(x). \end{aligned}$$

By Itô's formula,

$$\begin{aligned} \frac{1}{2} \int_0^t h(B_s) ds &= H(B_t) - H(B_0) - \int_0^t H'(B_s) dB_s \\ &= \int_{\mathbb{R}} \left[(B_t - a)^+ - (-u)^+ - \int_0^t \chi_{(u,\infty)}(B_s) dB_s \right] du \\ &= \int_{\mathbb{R}} h(u) M_t(u, \cdot) du. \end{aligned}$$

(c) In the last step one has to replace M by L. □

Formally, we also get the following:

$$\Gamma((a-\varepsilon, a+\varepsilon), \omega) = \int_{a-\varepsilon}^{a+\varepsilon} 2L_t(x,\omega)dx$$

and

$$\lim_{\varepsilon \downarrow 0} \frac{1}{4\varepsilon} \Gamma((a-\varepsilon, a+\varepsilon), \omega) = L_t(a,\omega).$$

Proposition 5.18 (Tanaka formulas)**.** *One has that, a.s.,*

$$L_t(a) = (B_t - a)^+ - (-a)^+ - \int_0^t \chi_{(a,\infty)}(B_s)dB_s$$

and

$$2L_t(a) = |B_t - a| - |-a| - \int_0^t \operatorname{sgn}(B_s - a)dB_s.$$

Proposition 5.19 (Itô's formula for convex functions)**.** *For a convex function f and its second derivative μ one has, a.s.,*

$$f(B_t) = f(0) + \int_0^t D^- f(B_s)dB_s + \int_{\mathbb{R}} L_t(x)d\mu(x)$$

where

$$D^- f(x) := \lim_{h \downarrow 0} \frac{1}{h}[f(x-h) - f(x)]$$

and μ is determined by

$$\mu([a,b)) := D^- f(b) - D^- f(a).$$

7. Lecture
Stochastic Differential Equations

Stochastic differential equations (SDE's) play an important role in stochastic modeling. For example, in Economics solutions of the SDE's considered below are used to model share prices. In Biology solutions of stochastic partial differential equations (not considered here) describe sizes of populations.

7.1 Strong solutions of stochastic differential equations

Stochastic differential equations are (for us) a formal abbreviation of *integral* equations as described now:

Definition 7.1. Let $x_0 \in \mathbb{R}$, $D \subseteq \mathbb{R}$ be an open set, and $\sigma, a : [0,\infty) \times D \to \mathbb{R}$ be continuous. A continuous and adapted stochastic process $X = (X_t)_{t \geq 0}$ is a solution of the *stochastic differential equation* (SDE)

$$dX_t = \sigma(t, X_t)B_t + a(t, X_t)dt \quad \text{with} \quad X_0 = x_0 \tag{3}$$

provided that the following conditions are satisfied:

(i) $X_t(\omega) \in D$ for all $t \geq 0$ and $\omega \in \Omega$.

(ii) $X_0 \equiv x_0$.

(iii) $X_t = x_0 + \int_0^t \sigma(u, X_u)dB_u + \int_0^t a(u, X_u)du$ for $t \geq 0$ a.s.

Let us give some examples of SDE's.

Example 7.2 (Geometric Brownian motion with drift)**.** If $X_t := x_0 e^{cB_t + bt}$ with $x_0, b, c \in \mathbb{R}$, then we obtain by Itô's formula that, a.s.,

$$\begin{aligned} X_t &= x_0 + \int_0^t cX_u dB_u + \int_0^t bX_u du + \frac{1}{2}\int_0^t c^2 X_u du \\ &= x_0 + \int_0^t cX_u dB_u + \int_0^t \left[b + \frac{1}{2}c^2\right] X_u du \\ &= x_0 + \int_0^t \sigma X_u dB_u + \int_0^t aX_u du \end{aligned}$$

with

$$\begin{aligned} \sigma &:= c, \\ a &:= b + \frac{1}{2}c^2. \end{aligned}$$

Going the other way round by starting with a and σ, we get that

$$\begin{aligned} c &= \sigma, \\ b &= a - \frac{1}{2}\sigma^2. \end{aligned}$$

Consequently, the SDE

$$dX_t = \sigma X_t dB_t + aX_t dt \quad \text{with} \quad X_0 = x_0$$

is solved by

$$X_t = x_0 e^{\sigma B_t + \left(a - \frac{1}{2}\sigma^2\right)t}.$$

We may use $D = \mathbb{R}$ for $\sigma(t,x) := \sigma x$ and $a(t,x) := ax$.

The following examples only provide the formal SDE's. We do not discuss solvability at this point.

Example 7.3 (Ornstein-Uhlenback process)**.** Here one considers the SDE

$$dX_t = -cX_t dt + \sigma dB_t \quad \text{with} \quad X_0 = x_0.$$

Example 7.4 (Vasicek interest rate model)**.** Here one considers that

$$dr_t = a(b - r_t)dt + \sigma dB_t \quad \text{with} \quad r_0 \geq 0,$$

$\sigma \geq 0$, and $a, b > 0$ models an interest rate in Stochastic Finance. The problem with this model is that r_t might be negative if $\sigma > 0$. If $\sigma = 0$, then one gets as one solution

$$r_t = r_0 e^{-at} + b(1 - e^{-at})$$

so that the meaning of a and b become more clear: the interest rate moves from its initial value r_0 to the value b as $t \to \infty$ with a speed determined by the parameter a. If $\sigma > 0$ one tries to add a random perturbation to that.

7.2 Uniqueness and existence of strong solutions

We shall start with a beautiful lemma, the GRONWALL lemma.

Lemma 7.5 (GRONWALL)**.** *Let $A, B, T \geq 0$ and $f : [0, T] \to \mathbb{R}$ be a continuous function such that*

$$f(t) \leq A + B \int_0^t f(s)ds$$

for all $t \in [0, T]$. Then one has that $f(T) \leq Ae^{BT}$.

Proof. Letting $g(t) := e^{-Bt} \int_0^t f(s)ds$ we deduce

$$\begin{aligned} g'(t) = -Be^{-Bt} \int_0^t f(s)ds + e^{-Bt} f(t) \\ = e^{-Bt} \left(f(t) - B \int_0^t f(s)ds \right) \leq Ae^{-Bt} \end{aligned}$$

and

$$g(T) = \int_0^T g'(t)dt \leq A \int_0^T e^{-Bt}dt \leq \frac{A}{B} \left(1 - e^{-BT}\right).$$

Consequently,

$$\begin{aligned} f(T) \leq A + B \int_0^T f(t)dt = A + Be^{BT} g(T) \\ \leq A + Be^{BT} \frac{A}{B} \left(1 - e^{-BT}\right) = Ae^{BT}. \end{aligned}$$

□

Proposition 7.6 (Strong uniqueness)**.** *Suppose that for all $n = 1, 2, ...$ there is a constant $C_n > 0$ such that*

$$|\sigma(t, x) - \sigma(t, y)| + |a(t, x) - a(t, y)| \leq c_n |x - y|$$

for $|x| \leq n$, $|y| \leq n$, and $t \geq 0$. Assume that $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ are solutions of the SDE (3). Then

$$\mathbb{P}\left(X_t = Y_t, t \geq 0\right) = 1.$$

Proof. We use the stopping times

$$\sigma_n := \inf\{t \geq 0 : |X_t| \geq n\} \quad \text{and} \quad \tau_n := \inf\{t \geq 0 : |Y_t| \geq n\}$$

where we assume that $n > |x_0|$. Letting $\rho_n := \min\{\sigma_n, \tau_n\}$ we obtain, a.s., that

$$\begin{aligned} & X_{t\wedge\rho_n} - Y_{t\wedge\rho_n} \\ & \quad = \int_0^{t\wedge\rho_n} [a(u,X_u) - a(u,Y_u)]\, du + \int_0^{t\wedge\rho_n} [\sigma(u,X_u) - \sigma(u,Y_u)]\, dB_u. \end{aligned}$$

Hence

$$\begin{aligned} \mathbb{E}\,|X_{t\wedge\rho_n} - Y_{t\wedge\rho_n}|^2 &\leq 2\mathbb{E}\left|\int_0^{t\wedge\rho_n} [a(u,X_u) - a(u,Y_u)]\, du\right|^2 \\ &\quad +2\mathbb{E}\left|\int_0^{t\wedge\rho_n} [\sigma(u,X_u) - \sigma(u,Y_u)]\, dB_u\right|^2 \\ &\leq 2t\mathbb{E}\int_0^{t\wedge\rho_n} |a(u,X_u) - a(u,Y_u)|^2\, du \\ &\quad +2\mathbb{E}\int_0^{t\wedge\rho_n} [\sigma(u,X_u) - \sigma(u,Y_u)]^2\, du \\ &\leq (2t+2)c_n^2\mathbb{E}\int_0^{t\wedge\rho_n} |X_u - Y_u|^2 du \\ &\leq (2t+2)c_n^2\mathbb{E}\int_0^t |X_{u\wedge\rho_n} - Y_{u\wedge\rho_n}|^2 du. \end{aligned}$$

