Non-Life Insurance Mathematics

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Chapter 1

Introduction

Insurance Mathematics might be divided into

- *life insurance*,
- *health insurance*,
- *non-life insurance*.

Life insurance includes for instance life insurance contracts and pensions, where long terms are covered. Non-life insurance comprises insurances against fire, water damage, earthquake, industrial catastrophes or car insurance, for example. Non-life insurances cover in general a year or other fixed time periods. Health insurance is special because it is differently organized in each country.

The course material is based on the textbook *Non-Life Insurance Mathematics* by Thomas Mikosch [7].

1.1 The ruin of an insurance company

1.1.1 Solvency II Directive

In the following we concentrate ourselves on *non-life insurance*. There is a the *Solvency II Directive* of the *European Union*.

- Published: 2009
- Taken into effect: 01/01/2016
- Contents: Defines requirements for insurance companies.

One of these requirements is the amount of capital an insurer should hold, or in other words, the **Solvency Capital Requirement**:

- The probability that the assets of an insurer are greater or equal to its liabilities (in other words, to avoid the *ruin*) has to be larger or equal than 99,5 %.

In the lecture we will treat exactly this problem. Here we slightly simplify the problem by only looking at one particular insurance contract instead of looking at the overall company (this is a common approach in research as well). What are the key parameters for a non-life insurance contract for a certain class of claims?

1. How often does this event occur?
   - Significant weather catastrophes in Europe: 2 per year
   - Accidents in public transportation in Berlin in 2016: 141.155
2. Amount of loss or the typical claim size:
   - Hurricane *Niklas*, 2015, Europe: 750.000.000 €
   - Storm *Ela*, 2014, Europe, 650.000.000 €
   - Damage on a parking area: 500-1500 €

Opposite to the *Solvency II Directive* an insurance company needs to have reasonable low premiums and fees to attract customers. So there has to be a balance between the *Solvency Capital Requirement* and the *premiums.*

1.1.2 Idea of the mathematical model

We will consider the following situation:

(1) Insurance contracts (or **policies**) are sold. The resulting premium (yearly or monthly payments of the customers for the contract) form the **income** of the insurance company.

(2) At times T_i, $0 \leq T_1 \leq T_2 \leq \ldots$ claims happen. The times T_i are called the **claim arrival times.**

(3) The i-th claim arriving at time T_i causes the **claim size** X_i.

Mathematical problem: Find a stochastic model for the T_i's and X_i's to compute or estimate how much an insurance company should demand for its contracts and how much initial capital of the insurance company is required to keep the probability of ruin below a certain level.

1.2 Some facts about probability

We shortly recall some definitions and facts from probability theory which we need in this course. For more information see [12], or [2] and [3], for example.

(1) A **probability space** is a triple $(\Omega, \mathcal{F}, \mathbb{P})$, where

- Ω is a non-empty set,
- $\mathcal{F}$ is a σ-algebra consisting of subsets of Ω, and
- $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$.

(2) A function $f : \Omega \to \mathbb{R}$ is called a **random variable** if and only if for all intervals (a, b), $-\infty < a < b < \infty$ we have that

$$f^{-1}((a, b)) := \{\omega \in \Omega : a < f(\omega) < b\} \in \mathcal{F}.$$

(3) By $\mathcal{B}(\mathbb{R})$ we denote the Borel σ-algebra. It is the smallest σ-algebra on $\mathbb{R}$ which contains all open intervals. The σ-algebra $\mathcal{B}(\mathbb{R}^n)$ is the Borel σ-algebra, which is the smallest σ-algebra containing all the open rectangles $(a_1, b_1) \times ... \times (a_n, b_n)$.

(4) The random variables $f_1, ..., f_n$ are **independent** if and only if

$$\mathbb{P}(f_1 \in B_1, ..., f_n \in B_n) = \mathbb{P}(f_1 \in B_1) \cdots \mathbb{P}(f_n \in B_n)$$

for all $B_k \in \mathcal{B}(\mathbb{R}), k = 1, ..., n$. If the f_i's have discrete values, i.e. $f_i : \Omega \to \{x_1, x_2, x_3, \ldots\}$, then the random variables $f_1, ..., f_n$ are independent if and only if

$$\mathbb{P}(f_1 = k_1, ..., f_n = k_n) = \mathbb{P}(f_1 = k_1) \cdots \mathbb{P}(f_n = k_n)$$

for all $k_i \in \{x_1, x_2, x_3 \ldots\}$.