Now fix $T > 0$. The above computation gives

$$\mathbb{E}\,|X_{t\wedge\rho_n} - Y_{t\wedge\rho_n}|^2 \leq (2T+2)c_n^2 \int_0^t \mathbb{E}\,|X_{u\wedge\rho_n} - Y_{u\wedge\rho_n}|^2\, du$$

for $t \in [0,T]$. For

$$f(t) := \mathbb{E}\,|X_{t\wedge\rho_n} - Y_{t\wedge\rho_n}|^2$$

we may apply Gronwall's lemma. The function f is continuous since for $t_k \to t$ one gets

$$\lim_k f(t_k) = \lim_k \mathbb{E}\,|X_{t_k\wedge\rho_n} - Y_{t_k\wedge\rho_n}|^2$$

$$
\begin{aligned}
&= \mathbb{E} \lim_k |X_{t_k \wedge \rho_n} - Y_{t_k \wedge \rho_n}|^2 \\
&= \mathbb{E} |X_{t \wedge \rho_n} - Y_{t \wedge \rho_n}|^2 \\
&= f(t)
\end{aligned}
$$

by dominated convergence as a consequence of (for example)

$$\mathbb{E} \sup_{t \in [0,T]} |X_{t \wedge \rho_n}|^2 \leq n^2$$

and the continuity of the processes X and Y. Exploiting GRONWALL's lemma with $A := 0$ and $B := (2T+2)c_n$ yields

$$f(T) \leq A e^{BT} = 0 \quad \text{and} \quad \mathbb{E} |X_{t \wedge \rho_n} - Y_{t \wedge \rho_n}|^2 = 0.$$

Since

$$\lim_n \rho_n = \infty$$

because X and Y are continuous processes, we get by FATOU's lemma that

$$\mathbb{E} |X_t - Y_t|^2 = \mathbb{E} \liminf_n |X_{t \wedge \rho_n} - Y_{t \wedge \rho_n}|^2 \leq \liminf_n \mathbb{E} |X_{t \wedge \rho_n} - Y_{t \wedge \rho_n}|^2 = 0.$$

Hence $\mathbb{P}(X_t = Y_t) = 1$ and, by the continuity of X and Y,

$$\mathbb{P}(X_t = Y_t, t \geq 0) = 1.$$

□

Sometimes the assumptions of the above criteria are too strong. There is a nice extension:

Proposition 7.7 (YAMADA-TANAKA). *Suppose that*

$$\sigma, a : [0,\infty) \times \mathbb{R} \to \mathbb{R}$$

are continuous such that

$$
\begin{aligned}
|\sigma(t,x) - \sigma(t,y)| &\leq h(|x-y|), \\
|a(t,x) - a(t,y)| &\leq K(|x-y|)
\end{aligned}
$$

for $x, y \in \mathbb{R}$, where $h : [0,\infty) \to [0,\infty)$ is strictly increasing with $h(0) = 0$ and $K : [0,\infty) \to \mathbb{R}$ is strictly increasing and concave with $K(0) = 0$, such that

$$\int_0^\varepsilon \frac{du}{K(u)} = \int_0^\varepsilon \frac{du}{h(u)^2} = \infty$$

for all $\varepsilon > 0$. Then any two solutions of (3) are indistinguishable until the hitting time of the boundary of D.

Example 7.8. *One can take $h(x) := x^\alpha$ for $\alpha \geq \frac{1}{2}$.*

However, there is also the following example:

Example 7.9. *Let $\sigma : \mathbb{R} \to [0,\infty)$ be continuous such that*

(i) $\sigma(x_0) = 0$,

(ii) $\int_{x_0-\varepsilon}^{x_0+\varepsilon} \frac{dx}{\sigma^2(x)} < \infty$ *and* $\sigma(x) \geq 1$ *if* $|x - x_0| > \varepsilon$ *for some* $\varepsilon > 0$.

Then the SDE

$$dX_t = \sigma(X_t)dB_t \quad \textit{with} \quad X_0 = x_0$$

has infinitely many solutions.

Proposition 7.10. *Suppose that $\sigma, a : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ are continuous such that*

$$\begin{aligned} |\sigma(t,x) - \sigma(t,y)| + |a(t,x) - a(t,y)| &\leq K|x-y|, \\ |\sigma(t,x)| + |a(t,x)| &\leq K(1+|x|) \end{aligned}$$

for all $t \geq 0$, $x \in \mathbb{R}$, and some $K > 0$. Then there exists a solution of the SDE (3).

Proof. (a) We define a sequence of processes $X^{(k)} = (X_t^{(k)})_{t\geq 0}$ which converges to our solution:

$$\begin{aligned} X_t^{(0)} &:= x_0, \\ X_t^{(k+1)} &:= x_0 + \int_0^t \sigma(u, X_u^{(k)})dB_u + \int_0^t a(u, X_u^{(k)})du. \end{aligned}$$

(b) Let us fix $T > 0$. By induction we show that

$$\sup_{t\in[0,T]} \mathbb{E}|X_t^{(k)}|^2 =: A_k < \infty.$$

For $k = 0$ this is clear so let us consider the step from k to $k+1$ where we get that

$$\mathbb{E}|X_t^{(k+1)}|^2$$

$$
\begin{aligned}
&\leq 4\left[|x_0|^2 + \mathbb{E}\int_0^T \sigma(u, X_u^{(k)})^2 du + \mathbb{E}\left|\int_0^t a(u, X_u^{(k)}) du\right|^2\right] \\
&\leq 4\left[|x_0|^2 + \int_0^T K^2(1+|X_u^{(k)}|)^2 du + T\int_0^T \mathbb{E}K^2(1+|X_u^{(k)}|)^2 du\right] \\
&=: A_{k+1} < \infty
\end{aligned}
$$

since $\mathbb{E}(1+|X_u^{(k)}|)^2 \leq 2(1+A_k)$.
(c) Now we show some kind of Cauchy sequence property for the sequence of processes $X^{(k)}$ to obtain an appropriate limit. First we decompose the difference $X_t^{(k+1)} - X_t^{(k)}$ almost surely as

$$
\begin{aligned}
X_t^{(k+1)} - X_t^{(k)} &= \int_0^t \left[\sigma(u, X_u^{(k)}) - \sigma(u, X_u^{(k-1)})\right] dB_u \\
&\quad + \int_0^t \left[a(u, X_i^{(k)}) - a(u, X_u^{(k-1)})\right] du \\
&=: M_t + A_t.
\end{aligned}
$$

Now

$$
\mathbb{E}\sup_{t\in[0,T]} |A_t|^2 \leq TK^2 \int_0^T \mathbb{E}|X_u^{(k)} - X_u^{(k-1)}|^2 du
$$

and

$$
\begin{aligned}
\mathbb{E}\sup_{t\in[0,T]} |M_t|^2 &\leq 4\mathbb{E}M_T^2 \\
&= 4\mathbb{E}\int_0^T \left|\sigma(u, X_u^{(k)}) - \sigma(u, X_u^{(k-1)})\right|^2 du \\
&\leq 4K^2 \int_0^T \mathbb{E}\left|X_u^{(k)} - X_u^{(k-1)}\right|^2 du
\end{aligned}
$$

by Doob's inequality and Itô's isometry. Consequently,

$$
\mathbb{E}\sup_{t\in[0,T]}\left|X_t^{(k+1)} - X_t^{(k)}\right|^2 \leq L\int_0^T \mathbb{E}|X_u^{(k)} - X_u^{(k-1)}|^2 du
$$

for $L := 2(4K^2 + TK^2)$. We iterate the last equation and get

$$
\mathbb{E}\sup_{t\in[0,T]}\left|X_t^{(k+1)} - X_t^{(k)}\right|^2
$$

$$
\begin{aligned}
&\leq \; L^2 \int_0^T \int_0^{u_1} \mathbb{E} \left| X_v^{(k-1)} - X_v^{(k-2)} \right|^2 dv du_1 \\
&\leq \; L^k \int_0^T \int_0^{u_{k-1}} \cdots \int_0^{u_1} \mathbb{E} \left| X_v^1 - x_0 \right|^2 dv du_1 \cdots du_{k-1} \\
&\leq \; L^k \frac{T^k}{k!} \sup_{v \in [0,T]} \mathbb{E} |X_v^{(1)} - x_0|^2 \\
&= \; c \frac{(LT)^k}{k!}.
\end{aligned}
$$