(5) If $f_1, \ldots, f_n$ are independent random variables such that f_i has the density function $h_i(x)$, i.e. $\mathbb{P}(f_i \in (a, b)) = \int_a^b h_i(x)dx$, then

$$\mathbb{P}((f_1, ..., f_n) \in B) = \int_{\mathbb{R}^n} \mathbb{1}_B(x_1, ..., x_n) h_1(x) \cdots h_n(x_n) dx_1 \cdots dx_n$$

for all $B \in \mathcal{B}(\mathbb{R}^n)$..

(6) The function $\mathbb{1}_B(x)$ is the **indicator function** for the set B, which is defined as

$$\mathbb{1}_B(x) = \begin{cases} 1 \text{ if } x \in B \\ 0 \text{ if } x \notin B. \end{cases}$$

(7) A random variable $f : \Omega \to \{0, 1, 2, ...\}$ is **Poisson distributed with parameter** $\lambda > 0$ if and only if

$$\mathbb{P}(f = k) = e^{-\lambda} \frac{\lambda^k}{k!}.$$

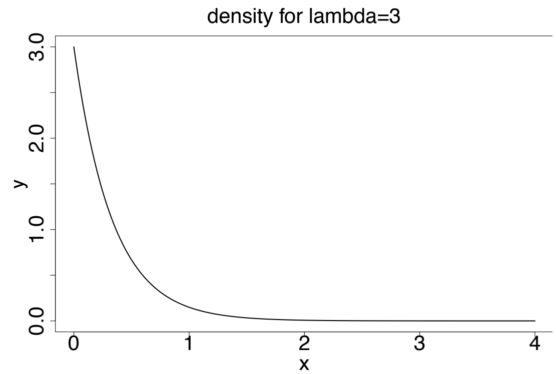
This is often written as $f \sim Pois(\lambda)$.

(8) A random variable $g : \Omega \to [0,\infty)$ is **exponentially distributed with parameter** $\lambda > 0$ if and only if for all $a < b$

$$\mathbb{P}(g \in (a,b)) = \lambda \int_a^b \mathbb{1}_{[0,\infty)}(x) e^{-\lambda x} dx.$$

This is often written as $f \sim Exp(\lambda)$.

The picture below shows the density $\lambda \mathbb{1}_{[0,\infty)}(x) e^{-\lambda x}$ for $\lambda = 3$.



(9) How to characterize distributions?

We briefly recall how to describe the distribution of a random variable. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

(a) The distribution of a random variable $f : \Omega \to \mathbb{R}$ can be uniquely described by its distribution function $F : \mathbb{R} \to [0,1]$,

$$F(x) := \mathbb{P}(\{\omega \in \Omega : f(\omega) \leq x\}), \ x \in \mathbb{R}.$$

(b) Especially, it holds for $g : \mathbb{R} \to \mathbb{R}$, such that $g^{-1}(B) \in \mathcal{B}(\mathbb{R})$ for all $B \in \mathcal{B}(\mathbb{R})$, that

$$\mathbb{E}g(f) = \int_{\mathbb{R}} g(x) dF(x)$$

in the sense that, if either side of this expression exists, so does the other, and then they are equal, see [8], pp. 168-169.

(c) The distribution of f can also be determined by its **characteristic function** (see [12])

$$\varphi_f(u) := \mathbb{E} e^{iuf}, \ u \in \mathbb{R},$$

or by its **moment-generating function**

$$m_f(h) := \mathbb{E}e^{hf},\ h \in (-h_0, h_0)$$

provided that $\mathbb{E}e^{h_0 f} < \infty$ for some $h_0 > 0$. We also recall that for independent random variables f and g it holds that

$$\varphi_{f+g}(u) = \varphi_f(u)\varphi_g(u).$$

Chapter 2

The Models for the claim number process $N(t)$

In the following we will introduce three processes which are used as claim number processes: the Poisson process, the renewal process and the inhomogeneous Poisson process.

2.1 The homogeneous Poisson process with parameter $\lambda > 0$

Definition 2.1.1 (homogeneous Poisson process)**.** Assume a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A stochastic process $N = (N(t))_{t\in[0,\infty)}$ is a map

$$N : [0, \infty) \times \Omega \to \mathbb{R}$$

such that for each fixed $t \in [0, \infty)$ the map $N(t, \cdot) : \Omega \to \mathbb{R}$ is a random variable. The process $N = (N(t))_{t\in[0,\infty)}$ is a **Poisson process** with intensity $\lambda > 0$ if $N(t, \omega) \in \{0, 1, 2, \ldots\}$, for each fixed ω the function $t \mapsto N(t, \omega)$ is non-decreasing and if the following conditions are fulfilled:

(P1) $N(0) = 0$ a.s. (almost surely), i.e. $\mathbb{P}(\{\omega \in \Omega : N(0, \omega) = 0\}) = 1$.

(P2) The process N has independent increments, i.e. for all $n \geq 1$ and $0 = t_0 < t_1 < ... < t_n < \infty$ the random variables $N(t_n) - N(t_{n-1}), N(t_{n-1}) - N(t_{n-2}), ..., N(t_1) - N(t_0)$ are independent.

(P3) For any $s \geq 0$ and $t > 0$ the random variable $N(t + s) - N(s)$ is Poisson distributed, i.e.

$$\mathbb{P}(N(t + s) - N(s) = m) = e^{-\lambda t} \frac{(\lambda t)^m}{m!}, \ m = 0, 1, 2, ...$$

(P4) The **paths** of N, i.e. the functions $(N(t,\omega))_{t\in[0,\infty)}$ for **fixed** ω, are almost surely right continuous and have left limits. One says N has **càdlàg** (**c**ontinue **à** **d**roite, **l**imite **à** **g**auche) paths.

Remark 2.1.2. One can show that the definition implies the following: there is a set $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ such that for all $\omega \in \Omega_0$ one has that

(1) $N(0,\omega) = 1$,

(2) the paths $t \to N(t,\omega)$ are càdlàg, take values in $\{0,1,2,\ldots\}$, are non-decreasing,

(3) all jumps have the size 1.

In the following we prove that the Poisson process does exist. To do so we need some preparations:

Lemma 2.1.3. *Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and let $W_1, W_2, \ldots : \Omega \to (0,\infty)$ be independent and exponentially distributed random variables with parameter $\lambda > 0$. Then, for any $t > 0$ we have*

$$\mathbb{P}(W_1 + \cdots + W_n \leq t) = 1 - e^{-\lambda t} \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!}.$$

Consequently, the law of $W_1 + \cdots + W_n$ has the density

$$p_{n,\frac{1}{\lambda}}(t) = \mathbb{1}_{(0,\infty)}(t) e^{-t\lambda} \frac{\lambda^n t^{n-1}}{(n-1)!},$$

i.e. the sum of n independent exponentially distributed random variables with parameter λ is a Gamma distributed with shape n and scale $1/\lambda$.

Proof. The first part is subject to an exercise, we only show the formula for the density. The density is obtained by differentiating the distribution function on $(0,\infty)$. Using the product rule, we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[1 - e^{-\lambda t}\sum_{k=0}^{n-1}\frac{(\lambda t)^k}{k!}\right] &= \lambda e^{-\lambda t}\left[\sum_{k=0}^{n-1}\frac{(\lambda t)^k}{k!}\right] + \left[-e^{-\lambda t}\sum_{k=1}^{n-1} k\frac{\lambda^k t^{k-1}}{k!}\right] \\
&= \lambda e^{-\lambda t}\left[\sum_{k=0}^{n-1}\frac{(\lambda t)^k}{k!}\right] + \left[-\lambda e^{-\lambda t}\sum_{k=0}^{n-2}\frac{(\lambda t)^k}{k!}\right] \\
&= \lambda e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!}.
\end{aligned}$$

□

Definition 2.1.4. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $W_1, W_2, ... : \Omega \to (0, \infty)$ be random variables that are independent and exponentially distributed with parameter $\lambda > 0$. Define $T_0(\omega) \equiv 0$ and, for $n \geq 1$, $t \geq 0$, and $\omega \in \Omega$,

$$\begin{aligned} T_n(\omega) &:= W_1(\omega) + \cdots + W_n(\omega), \\ \hat{N}(t,\omega) &:= \#\{i \geq 1 : T_i(\omega) \leq t\},\ t \geq 0. \end{aligned}$$

We say that

- $W_1, W_2, \ldots$ are the **waiting times**,
- $T_1, T_2, \ldots$ are the **arrival** times.