Using CHEBYCHEV's inequality we continue to

$$
\mathbb{P}\left(\sup_{t \in [0,T]} \left| X_t^{(k+1)} - X_t^{(k)} \right| > \frac{1}{2^{k+1}} \right) \leq 4^{k+1} c \frac{(LT)^k}{k!} = (4c) \frac{(4LT)^k}{k!}.
$$

Now we use the BOREL-CANTELLI lemma: letting

$$
A_k := \left\{ \sup_{t \in [0,T]} \left| X_t^{(k+1)} - X_t^{(k)} \right| > \frac{1}{2^{k+1}} \right\}
$$

we get

$$
\sum_{k=1}^{\infty} \mathbb{P}(A_k) < \infty \quad \text{so that} \quad \mathbb{P}\left(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n \right) = 0
$$

or

$$
\mathbb{P}\left(\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n^c \right) = 1.
$$

Assuming

$$
\omega \in \Omega_0^T := \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n^c
$$

we have

$$
\sup_{t \in [0,T]} \left| X_t^{(n+1)}(\omega) - X_t^{(n)}(\omega) \right| \leq \frac{1}{2^{n+1}}
$$

for $n \geq k(\omega)$. Define

$$
Y_t^{(T)}(\omega) := \begin{cases} \lim_k X_t^{(k)}(\omega) & : \; \omega \in \Omega_0^T \\ 0 & : \; \omega \notin \Omega_0^T \end{cases}.
$$

For this process one can show that it satisfies

$$Y_t^{(T)} = x_0 + \int_0^t \sigma(u, Y_u^{(T)}) dB_u + \int_0^t a(u, Y_u^{(T)}) du$$

on $[0, T]$. By the uniqueness argument for the strong solutions we also get that

$$\mathbb{P}(Y_t^{(T_1)} = Y_t^{(T_2)}) = 1$$

for $t \in [0, \min\{T_1, T_2\}]$. Hence, as already carried out earlier we may find a continuous and adapted process $X = (X_t)_{t \geq 0}$ such that

$$\mathbb{P}(X_t = Y_t^{(n)}) = 1 \quad \text{for all} \quad t \in [0, n]$$

which turns out to be our solution. $\square$

8. Lecture

Transformation of drift

To perform a so-called *transformation of drift* we need the following result of Girsanov:

Proposition 8.1 (Girsanov). *Let $L = (L_t)_{t\geq 0} \in \mathcal{L}_2(B)$ and assume that the process $(\mathcal{E}_t)_{t\geq 0}$ defined by*

$$\mathcal{E}_t := \exp\left(-\int_0^t L_u dB_u - \frac{1}{2}\int_0^t L_u^2 du\right)$$

is a martingale. Let $T > 0$ and

$$dQ_T := \mathcal{E}_T d\mathbb{P}.$$

Then $(W_t)_{t\in[0,T]}$ with

$$W_t := B_t + \int_0^t L_u du$$

defines a Brownian motion $(W_t)_{t\in[0,T]}$ with respect to $(\Omega, \mathcal{F}_T, Q_T, (\mathcal{F}_t)_{t\in[0,T]})$.

Lemma 8.2. *Let $0 \leq t \leq T < \infty$.*

(i) *The measures Q_t and Q_T coincide on $\mathcal{F}_t$.*

(ii) *Assume that $Z : \Omega \to \mathbb{R}$ is $\mathcal{F}_T$-measurable such that $\mathbb{E}_{Q_T}|Z| < \infty$. Then*

$$\mathbb{E}_{Q_T}(Z|\mathcal{F}_t) = \frac{\mathbb{E}(Z\mathcal{E}_T|\mathcal{F}_t)}{\mathcal{E}_t} \quad a.s.$$

Proof. (i) For $B \in \mathcal{F}_t$ one has

$$Q_T(B) = \int_B \mathcal{E}_T d\mathbb{P} = \int_B \mathbb{E}(\mathcal{E}_T|\mathcal{F}_t) d\mathbb{P} = \int_B \mathcal{E}_t d\mathbb{P} = Q_t(B)$$

where we have used that $(\mathcal{E}_t)_{t\geq 0}$ is a martingale.
(ii) We show that

$$\mathbb{E}\mathcal{E}_t \mathbb{E}_{Q_T}(|Z||\mathcal{F}_t) < \infty \quad \text{and} \quad \int_B \mathcal{E}_t \mathbb{E}_{Q_T}(Z|\mathcal{F}_t) d\mathbb{P} = \int_B Z\mathcal{E}_T d\mathbb{P}$$

for all $B \in \mathcal{F}_t$, which follows from

$$
\begin{aligned}
\infty &> \int_\Omega |Z| dQ_T \\
&= \int_\Omega \mathbb{E}_{Q_T}(|Z||\mathcal{F}_t) dQ_T \\
&= \int_\Omega \mathbb{E}_{Q_T}(|Z||\mathcal{F}_t) dQ_t \\
&= \int_\Omega \mathcal{E}_t \mathbb{E}_{Q_T}(|Z||\mathcal{F}_t) d\mathbb{P},
\end{aligned}
$$

and, the other way round, by

$$
\begin{aligned}
\int_B \mathcal{E}_t \mathbb{E}_{Q_T}(Z|\mathcal{F}_t) d\mathbb{P} &= \int_B \mathbb{E}_{Q_T}(Z|\mathcal{F}_t) dQ_t \\
&= \int_B \mathbb{E}_{Q_T}(Z|\mathcal{F}_t) dQ_T \\
&= \int_B Z dQ_T \\
&= \int_B Z \mathcal{E}_T d\mathbb{P}.
\end{aligned}
$$

□

Proof of Proposition 8.1. We restrict ourselves to $L_t \equiv \mu \in \mathbb{R}$. In this case we have that $W_t = B_t + \mu t$ which is a Brownian motion with a deterministic drift. Moreover, using the same argument as in Example 4.2 we derive that $(\mathcal{E}_t)_{t \geq 0}$ is a martingale.

(a) First we show that $(W_t)_{t \in [0,T]}$ is a Q_T-martingale, that means $\mathbb{E}_{Q_T}|W_t| < \infty$ and $\mathbb{E}_{Q_T}(W_T|\mathcal{F}_t) = W_t$ a.s. for $0 \leq t \leq T$. The integrability is a consequence of

$$
\begin{aligned}
\mathbb{E}_{Q_T}|W_t| &= \int_\Omega |B_t + \mu t| \mathcal{E}_T d\mathbb{P} \\
&= \int_\Omega |B_t + \mu t| e^{-\mu B_T - \frac{\mu^2}{2}T} d\mathbb{P} \\
&\leq \int_\Omega |B_t| e^{-\mu B_T - \frac{\mu^2}{2}T} d\mathbb{P} + \mu t \int_\Omega e^{-\mu B_T - \frac{\mu^2}{2}T} d\mathbb{P}
\end{aligned}
$$

$$
\begin{aligned}
&= \int_\Omega \sqrt{t}|g_1|e^{-\mu\sqrt{t}g_1-\mu\sqrt{T-t}g_2-\frac{\mu^2}{2}T}d\mathbb{P}+\mu t \\
&\leq \sqrt{t}\int_\Omega |g_1|e^{-\mu\sqrt{t}g_1}d\mathbb{P}\int_\Omega e^{-\mu\sqrt{T-t}g_2}d\mathbb{P}e^{-\frac{\mu^2}{2}T}+\mu t
\end{aligned}
$$

where g_1 and g_2 are independent standard Gaussian random variables. So we have to estimate, for an integer $n \geq 0$ and $c \in \mathbb{R}$, that

$$\int_\Omega |g|^n e^{cg}d\mathbb{P} = \int_{\mathbb{R}} |\xi|^n e^{c\xi}e^{-\frac{\xi^2}{2}}\frac{d\xi}{\sqrt{2\pi}} < \infty$$

which follows by an easy computation. Now we turn to the martingale property. Using Lemma 8.2 we need to show that

$$\frac{\mathbb{E}(W_T\mathcal{E}_T|\mathcal{F}_t)}{\mathcal{E}_t} = W_t \text{ a.s.}$$

which is equivalent to

$$\mathbb{E}\left((W_T-W_t)\frac{\mathcal{E}_T}{\mathcal{E}_t}|\mathcal{F}_t\right)+W_t\mathbb{E}\left(\frac{\mathcal{E}_T}{\mathcal{E}_t}|\mathcal{F}_t\right) = W_t \;\text{ a.s.}$$