What is the idea behind using exponentially distributed waiting times $W_1, W_2, \ldots$? Assume that $W : \Omega \to \mathbb{R}$ is exponentially distributed with parameter $\lambda > 0$ and that $s, t > 0$. Then one has

$$\mathbb{P}(W \geq s + t | W \geq s) = \mathbb{P}(W \geq t).$$

In other words, the distribution does not have a memory. The distribution of the counting process $(\hat{N}(t, \cdot))_{t \geq 0}$ is given by the next statement, which also explains the name *Poisson process*:

Lemma 2.1.5. *For each $n = 0, 1, 2, ...$ and for all $t > 0$ it holds*

$$\mathbb{P}(\{\omega \in \Omega : \hat{N}(t, \omega) = n\}) = e^{-\lambda t} \frac{(\lambda t)^n}{n!},$$

i.e. $\hat{N}(t)$ is Poisson distributed with parameter λt.

Proof. From the definition of $\hat{N}$ it can be concluded that

$$\begin{aligned} \{\omega \in \Omega : \hat{N}(t, \omega) = n\} &= \{\omega \in \Omega : T_n(\omega) \leq t < T_{n+1}(\omega)\} \\ &= \{\omega \in \Omega : T_n(\omega) \leq t\} \setminus \{\omega \in \Omega : T_{n+1}(\omega) \leq t\}. \end{aligned}$$

Because of $T_n \leq T_{n+1}$ we have the inclusion $\{T_{n+1} \leq t\} \subseteq \{T_n \leq t\}$. This implies

$$\begin{aligned} \mathbb{P}(\hat{N}(t) = n) &= \mathbb{P}(T_n \leq t) - \mathbb{P}(T_{n+1} \leq t) \\ &= 1 - e^{-\lambda t} \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} - 1 + e^{-\lambda t} \sum_{k=0}^{n} \frac{(\lambda t)^k}{k!} \\ &= e^{-\lambda t} \frac{(\lambda t)^n}{n!}. \end{aligned}$$

□

Now we can prove the existence of the Poisson process which is one of the important processes in stochastic process theory:

Theorem 2.1.6 (Existence of the Poisson process)**.**

(1) $\hat{N}(t)_{t\in[0,\infty)}$ *is a Poisson process with parameter* $\lambda > 0$.

(2) *Any Poisson process* $N(t)$ *with parameter* $\lambda > 0$ *can be written as*

$$N(t) = \#\{i \geq 1, T_i \leq t\},\ t \geq 0,$$

where $T_n = W_1 + ... + W_n,\ n \geq 1,$ *and* $W_1, W_2, ...$ *are independent and exponentially distributed with* $\lambda > 0$.

Proof. (1) We check the properties of Definition 2.1.1.

(P1) According to our construction we have that $W_1(\omega) > 0$ for all $\omega \in \Omega$. Because $\hat{N}(0,\omega) = 0$ if and only if $0 < T_1(\omega) = W_1(\omega)$, we deduce $\hat{N}(0,\omega) = 0$ for all $\omega \in \Omega$.

(P2) We only show that $\hat{N}(s)$ and $\hat{N}(t) - \hat{N}(s)$ are independent, i.e.

$$\mathbb{P}(\hat{N}(s) = l, \hat{N}(t) - \hat{N}(s) = m) = \mathbb{P}(\hat{N}(s) = l)\mathbb{P}(\hat{N}(t) - \hat{N}(s) = m) \tag{1}$$

for $l, m \geq 0$. The independence of arbitrary many increments can be shown similarly. It holds for $l \geq 0$ and $m \geq 1$ that

$$\begin{aligned}\mathbb{P}(\hat{N}(s) = l,\ \hat{N}(t) - \hat{N}(s) = m) &= \mathbb{P}(\hat{N}(s) = l,\ \hat{N}(t) = m + l)\\ &= \mathbb{P}(T_l \leq s < T_{l+1},\ T_{l+m} \leq t < T_{l+m+1}).\end{aligned}$$