Since

$$\mathbb{E}\left(\frac{\mathcal{E}_T}{\mathcal{E}_t}|\mathcal{F}_t\right) = 1 \text{ a.s.}$$

because $(\mathcal{E}_t)_{t\geq 0}$ is a martingale, we end up by checking that

$$\mathbb{E}\left((W_T-W_t)\frac{\mathcal{E}_T}{\mathcal{E}_t}|\mathcal{F}_t\right) = 0 \;\text{ a.s.}$$

Again, by independence,

$$\mathbb{E}\left((W_T-W_t)\frac{\mathcal{E}_T}{\mathcal{E}_t}|\mathcal{F}_t\right) = \mathbb{E}\left((W_T-W_t)\frac{\mathcal{E}_T}{\mathcal{E}_t}\right) \;\text{ a.s.}$$

Finally, we observe, for $s := T-t$, that

$$\mathbb{E}\left((W_T-W_t)\frac{\mathcal{E}_T}{\mathcal{E}_t}\right) = \int_{\mathbb{R}}(\sqrt{s}\xi-\mu s)e^{-\mu\sqrt{s}\xi-\frac{s\mu^2}{2}}\frac{d\xi}{\sqrt{2\pi}} = 0.$$

(b) Now we compute the quadratic variation of W. It should be the same as the quadratic variation of the Brownian motion B, since the only difference is a process of 1-variation. Let us take a sequence of nets

$$0 = t_0^n \leq \cdots \leq t_n^n = t$$

with

$$\lim_n \max_{1\leq i\leq n} |t_i^n - t_{i-1}^n| = 0.$$

We find a subsequence $(n_k)_{k=1}^{\infty}$ such that

$$\lim_k \sum_{i=1}^{n} |W_{t_i^n} - W_{t_{i-1}^n}|^2 = \langle W\rangle_t \quad \text{Q-a.s.}$$

Since $\mathbb{P} \sim Q$ we may replace Q-a.s. by $\mathbb{P}$-a.s. Finally, we have that

$$\begin{aligned}
\left(\sum_{i=1}^{n} |B_{t_i^n} - B_{t_{i-1}^n}|^2\right)^{\frac{1}{2}} &- \mu \left(\sum_{i=1}^{n} |t_i^n - t_{i-1}^n|^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{i=1}^{n} |W_{t_i^n} - W_{t_{i-1}^n}|^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{i=1}^{n} |B_{t_i^n} - B_{t_{i-1}^n}|^2\right)^{\frac{1}{2}} + \mu \left(\sum_{i=1}^{n} |t_i^n - t_{i-1}^n|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Since

$$\begin{aligned}
\lim_n \left(\sum_{i=1}^{n} |t_i^n - t_{i-1}^n|^2\right) &\leq \lim_n \left[\sup_i |t_i^n - t_{i-1}^n|\right] \left(\sum_{i=1}^{n} |t_i^n - t_{i-1}^n|\right) \\
&= \lim_n t \sup_i |t_i^n - t_{i-1}^n| = 0
\end{aligned}$$

it follows that, a.s.,

$$\lim_k \left(\sum_{i=1}^{n_k} |B_{t_i^{n_k}} - B_{t_{i-1}^{n_k}}|^2\right) = \lim_k \left(\sum_{i=1}^{n_k} |W_{t_i^{n_k}} - W_{t_{i-1}^{n_k}}|^2\right).$$

Because of $\langle B\rangle_t = t$, $t \geq 0$, a.s., this implies that

$$\langle W\rangle_t = t, t \geq 0, \text{ a.s.}$$

Applying Proposition 5.15 we can conclude the proof. □

Now we turn to the Novikov condition, an important condition to decide whether $(\mathcal{E}_t)_{t\geq 0}$ is a martingale.

Proposition 8.3. *Assume that $M = (M_t)_{t\in[0,T]}$ is a continuous martingale such that*

$$\mathbb{E}e^{\frac{1}{2}\langle M\rangle_T} < \infty.$$

Then $\mathcal{E} = (\mathcal{E}_t)_{t\in[0,T]}$ with

$$\mathcal{E}_t := e^{M_t - \frac{1}{2}\langle M\rangle_t}$$

is a martingale.

Now we come back to our SDE's and show how the method of the transformation of drift works.

Proposition 8.4 (Transformation of drift)**.** *Let $\sigma, a : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ be continuous such that*

$$\begin{aligned} |\sigma(t,x) - \sigma(t,y)| + |a(t,x) - a(t,y)| &\leq K|x-y| \\ |\sigma(t,x)| + |a(t,x)| &\leq K(1+|x|) \end{aligned}$$

for all $x, y \in \mathbb{R}$ and $t \geq 0$. Let $X = (X_t)_{t\geq 0}$ be the unique strong solution of

$$dX_t = \sigma(t, X_t)dB_t + a(t, X_t)dt$$

with $X_0 \equiv x_0 \in \mathbb{R}$. Let $T > 0$ and $L : [0,T] \times \mathbb{R} \to \mathbb{R}$ be continuous such that

$$\mathbb{E}e^{\frac{1}{2}\int_0^T L(u,X_u)^2 du} < \infty$$

and let

$$W_t := B_t + \int_0^t L(u, X_u)du$$

for $u \in [0,T]$. Then, under Q_T with

$$dQ_T := \mathcal{E}_T d\mathbb{P} \quad \textit{where} \quad \mathcal{E}_t := e^{-\int_0^t L(u,X_u)dB_u - \frac{1}{2}\int_0^t L(u,X_u)^2 du},$$

one has that

$$dX_t = \sigma(t,X_t)dW_t + [a(t,X_t) - \sigma(t,X_t)L(t,X_t)]\, dt \quad \textit{for} \quad t \in [0,T].$$

What is our philosophy in this case? We wish to solve

$$dX_t = \sigma(t,X_t)dW_t + [a(t,X_t) - \sigma(t,X_t)L(t,X_t)]\, dt \quad \text{for} \quad t \in [0,T].$$

For this purpose we construct a specific Brownian motion $W = (W_t)_{t\in[0,T]}$ on an appropriate stochastic basis $(\Omega, \mathcal{F}, Q_T; (\mathcal{F}_t)_{t\in[0,T]})$ so that this problem has the solution $X = (X_t)_{t\in[0,T]}$ which is called *weak solution*.

Proof of Proposition 8.4. By Propositions 7.6 and 7.10 there is a unique strong solution $X = (X_t)_{t\geq 0}$. Setting

$$M_t := \int_0^t (-L_u) dB_u$$

we get that $(\mathcal{E}_t)_{t\geq 0}$ is a martingale by NOVIKOV's condition (Proposition 8.3). The GIRSANOV Theorem (Proposition 8.1) gives that $(W_t)_{t\in[0,T]}$ is a Brownian motion with respect to Q_T. And finally (and also a bit formally)

$$\begin{aligned} dX_t &= \sigma(t, X_t)dB_t + a(t, X_t)dt \\ &= \sigma(t, X_t)(dB_t + L(t, X_t)dt) - \sigma(t, X_t)L(t, X_t)dt + a(t, X_t)dt \\ &= \sigma(t, X_t)dW_t + (a(t, X_t) - \sigma(t, X_t)L(t, X_t))dt. \end{aligned}$$

$\square$

Example 8.5. Let $\sigma(t, x) = x$, $a \equiv 0$, $x_0 = 1$, $S_t = e^{B_t - \frac{t}{2}}$, and

$$\mathbb{E}e^{\frac{1}{2}\int_0^T L(u,S_u)^2 du} < \infty.$$

Then

$$dS_t = S_t dW_t - S_t L(t, S_t)dt, \quad t \in [0, T], \quad \text{under} \quad Q_T.$$

A. For extended reading: weak solutions

In the previous section we already indicated the principle of weak solutions: we do not start with a stochastic basis, we constructed a particular Brownian motion to our problem. The formal definition is as follows:

Definition 8.6. Assume that $\sigma, a : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ are measurable and bounded. A pair $(X_t, W_t)_{t\geq 0}$ is a *weak solution* of

$$dX_t = \sigma(t, X_t)dW_t + a(t, X_t)dt \quad \text{with} \quad X_0 \equiv x_0$$

if there exits a stochastic basis $(\Omega, \mathcal{F}, Q, (\mathcal{F}_t)_{t\geq 0})$ satisfying the usual conditions such that

(i) $(W_t)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion,

(ii) $X_t = x_0 + \int_0^t \sigma(u, X_u)dW_u + \int_0^t a(u, X_u)du$, $t \geq 0$, a.s., and $X_0 \equiv x_0$.

Let us start with an example of an SDE that has a weak solution, but not a strong solution.