By defining functions f_1, f_2, f_3 and f_4 as

$$\begin{aligned}f_1 &:= T_l\\ f_2 &:= W_{l+1}\\ f_3 &:= W_{l+2} + ... + W_{l+m}\\ f_4 &:= W_{l+m+1},\end{aligned}$$

and $h_1, ..., h_4$ as the corresponding densities, it follows that

$$\begin{aligned}&\mathbb{P}(T_l \leq s < T_{l+1},\ T_{l+m} \leq t < T_{l+m+1})\\ =\ &\mathbb{P}(f_1 \leq s < f_1 + f_2,\ f_1 + f_2 + f_3 \leq t < f_1 + f_2 + f_3 + f_4)\\ =\ &\mathbb{P}(0 \leq f_1 < s, s - f_1 < f_2 < \infty, 0 \leq f_3 < t - f_1 - f_2,\\ &\qquad\qquad t - (f_1 + f_2 + f_3) < f_4 < \infty)\\ =\ &\underbrace{\int_0^s \underbrace{\int_{s-x_1}^{\infty} \underbrace{\int_0^{t-x_1-x_2} \underbrace{\int_{t-x_1-x_2-x_3}^{\infty} h_4(x_4)dx_4}_{I_4(x_1,x_2,x_3)}\, h_3(x_3)dx_3}_{I_3(x_1,x_2)}\, h_2(x_2)dx_2}_{I_2(x_1)}\, h_1(x_1)dx_1\\ =:\ &I_1\end{aligned}$$

By direct computation and rewriting the density function of $f_4 = W_{l+m+1}$,

$$I_4(x_1, x_2, x_3) = \int_{t-x_1-x_2-x_3}^{\infty} \lambda e^{-\lambda x_4} \mathbb{1}_{[0,\infty)}(x_4) dx_4 = e^{-\lambda(t-x_1-x_2-x_3)}.$$

Here we used $t - x_1 - x_2 - x_3 \geq 0$. This is true because the integration w.r.t. x_3 implies $0 \leq x_3 \leq t - x_1 - x_2$. The density of $f_3 = W_{l+2} + ... + W_{l+m}$ is

$$h_3(x_3) = \lambda^{m-1} \frac{x_3^{m-2}}{(m-2)!} \mathbb{1}_{[0,\infty)}(x_3) e^{-\lambda x_3}$$

according to Lemma 2.1.3. Therefore,

$$\begin{aligned} I_3(x_1, x_2) &= \int_0^{t-x_1-x_2} \lambda^{m-1} \frac{x_3^{m-2}}{(m-2)!} e^{-\lambda x_3} e^{-\lambda(t-x_1-x_2-x_3)} dx_3 \\ &= \mathbb{1}_{[0,t-x_1)}(x_2) e^{-\lambda(t-x_1-x_2)} \lambda^{m-1} \frac{(t-x_1-x_2)^{m-1}}{(m-1)!}. \end{aligned}$$

The density of $f_2 = W_{l+1}$ is

$$h_2(x_2) = \mathbb{1}_{[0,\infty)}(x_2) \lambda e^{-\lambda x_2}.$$

This implies

$$\begin{aligned} I_2(x_1) &= \int_{s-x_1}^{\infty} \mathbb{1}_{[0,t-x_1)}(x_2) e^{-\lambda(t-x_1-x_2)} \lambda^{m-1} \frac{(t-x_1-x_2)^{m-1}}{(m-1)!} \lambda e^{-\lambda x_2} dx_2 \\ &= \lambda^m e^{-\lambda(t-x_1)} \frac{(t-s)^m}{m!}. \end{aligned}$$

Finally, from Lemma 2.1.5 we conclude

$$\begin{aligned} I_1 &= \int_0^s \lambda^m e^{-\lambda(t-x_1)} \frac{(t-s)^m}{m!} \lambda^l \frac{x_1^{l-1}}{(l-1)!} \mathbb{1}_{[0,\infty)}(x_1) e^{-\lambda x_1} dx_1 \\ &= \lambda^m \lambda^l e^{-\lambda t} \frac{(t-s)^m}{m!} \frac{s^l}{l!} \\ &= \left(\frac{(\lambda s)^l}{l!} e^{-\lambda s} \right) \left(\frac{(\lambda(t-s))^m}{m!} e^{-\lambda(t-s)} \right) \\ &= \mathbb{P}(\hat{N}(s) = l) \mathbb{P}(\hat{N}(t-s) = m). \end{aligned}$$

If we sum

$$\mathbb{P}(\hat{N}(s) = l,\ \hat{N}(t) - \hat{N}(s) = m) = \mathbb{P}(\hat{N}(s) = l) \mathbb{P}(\hat{N}(t-s) = m)$$

over $l \in \mathbb{N}$ we get

$$\mathbb{P}(\hat{N}(t) - \hat{N}(s) = m) = \mathbb{P}(\hat{N}(t-s) = m) \tag{2}$$

and hence (1) for $l \geq 0$ and $m \geq 1$. The case $m = 0$ can deduced from that above by exploiting

$$\mathbb{P}(\hat{N}(s) = l,\ \hat{N}(t) - \hat{N}(s) = 0) \\ = \mathbb{P}(\hat{N}(s) = l) - \sum_{m=1}^{\infty} \mathbb{P}(\hat{N}(s) = l,\ \hat{N}(t) - \hat{N}(s) = m)$$

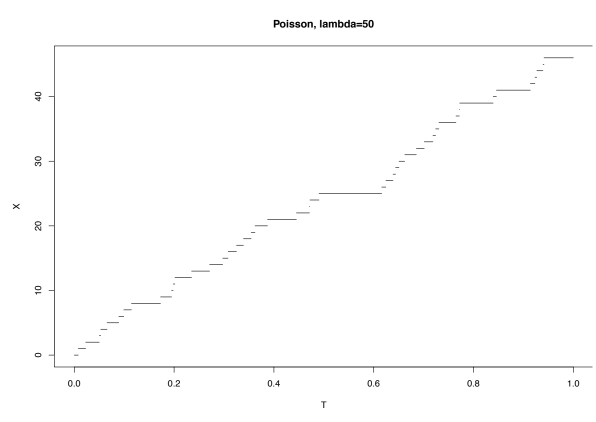
and

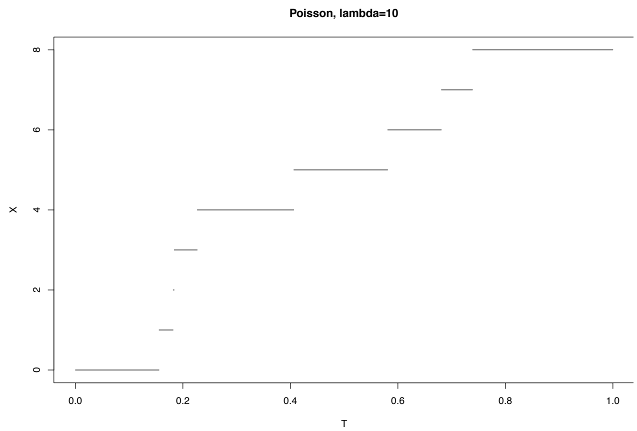
$$\mathbb{P}(\hat{N}(t-s) = 0) = 1 - \sum_{m=1}^{\infty} \mathbb{P}(\hat{N}(t-s) = m).$$

(P3) follows from Lemma 2.1.5 and (2).

(P4) follows from the construction.

(2) This part is subject to an exercise. □





2.2 A generalization of the Poisson process: the renewal process

To model windstorm claims, for example, it is not appropriate to use the Poisson process because windstorm claims happen rarely, sometimes with years in between. The Pareto distribution, for example, which has the distribution function

$$F(x) = 1 - \left(\frac{\kappa}{\kappa + x}\right)^{\alpha} \quad \text{for} \quad x \geq 0$$

with parameters $\alpha, \kappa > 0$ will fit better when we use this as distribution for the waitig times, i.e.

$$\mathbb{P}(W_i \geq x) = \left(\frac{\kappa}{\kappa + x}\right)^{\alpha} \quad \text{for} \quad x \geq 0.$$

For a Pareto distributed random variable it is more likely to have large values than for an exponential distributed random variable.

Definition 2.2.1 (Renewal process)**.** Assume a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and independent and identically distributed random variables $W_1, W_2, ... : \Omega \to (0, \infty)$. Then

$$\begin{cases} T_0 & := & 0 \\ T_n & := & W_1 + ... + W_n, \; n \geq 1, \end{cases}$$

is a **renewal sequence** and $N : [0, \infty) \times \Omega \to \{0, 1, \ldots\}$ with

$$N(t) := \#\{i \geq 1 : T_i \leq t\}, \; t \geq 0,$$

is the corresponding **renewal process**.