Example 8.7 (Tanaka)**.** Assume that $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_t)_{t\geq 0})$ satisfies the usual conditions and that B is an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion and that the filtration is the augmentation of the natural filtration of B. Then the SDE

$$dX_t = \operatorname{sgn}(X_t)dB_t \quad \text{with} \quad X_0 \equiv 0,$$

with $\operatorname{sgn}(x) = -1$ for $x < 0$ and $\operatorname{sgn}(x) = 1$ for $x \geq 0$, has no strong solution, but a weak solution.

Proof. (a) Assume that we have a strong solution: By Levy's theorem the process X is a Brownian motion as well because,

$$\langle X \rangle_t = t$$

for $t \geq 0$ a.s. Applying Tanaka's formula gives that

$$\begin{aligned} |X_t| &= \int_0^t \operatorname{sgn}(X_s)dX_s + 2L_t(0) \\ &= \int_0^t \operatorname{sgn}(X_s)\operatorname{sgn}(X_s)dB_s + 2L_t(0) \end{aligned}$$

$$= \quad B_t + 2L_t(0).$$

Consequently, B_t can be expressed (modulo null sets) via $|X_t|$, so that

$$\mathcal{F}_t^X \vee \mathcal{N} \subseteq \mathcal{F}_t^B \vee \mathcal{N} \subseteq \mathcal{F}_t^{|X|} \vee \mathcal{N}$$

which can be shown to be a contradiction, where $\mathcal{N}$ is the system of all null sets of $(\Omega, \mathcal{F}, \mathbb{P})$.

(b) The SDE has a weak solution. We start with a Brownian motion X on $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_t)_{t \geq 0})$ and let

$$W_t := \int_0^t \operatorname{sgn}(X_s) dX_s.$$

By LEVY's theorem the process W is a Brownian motion as well and we have that, a.s.,

$$X_t = \int_0^t \operatorname{sgn}(X_s)^2 dX_s = \int_0^t \operatorname{sgn}(X_s) dW_s.$$

□

Proposition 8.8. *Assume that* $\sigma, a : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ *are continuous and bounded. Then the SDE*

$$dX_t = \sigma(t, X_t) dW_t + a(t, X_t) dt \quad \textit{with} \quad X_0 \equiv x_0$$

has a weak solution.

Proof. (Idea) Let us define

$$Af(t, x) := \frac{\sigma(t, x)^2}{2} f''(x) + a(t, x) f'(x)$$

for $f \in C^2$. Using ITÔ's formula and assuming that one has a solution X one can show that

$$\left(f(X_t) - f(X_0) - \int_0^t Af(s, X_s) ds \right)_{t \geq 0} \tag{4}$$

is a local martingale for $f(x) = x$ and $f(x) = x^2$. The idea is as follows:
Step 1: By an approximation scheme one constructs a continuous stochastic process X which solves the martingale problem (4) as above.

Step 2: Letting

$$M_t := X_t - X_0 - \int_0^t a(s, X_s)ds$$

one proves (by using $f(x) = x$ and $f(x) = x^2$) that

$$\langle M \rangle_t = \int_0^t \sigma(s, X_s)^2 ds$$

for $t \geq 0$ a.s. From that one constructs a Brownian motion W such that

$$M_t = \int_0^t \sigma(s, X_s) dW_s$$

and checks that this is the desired solution. $\square$

To formulate a connection of weak solutions to strong solutions we introduce the notion of *path-wise uniqueness*.

Definition 8.9. The SDE

$$dX_t = \sigma(u, X_u)dB_u + a(u, X_u)du \quad \text{with} \quad X_0 \equiv x_0,$$

$t \geq 0$, a.s., $X_0 \equiv x_0$, satisfies the *path-wise uniqueness* if any two solutions with respect to the same stochastic basis and Brownian motion are indistinguishable.

The application of this concept consists in

Proposition 8.10. *The existence of weak solutions together with the path-wise uniqueness implies the existence of strong solutions.*

9. Lecture

A connection to PDEs

Throughout this section we assume a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,1]})$ and a standard d-dimensional Brownian motion $B = (B_t)_{t\in[0,1]}$, where $(\mathcal{F}_t)_{t\in[0,1]}$ is the augmentation of the natural filtration and $\mathcal{F} = \mathcal{F}_1$. We consider the parabolic backwards PDE

$$\frac{\partial F}{\partial t} + \frac{1}{2}\Delta F + g = kF \quad \text{with} \quad F(1,x) = f(x)$$

where

- the potential $k : \mathbb{R}^d \to [0,\infty)$ is continuous,
- the Lagrangian $g : [0,1] \times \mathbb{R}^d \to \mathbb{R}$ is continuous,
- the terminal condition $f : \mathbb{R}^d \to \mathbb{R}$ is continuous,
- $F : [0,1] \times \mathbb{R}^d \to \mathbb{R}$ is continuous and in $C^{1,2}$ on $[0,1) \times \mathbb{R}^d$.

Proposition 9.1 (Feynman-Kac). *Assume that*

$$\max_{t\in[0,1]} |F(t,x)| + \max_{t\in[0,1]} |g(t,x)| \le ce^{a|x|^2}$$

and that $0 < a < \frac{1}{2d}$. *Then*

$$F(t,x) =$$
$$\mathbb{E}\left[f(x+B_{1-t})e^{-\int_0^{1-t} k(x+B_s)ds} + \int_0^{1-t} g(t+\theta, x+B_\theta)e^{-\int_0^\theta k(x+B_s)ds} d\theta\right].$$

Proof. Let us consider the case $d = 1$ and let us fix $x \in \mathbb{R}$ and $0 < r < 1-t$. Define the process

$$A_\theta := \int_0^\theta k(x+B_s)ds$$

and the function

$$G(\theta, y, a) := F(t+\theta, x+y)e^{-a}.$$

Define the stopping time

$$\tau_n := \inf\{s \geq 0 : |x + B_s| \geq n\}.$$

An application of Itô's formula gives

$$\begin{aligned}
& F(t + (r \wedge \tau_n), x + B_{r\wedge\tau_n}) e^{-\int_0^{r\wedge\tau_n} k(x+B_s)ds} \\
= \;& G(r \wedge \tau_n, B_{r\wedge\tau_n}, A_{r\wedge\tau_n}) \\
= \;& G(0, B_0, A_0) + \int_0^{r\wedge\tau_n} \left[\frac{\partial G}{\partial \theta} + \frac{\partial G}{\partial a} k(x + B_\theta) + \frac{1}{2} \frac{\partial G^2}{\partial x^2} \right] (\theta, B_\theta, A_\theta) d\theta \\
& + \int_0^{r\wedge\tau_n} \frac{\partial G}{\partial x} (\theta, B_\theta, A_\theta) dB_\theta \\
= \;& F(t, x) - \int_0^{r\wedge\tau_n} e^{-\int_0^\theta k(x+B_s)ds} g(t + \theta, x + B_\theta) d\theta \\
& + \int_0^{r\wedge\tau_n} e^{-\int_0^\theta k(x+B_s)ds} \frac{\partial F}{\partial x} (t + \theta, x + B_\theta) dB_\theta.
\end{aligned}$$

Taking the expectation and rearranging gives that

$$\begin{aligned}
F(t, x) = \;& \mathbb{E} F(t + (r \wedge \tau_n), x + B_{r\wedge\tau_n}) e^{-\int_0^{r\wedge\tau_n} k(B_s)ds} \\
& + \mathbb{E} \int_0^{r\wedge\tau_n} e^{-\int_0^\theta k(B_s)ds} g(t + \theta, x + B_\theta) d\theta.
\end{aligned}$$

Now we let $n \to \infty$ and $r \uparrow 1 - t$ so that

$$\begin{aligned}
F(t, x) = \;& \mathbb{E} F(1, x + B_{1-t}) e^{-\int_0^{1-t} k(B_s)ds} \\
& + \mathbb{E} \int_0^{1-t} e^{-\int_0^\theta k(B_s)ds} g(t + \theta, x + B_\theta) d\theta.
\end{aligned}$$

Finally, we observe that $F(1, x) = f(x)$. □

If we assume $d = 1$, $g = 0$, and $k = 0$ we get that

$$F(t, x) = \mathbb{E} f(x + B_{1-t})$$

solves the backwards heat equation

$$\frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} = 0 \quad \text{with} \quad F(1, x) = f(x).$$

Another variant of the above result is the famous Black-Scholes PDE in option pricing:

Proposition 9.2. *Let $S_t := e^{B_t - \frac{t}{2}}$ be the geometric Brownian motion, $f : (0,\infty) \to \mathbb{R}$ be a Borel function such that $\mathbb{E}f(S_1)^2 < \infty$, and*

$$G(t,y) := \mathbb{E}f(yS_{1-t}).$$

Then

$$\frac{\partial G}{\partial t}(t,y) + \frac{x^2}{2}\frac{\partial^2 G}{\partial y^2}(t,y) = 0 \quad \text{and} \quad \int_0^1 \mathbb{E}\left|\frac{\partial G}{\partial y}(t,S_t)\right|^2 S_t^2 dt < \infty,$$

$$G(t,S_t) = \mathbb{E}f(S_1) + \int_0^t \frac{\partial G}{\partial y}(u,S_u)dS_u \quad a.s.$$

for $t \in [0,1)$, and

$$f(S_1) = \mathbb{E}f(S_1) + \int_0^1 \frac{\partial G}{\partial y}(t,S_t)dS_t \quad a.s.$$

Before we prove the proposition, let us give an interpretation from option pricing in Stochastic Finance: The random variable S_t describes the share price at time t, f is a pay-off function, $G(t,y)$ is the option price at time t if the underlying share price equals y, the integrand $\frac{\partial G}{\partial y}$ describes the so-called δ-hedging strategy.