By Theorem 2.1.6 we know that a renewal process with $W_1, W_2, \ldots \sim Exp(\lambda)$ is a Poisson process with intensity $\lambda > 0$.
In order to study the limit behavior of N we need the **Strong Law of Large Numbers** (SLLN):

Theorem 2.2.2 (SLLN)**.** *If the random variables $X_1, X_2, ...$ are i.i.d. with $\mathbb{E}|X_1| < \infty$, then*

$$\lim_{n\to\infty} \frac{X_1 + X_2 + ... + X_n}{n} = \mathbb{E}X_1 \ a.s.$$

Theorem 2.2.3 (SLLN for renewal processes)**.** *Assume $N(t)$ is a renewal process. If $\mathbb{E}W_1 < \infty$, then*

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mathbb{E}W_1} \ a.s.$$

Proof. Because of

$$\{\omega \in \Omega : N(t,\omega) = n\} = \{\omega \in \Omega : T_n(\omega) \le t < T_{n+1}(\omega)\} \quad \text{for} \quad n \in \mathbb{N}$$

we have for $N(t)(\omega) > 0$ that

$$\frac{T_{N(t,\omega)}(\omega)}{N(t,\omega)} \le \frac{t}{N(t,\omega)} < \frac{T_{N(t,\omega)+1}(\omega)}{N(t,\omega)} = \frac{T_{N(t,\omega)+1}(\omega)}{N(t,\omega)+1} \frac{N(t,\omega)+1}{N(t,\omega)}. \tag{3}$$

Note that

$$\Omega = \{\omega \in \Omega : T_1(\omega) < \infty\} = \{\omega \in \Omega : \sup_{t\ge 0} N(t) > 0\}.$$

Theorem 2.2.2 implies that

$$\frac{T_n}{n} \to \mathbb{E}W_1 \tag{4}$$

holds on a set Ω_0 with $\mathbb{P}(\Omega_0) = 1$. Hence $\lim_{n\to\infty} T_n \to \infty$ on Ω_0 and by definition of N also $\lim_{t\to\infty} N(t) \to \infty$ on Ω_0. From (4) we get

$$\lim_{\substack{t\to\infty \\ N(t,\omega)>0}} \frac{T_{N(t,\omega)}}{N(t,\omega)} = \mathbb{E}W_1 \quad \text{for } \omega \in \Omega_0.$$

Finally (3) implies that

$$\lim_{\substack{t\to\infty \\ N(t,\omega)>0}} \frac{t}{N(t,\omega)} = \mathbb{E}W_1 \quad \text{for } \omega \in \Omega_0.$$

□

In the following we will investigate the behavior of $\mathbb{E}N(t)$ as $t \to \infty$.

Theorem 2.2.4 (Elementary renewal theorem)**.** *Assume that* $(N(t))_{t\geq 0}$ *is a renewal process and that* $0 < \mathbb{E}W_1 < \infty$. *Then*

$$\lim_{t\to\infty} \frac{\mathbb{E}N(t)}{t} = \frac{1}{\mathbb{E}W_1}. \tag{5}$$

Remark 2.2.5. If the W_i's are exponentially distributed with parameter $\lambda > 0$, $W_i \sim Exp(\lambda),\ i \geq 1$, then $N(t)$ is a Poisson process and

$$\mathbb{E}N(t) = \lambda t.$$

Since $\mathbb{E}W_i = \frac{1}{\lambda}$, it follows that for all $t > 0$ that

$$\frac{\mathbb{E}N(t)}{t} = \frac{1}{\mathbb{E}W_1}. \tag{6}$$

If the W_i's are not exponentially distributed, then the equation (6) holds only for the limit $t \to \infty$.

In order to prove Theorem 2.2.4 we formulate the following Lemma of Fatou type:

Lemma 2.2.6. *Let* $Z = (Z_t)_{t\in[0,\infty)}$ *be a stochastic process such that*

$$Z_t : \Omega \to [0,\infty) \quad \text{for all } t \geq 0$$

and $\inf_{s\geq t} Z_s : \Omega \to [0,\infty)$ *is measurable for all* $t \geq 0$. *Then*

$$\mathbb{E} \liminf_{t\to\infty} Z_t \leq \liminf_{t\to\infty} \mathbb{E}Z_t.$$

Proof. By monotone convergence, since $t \mapsto \inf_{s\geq t} Z_s$ is non-decreasing, we have

$$\mathbb{E} \lim_{t\to\infty} \inf_{s\geq t} Z_s = \lim_{t\to\infty} \mathbb{E} \inf_{s\geq t} Z_s.$$