Proof of Proposition 9.2. (a) Let $h : \mathbb{R} \to \mathbb{R}$ be Borel-measurable and $\theta > 0$. Assume that

$$\int_{\mathbb{R}} e^{-\theta x^2} |h(x)|\, dx < \infty.$$

Then $\psi(s,x) := \mathbb{E}h(x + W_s)$ exists (and is finite) for $(s,x) \in \left(0, \frac{1}{2\theta}\right) \times \mathbb{R}$, has partial derivatives of all orders, and satisfies

$$\frac{\partial \psi}{\partial s} = \frac{1}{2}\frac{\partial^2 \psi}{\partial x^2} \quad \text{on} \quad \left(0, \frac{1}{2\theta}\right) \times \mathbb{R}.$$

(b) Let $1 < q, r < \infty$ be such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, $\theta := \frac{1}{2r}$, and $h(x) := f\left(e^{x-\frac{1}{2}}\right)$ for $x \in \mathbb{R}$. Assume that $f(S_1) = h(B_1) \in L_p$. Then

$$\begin{aligned}\int_{\mathbb{R}} e^{-\theta x^2} |h(x)| dx &= \int_{\mathbb{R}} \left(e^{-\frac{x^2}{2q}}\right)\left(e^{-\frac{x^2}{2p}} \left|f\left(e^{x-\frac{1}{2}}\right)\right|\right) dx \\ &\leq \left(\int_{\mathbb{R}} e^{-\frac{x^2 p'}{2q}} dx\right)^{\frac{1}{p'}} \left(\int_{\mathbb{R}} e^{-\frac{x^2}{2}} \left|f\left(e^{x-\frac{1}{2}}\right)\right|^p dx\right)^{\frac{1}{p}}\end{aligned}$$

$$= \left(\int_{\mathbb{R}} e^{-\frac{x^2 p'}{2q}} dx \right)^{\frac{1}{p'}} (2\pi)^{\frac{1}{2p}} \; \|f(S_1)\|_{L_p} < \infty$$

where $1 = 1/p + 1/p'$. Now one can apply (a) to obtain that $G \in C^\infty((-\varepsilon,1),(0,\infty))$ for some $\varepsilon > 0$ and satisfies the claimed PDE. It can be also shown that

$$\mathbb{E} \sup_{t\in[0,b]} \left| \frac{\partial G}{\partial x}(t,S_t)S_t \right|^2 < \infty \quad \text{and} \quad \int_0^1 \mathbb{E} \left| \frac{\partial G}{\partial y}(t,S_t)S_t \right|^2 dt < \infty$$

for all $b \in [0,1)$. From this and Itô's formula we may deduce that

$$\mathbb{E}(f(S_1)|\mathcal{F}_t) = G(t,S_t) = \mathbb{E}f(S_1) + \int_0^t \frac{\partial G}{\partial y}(t,S_t)dS_t \quad a.s.$$

By $t \uparrow 1$ the assertion follows because $\vee_{t\in[0,1)}\mathcal{F}_t = \mathcal{F}_1$ and a martingale convergence theorem. $\square$

10. Lecture

The Cox-Ingersoll-Ross SDE

Here we want to discuss the Cox-Ingersoll-Ross SDE

$$dX_t = (a - bX_t)dt + \sigma\sqrt{X_t}dB_t \quad \text{with} \quad X_0 \equiv x_0 > 0 \tag{5}$$

on $[0, \tau]$ where $a, \sigma > 0$, $b \in \mathbb{R}$, and

$$\tau(\omega) := \inf\{t \geq 0 : X_t(\omega) = 0\}.$$

Proposition 10.1. *There exists a unique continuous and adapted solution of the SDE (5).*

We are not in a position to prove this proposition. The uniqueness can be deduced by exploiting the uniqueness criteria of Yamada-Tanaka presented in Proposition 7.7 since

$$|\sqrt{|x|} - \sqrt{|y|}| \leq h(|x - y|)$$

with $h(x) := \sqrt{x}$ and

$$\int_0^\infty \frac{dx}{h(x)^2} = \int_0^\infty \frac{dx}{x} = \infty.$$

What we can do in more detail is to study the quantitative behavior of this equation in one case.

Proposition 10.2. *If $a \geq \frac{\sigma^2}{2}$, then $\mathbb{P}(\tau = \infty) = 1$.*

Proof. For $x, M > 0$ we let $(X_t^x)_{t\geq 0}$ be the solution of the Cox-Ingersoll-Ross SDE starting in $x > 0$ and

$$\tau_M^x(\omega) := \inf\{t \geq 0 : X_t^x(\omega) = M\}.$$

(a) Define

$$s(x) := \int_1^x e^{\frac{2by}{\sigma^2}} y^{-\frac{2a}{\sigma^2}} dy.$$

Then

$$\frac{\sigma^2}{2} x s''(x) + (a - bx)s'(x) = 0$$

by an easy computation.
(b) Let $0 < \varepsilon < x < M$ and $\tau^x_{\varepsilon,M} := \tau^x_\varepsilon \wedge \tau^x_M$. By Itô's formula

$$\begin{aligned}
s(X_{t\wedge\tau^x_{\varepsilon,M}}) &= s(x) + \int_0^{t\wedge\tau^x_{\varepsilon,M}} s'(X^x_s)dX_s + \frac{1}{2}\int_0^{t\wedge\tau^x_{\varepsilon,M}} s''(X^x_s)\sigma^2 X_s ds \\
&= s(x) + \int_0^{t\wedge\tau^x_{\varepsilon,M}} s'(X^x_s)\sigma\sqrt{X_s}dB_s \\
&\quad + \int_0^{t\wedge\tau^x_{\varepsilon,M}} \left[(a - bX^x_s)s'(X^x_s) + \frac{1}{2}s''(X^x_s)\sigma^2 X_s\right] ds \\
&= s(x) + \int_0^{t\wedge\tau^x_{\varepsilon,M}} s'(X^x_s)\sigma\sqrt{X_s}dB_s.
\end{aligned}$$

(c) Since $X_{t\wedge\tau^x_{\varepsilon,M}} \in [\varepsilon, M]$ for all $t \geq 0$ we have that

$$\begin{aligned}
\mathbb{E}\int_0^{t\wedge\tau^x_{\varepsilon,M}} s'(X^x_s)^2\sigma^2 X_s ds &= \mathbb{E}\left|s\left(X_{t\wedge\tau^x_{\varepsilon,M}}\right) - s(x)\right|^2 \\
&\leq \sup_{y\in[\varepsilon,M]} |s(y)|^2 \\
&=: c \\
&< \infty.
\end{aligned}$$

Letting $t \to \infty$ gives that

$$\mathbb{E}\int_0^{\tau^x_{\varepsilon,M}} s'(X^x_s)^2 X_s\sigma^2 ds < \infty.$$

Since $X_s \geq \varepsilon$ for $s \in [0, \tau^x_{\varepsilon,M}]$ by definition and since

$$s'(x) = e^{\frac{2bx}{\sigma^2}} x^{-\frac{2a}{\sigma^2}} \geq e^{-2\frac{|b|M}{\sigma^2}} M^{-2\frac{a}{\sigma^2}} =: d > 0$$

we get that

$$\mathbb{E}\int_0^{\tau^x_{\varepsilon,M}} ds < \infty$$

so that $\mathbb{E}\tau^x_{\varepsilon,M} < \infty$ and $\tau^x_{\varepsilon,M} < \infty$ a.s.
(d) Now

$$s(x) = \mathbb{E}\left(s(X^x_{\tau^x_{\varepsilon,M}\wedge t}) - \int_0^{\tau^x_{\varepsilon,M}\wedge t} s'(X^x_s)\sigma\sqrt{X^x_s}dB_s\right)$$

and the boundedness of the integrand of the stochastic integral on $[0, \tau^x_{\varepsilon,M} \wedge t]$ yields that

$$s(x) = \mathbb{E}s(X^x_{\tau^x_{\varepsilon,M} \wedge t}).$$

By $t \to \infty$, dominated convergence, and the fact that $\tau^x_{\varepsilon,M}$ is almost surely finite, we conclude that

$$s(x) = \mathbb{E}s(X^x_{\tau^x_{\varepsilon,M}}) = s(M)\mathbb{P}(\tau^x_M < \tau^x_\varepsilon) + s(\varepsilon)\mathbb{P}(\tau^x_M > \tau^x_\varepsilon).$$

(e) Now we can prove our assertion. First we observe that

$$\begin{aligned}
\lim_{\varepsilon \downarrow 0} s(\varepsilon) &= \lim_{\varepsilon \downarrow 0} \int_1^\varepsilon e^{\frac{2by}{\sigma^2}} y^{-\frac{2a}{\sigma^2}} dy \\
&= -\lim_{\varepsilon \downarrow 0} \int_\varepsilon^1 \left[e^{2b\frac{y}{\sigma^2}} \right] y^{-\theta} dy \\
&= -\infty
\end{aligned}$$

since $\theta = 2\frac{a}{\sigma^2} \geq 1$. Moreover, $\mathbb{P}(\tau^x_\varepsilon < \tau^x_M)$ decreases as $\varepsilon \downarrow 0$. Assume that

$$\lim_{\varepsilon \downarrow 0} \mathbb{P}(\tau^x_\varepsilon < \tau^x_M) = \delta > 0.$$

Then

$$s(x) \leq |s(M)| - \lim_{\varepsilon \downarrow 0} s(\varepsilon)\delta = -\infty$$

which is a contradiction. Hence

$$\lim_{\varepsilon \downarrow 0} \mathbb{P}(\tau^x_\varepsilon < \tau^x_M) = 0.$$

Since τ^x_ε is non-decreasing in ε we conclude that

$$\mathbb{P}(\tau^x_0 < \tau^x_M) = 0.$$

Letting $M \to \infty$ gives $\tau_M(\omega) \uparrow \infty$ so that

$$\mathbb{P}(\tau^x_0 < \infty) = \lim_{M \uparrow \infty} \mathbb{P}(\tau^x_0 < \tau^x_M) = 0.$$

Hence $\mathbb{P}(\tau^x_0 = \infty) = 1$. $\square$

References

[1] A.N. Borodin and P. Salminen, Handbook of Brownian motion – facts and formulae, Birkhäuser, 1996.

[2] K.L. Chung abd R.J. Williams, Introduction to Stochastic integration, 1983.

[3] A.S. Cherny and H.J. Engelbert, Singular Stochastic Differential Equations. LNM 1858, Springer.

[4] S. Geiss: Stochastic differential equations (lecture notes), www.maths.jyu.fi/∼ geiss/sripts.html.

[5] K. Itô and H.P. McKean, Diffusion processes and their sample paths, Springer, 1965.

[6] N. Ikeda and S. Watanabe, Stochastic Differential Equations and Diffusion Processes, North-Holland, 1989, second Edition.

[7] I. Karatzas and S. Shreve, Brownian Motion and Stochastic Calculus, 1988, Springer.

[8] P.E. Kloeden and E. Platen, Numerical Solution of Stochastic Differential Equations, Springer, 1992, 1995.

[9] D. Lamberton and B. Lapeyre, Stochastic Calculus Applied to Finance, Chapman & Hall, 1996.

[10] H.P. McKean, Stochastic Integrals, Academic Press, 1969.

[11] D. Nualart, The Malliavin Calculus and Related Topics, Springer, 1995.

[12] P. Protter, Stochastic integration and differential equations, Springer, second edition.

[13] D. Revuz and M. Yor, Continuous Martingales and Brownian Motion, Springer, 1994.

[14] L. C. G. Rogers and D. Williams, Diffusions, Markov Processes and Martingales, Foundations, Cambridge University Press, 2000.

[15] L. C. G. Rogers and D. Williams, Diffusions, Markov Processes and Martingales. Ito Calculus, Cambridge University Press, 2000.

1. Day: Gaussian processes and Brownian motion

(1) Let $W = (W_t)_{t\geq 0}$ be a Gaussian process such that

$$\mathbb{E}W_t = 0 \quad \text{and} \quad \Gamma(s,t) = \mathbb{E}W_sW_t = \min(s,t).$$

(a) Show that $\mathbb{E}(W_d - W_c)(W_b - W_a) = 0$ if $0 \leq a \leq b \leq c \leq d$. Are the random variables $W_d - W_c$ and $W_b - W_a$ independent?

(b) Define $Y_t := tW_{\frac{1}{t}}$ if $t > 0$ and $Y_0 := 0$. Show that $(Y_t)_{t\geq 0}$ is a Gaussian process and compute $\mathbb{E}Y_sY_t$. Do you recognize the process $(Y_t)_{t\geq 0}$?

(2) Let $W^H = (W_t^H)_{t\geq 0}$ be a fractional Brownian motion with Hurst index $H \in (0,1)$.

(a) Let $0 \leq s < t < \infty$. Show that

$$\mathbb{E}(W_t^H - W_s^H)W_s^H = \begin{cases} >0 & : \quad H > 1/2 \\ <0 & : \quad H < 1/2 \end{cases}.$$

This means that the increments of the process W^H correlate negatively if $H < 1/2$, and positively, if $H > 1/2$.

(b) Is there a continuous modification $Y = (Y_t)_{t\geq 0}$ of the process $W^H = (W_t^H)_{t\geq 0}$ such that all paths of Y are continuous?

(3) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ a filtration and $X = (X_t)_{\geq 0}$ a stochastic process $\mathbb{F}$-adapted whose paths are continuous for all $\omega \in \Omega$. Let $t_0 > 0$ and $A \subseteq \Omega$ the set of all $\omega \in \Omega$ such that

$$\sup_{|t-t_0|\leq \varepsilon(\omega)} X_t(\omega) \leq X_{t_0}(\omega)$$

for some $\varepsilon(\omega) > 0$. Is it true that $A \in \bigcap_{\varepsilon > 0} \mathcal{F}_{t_0+\varepsilon}$?

Homework for the 2nd day:

(1) Is there a modification $X = (X_t)_{t\in[0,1]}$ of the Brownian bridge $Y = (Y_t)_{t\geq 0}$ such that all the paths of X are continuous?

(2) Let $X = (X_t)_{t\in[0,1]}$ be a Brownian bridge. Define

$$Y_t := (t+1)X_{\frac{t}{t+1}}, \quad t \geq 0.$$

Do you recognize the process $Y = (Y_t)_{t\geq 0}$?

2. Day: Stopping times and stochastic integrals

(1) Let $X = (X_t)_{t\geq 0}$ be continuous and adapted, $\Gamma \subseteq \mathbb{R}$ be non-empty and closed, and define the hitting time

$$\tau_\Gamma := \inf\{t \geq 0 \,:\, X_t \in \Gamma\}$$

with the convention that $\inf \emptyset := \infty$. Show that τ_Γ is a stopping time.

(2) Let $S = (S_t)_{t\geq 0}$ be a geometric Brownian motion, defined by

$$S_t := e^{B_t - \frac{t}{2}},$$

where $B = (B_t)_{t\geq 0}$ is a standard Brownian motion. Show that S is a martingale.

(3) Suppose $\Omega = [0,1)$, $\mathcal{F} = \mathcal{B}(\mathbb{R})$, and $\mathbb{P} = \lambda$, and define

$$\mathcal{F}_n := \sigma\left(\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right) : k = 1, \ldots, 2^n\right)$$

Assume $f : [0,1) \to \mathbb{R}$ is continuous (and can be continuously extended to $[0,1]$).

(a) Show that g is $\mathcal{F}_n$-measurable if and only if g is constant on the intervals $\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)$.

(b) Define

$$M_n := \mathbb{E}(f \mid \mathcal{F}_n) = \sum_{k=1}^{2^n} \left(2^n \int_{\frac{k-1}{2^n}}^{\frac{k}{2^n}} f(u)du\right) \chi_{\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)}$$

Show that $(M_n)_{n=0}^\infty$ is a martingale wrt. the filtration $(\mathcal{F}_n)_{n=0}^\infty$.

Homework for the 3rd day:

(1) Suppose that $\tau, \sigma : \Omega \to [0, \infty]$ are two stopping times. Show that the minimum of the stopping times, $\tau \wedge \sigma$, is a stopping time.

(2) (Wiener integral) Suppose that $L : [0, \infty) \to \mathbb{R}$ is a bounded measurable function. Define the process $X = (X_t)_{t\geq 0}$ as follows

$$X_t := \int_0^t L_u dB_u$$

(a) Show that X is a Gaussian process, in the case L is a simple function.

(b) Compute the mean and covariance of the process X.

3. Day: Itô's formula

(1) Show that
$$\int_0^t B_u dB_u = \frac{1}{2}(B_t^2 - t) \quad \text{for} \quad t \geq 0 \, a.s.$$
where $B = (B_t)_{t\geq 0}$ is a standard Brownian motion.

(2) Let $\sigma, c > 0$ and $x \in \mathbb{R}$. Show that
$$X_t = xe^{-ct} + \sigma e^{-ct} \int_0^t e^{cs} dB_s$$
solves the SDE
$$dX_t = -cX_t dt + \sigma dB_t, \quad \text{where} \quad X_0 = x.$$
Compute the mean $\mathbb{E}X_t$ and variance $\mathbb{E}(X_t - \mathbb{E}X_t)^2$. Is the process $X = (X_t)_{t\geq 0}$ Gaussian?

Hint: You may use the formula
$$\int_0^t e^{-cu} \int_0^u e^{cs} dB_s du = \int_0^t \int_u^t e^{-cs} e^{cu} ds dB_u \, a.s.$$

(3) Let $S_t = e^{B_t - \frac{t}{2}}$. Find a SDE for the process $X_t := 1/S_t$.

(4) Let $b \in \mathbb{R}$. Solve the following SDE:
$$dX_t = X_t dB_t + bX_t dt \quad \text{missä} \quad X_0 = 1.$$
Hint: You can use the fact $X_t = e^{\alpha B_t - \beta t}$, where $\alpha, \beta \in \mathbb{R}$ and Itô's formula.

Homework for the 4th day:

(1) Let $L \in C[0, \infty)$ and $X_t := \int_0^t L_u dB_u$. Then
$$\mathcal{E}(X)_t := e^{X_t - \frac{1}{2}\int_0^t L_u^2 du}$$
is a martingale and called *exponential martingale*.

Hint: You can assume that $\int_0^t \mathbb{E}(\mathcal{E}(X)_t)^2 du < \infty$ for all $t \geq 0$.

4. Day: SDEs

(1) Let $x \in \mathbb{R}$ and

$$X_t := xt + (1-t)\int_0^t \frac{dB_s}{1-s},$$

where $t \in [0,1)$. Show that

(a) $\lim_{t\uparrow 1} \mathbb{E}X_t^2 = x^2$, and

(b) $dX_t = dB_t + \frac{x-X_t}{1-t}dt$, if $t \in [0,1)$.

Hint: You may use the identity $'dB_u ds = ds dB_u'$.

(2) Let $\sigma : (0,\infty) \to (0,\infty)$ and define $\widehat{\sigma} : \mathbb{R} \to (0,\infty)$ by

$$\widehat{\sigma}(x) := \frac{\sigma(e^x)}{e^x} \quad \text{and} \quad \widehat{b}(x) := -\frac{1}{2}\widehat{\sigma}^2(x).$$

Assume that $\widehat{\sigma}$ and $\widehat{\sigma}'$ are continuous and bounded. Prove that:

(a) The SDE

$$dX_t = \widehat{\sigma}(X_t)dB_t + \widehat{b}(X_t)dt \quad \text{with} \quad X_0 = x_0$$

has a unique solution.

(b) The process $Y = (Y_t)_{t\ge 0}$ with $Y_t := e^{X_t}$ satisfies the SDE

$$dY_t = \sigma(Y_t)dB_t \quad \text{with} \quad Y_0 = e^{x_0}.$$

(3) Prove that the SDE

$$dX_t = a(b - X_t)dt + \sigma\sqrt{|X_t|}dB_t \quad \text{with} \quad X_0 = x_0$$

and $\sigma > 0$ has at most one solution.

Hint: Use the criteria of Yamada and Tanaka (Proposition 7.7 of the lecture notes).

Homework for the 5th day:

(1) ORNSTEIN-UHLENBECK process: Prove that the SDE

$$dX_t = -cX_t dt + \sigma dB_t \quad \text{with} \quad X_0 = x_0$$

and $\sigma, c > 0$ has a unique solution. Prove by ITÔ's formula that the solution can be written as

$$X_t = x_0 e^{-ct} + \sigma \int_0^t e^{-c(t-s)} dB_s.$$

Hint: $f(t, x) := (x_0 + \sigma x)e^{-ct}$.

5. Day: SDEs

(1) Consider the differential equation

$$dX_t = 2X_t dB_t + 5X_t dt \quad \text{with} \quad x_0 = 1.$$

  (a) Find a measure Q_T equivalent to P such that the solution X is a martingale on $[0,T]$ with respect to Q_T.
  (b) Given an arbitrary solution X. Does X have almost surely positive trajectories?

(2) Solve the SDE

$$dS_t = S_t dW_t + (\mu S_t - tS_t)dt \quad \text{where} \quad S_0 = 1$$

where $t \in [0,T]$ and W is a Brownian motion.
Hint: Use the transformation of drift.

Check Your solution by Itô's formula.

(3) Let $0 < \varepsilon < r < c < \infty$ and let $W_t = (B_{t,1}, B_{t,2})$ be a two-dimensional standard Brownian motion starting at $(0,r)$. Let A be the event that the Brownian motion hits first the circle around zero with radius ε and B be the event that the Brownian motion hits first the circle around zero with radius c. Prove that

$$\ln r = \mathbb{P}(A) \ln \varepsilon + \mathbb{P}(B) \ln c.$$

Hint: Itô's formula.

(4) Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ be a stochastic basis and the process $(X_t)_{t\geq 0}$ adapted with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$. Suppose $X_t - X_s$ and $\mathcal{F}_s$ are independent for all $0 \leq s \leq t < \infty$ [1]. Show that

$$\mathbb{P}(X_{t+s} \in B | \mathcal{F}_s) = \Psi(X_s, B)\ a.s.$$

for an appropriate function Ψ.

Remark: From this one can deduce the Markov property

$$\mathbb{P}(X_{t+s} \in B | \mathcal{F}_s) = \mathbb{P}(X_{t+s} \in B | \sigma(X_s))\ a.s.$$

for all $s, t \geq 0$.

[1] $\mathbb{P}(\{X_t - X_s \in B\} \cap A) = \mathbb{P}(X_t - X_s \in B)\mathbb{P}(A)$ for all $B \in \mathcal{B}(\mathbb{R})$ and $A \in \mathcal{F}_s$.

Examination

General remarks

- Date for examination: 13/06/2007
- Time: 6h (exercises are for 4h)
- Send an e-mail to geiss at maths.jyu.fi with the name and e-mail of a person I can send the exercises to and which will 'supervise' You.
- The formulations from the script are the 'official' ones. Only quadratic integrable continuous martingales are used.
- In addition to the topics below the exercises are included.
- In case, the examination will not be passed, a retake is possible.

Topics for the examination

1. Gaussian processes and Brownian motion
   (a) Definition of a Gaussian process.
   (b) Criteria for the existence of a Gaussian process (Proposition 1.3).
   (c) Examples: Brownian motion, Brownian bridge, fractional Brownian motion (not the proof of the existence).
   (d) Kolmogorov's criteria for the existence of a continuous modification (Proposition 1.8) and its applications (Brownian motion has Hölder continuous paths with exponent $\alpha \in (0, 1/2)$, Brownian bridge, fractional Brownian motion).
   (e) Definition of an $(\mathcal{F}_t)_{t\geq 0}$ Brownian motion. Proof that the covariance structure implies independent increments.
   (f) Paths of the Brownian motion are not differentiable.
   (g) Reflection principle.
2. Stopping times
   (a) Hitting time of a closed set is a stopping time.

(b) The minimum and maximum of two stopping times is a stopping time.

3. Stochastic integrals

   (a) Definition and properties of the conditional expectation (Proposition 3.1.8).
   (b) Brownian motion and geometric Brownian motion are martingales.
   (c) Quadratic variation of a martingale and equivalent formulation (Proposition 4.3).
   (d) Quadratic variation of the Brownian motion and of the stochastic integral $\int_0^t L_u dB_u$.
   (e) Definition of the stochastic integral (simple and extension) and properties (Proposition 4.14: i.p. Itô's isometry).

4. Itô's formula

   (a) General formulation and formulation for Itô processes.
   (b) Applications: partial integration, compensator, Levy's characterization of the Brownian motion.

5. Stochastic differential equations

   (a) Criterion for uniqueness and existence (Propositions 7.6 and 7.10).
   (b) Criteria of Yamada and Tanaka and its application to the Cox-Ingersoll-Ross SDE.
   (c) Girsanov theorem (proposition 8.1) and transformation of drift (proposition 8.4).
   (d) Connection to PDEs.