Obviously, $\mathbb{E}\inf_{s\geq t} Z_s \leq \mathbb{E}Z_u$ for all $u \geq t$ which allows us to write

$$\mathbb{E} \inf_{s\geq t} Z_s \leq \inf_{u\geq t} \mathbb{E}Z_u.$$

This implies the assertion. □

Proof of Theorem 2.2.4. Let $\lambda = \frac{1}{\mathbb{E}W_1}$. From Theorem 2.2.3 we conclude

$$\lambda = \lim_{t\to\infty} \frac{N(t)}{t} = \lim_{t\to\infty} \inf_{s\geq t} \frac{N(s)}{s} \quad a.s.$$

Since $Z_t := \frac{N(t)}{t}$ for $t > 0$ and $Z_0 := 0$ fulfills the requirements of Lemma 2.2.6 we have

$$\lambda = \mathbb{E} \lim_{t\to\infty} \inf_{s\geq t} \frac{N(s)}{s} \leq \liminf_{t\to\infty} \mathbb{E}\frac{N(t)}{t}.$$

So, we only have to verify that $\limsup_{t\to\infty} \mathbb{E}\frac{N(t)}{t} \le \lambda$. Let $t > 0$. Recall that $N(t) = \#\{i \ge 1 : T_i \le t\}$. We introduce the filtration $(\mathcal{F}_n)_{n\ge 0}$ given by

$$\mathcal{F}_n := \sigma(W_1, ..., W_n), \quad n \ge 1, \quad \mathcal{F}_0 := \{\emptyset, \Omega\}.$$

Then the random variable $\tau_t := N(t) + 1$ is a stopping time w.r.t. $(\mathcal{F}_n)_{n\ge 0}$ i.e. it holds

$$\{\tau_t = n\} \in \mathcal{F}_n, \quad n \ge 0.$$

Let us verify this.

$\underline{n=0}$ yields to $\{\tau_t = 0\} = \{N(t) = -1\} = \emptyset \in \mathcal{F}_0$,
$\underline{n=1}$ yields to $\{\tau_t = 1\} = \{N(t) = 0\} = \{t < W_1\} \in \mathcal{F}_1$, and for
$\underline{n \ge 2}$ we have

$$\{\tau_t = n\} = \{T_{n-1} \le t < T_n\} = \{W_1 + ... + W_{n-1} \le t < W_1 + ... + W_n\} \in \mathcal{F}_n.$$

By definition of $N(t)$ we have that $T_{N(t)} \le t$. Hence we get

$$\mathbb{E}T_{N(t)+1} = \mathbb{E}(T_{N(t)} + W_{N(t)+1}) \le t + \mathbb{E}W_1 < \infty. \tag{7}$$

On the other hand it holds

$$\mathbb{E}T_{N(t)+1} = \mathbb{E}\sum_{i=1}^{\tau_t} W_i = \lim_{K\to\infty} \mathbb{E}\sum_{i=1}^{\tau_t\wedge K} W_i$$

by monotone convergence. Since $\mathbb{E}\tau_t \wedge K < \infty$ and the $W_i's$ are i.i.d with $\mathbb{E}W_1 < \infty$ we may apply Wald's identity

$$\mathbb{E}\sum_{i=1}^{\tau_t\wedge K} W_i = \mathbb{E}(\tau_t \wedge K)\mathbb{E}W_1.$$

This implies

$$\infty > \mathbb{E}T_{N(t)+1} = \lim_{K\to\infty} \mathbb{E}(\tau_t \wedge K)\mathbb{E}W_1 = \mathbb{E}\tau_t \, \mathbb{E}W_1.$$

This relation is used in the following computation to substitute $\mathbb{E}\tau_t = \mathbb{E}N(t)+1$:

$$\begin{aligned}
\limsup_{t\to\infty} \frac{\mathbb{E}N(t)}{t} &= \limsup_{t\to\infty} \frac{\mathbb{E}N(t)+1}{t} \\
&= \limsup_{t\to\infty} \frac{\mathbb{E}\tau_t}{t} \\
&= \limsup_{t\to\infty} \frac{\mathbb{E}T_{N(t)+1}}{t\,\mathbb{E}W_1} \\
&\le \limsup_{t\to\infty} \frac{t+\mathbb{E}W_1}{t\,\mathbb{E}W_1} = \frac{1}{\mathbb{E}W_1},
\end{aligned}$$

where (7) was used for the last estimate. □

2.3 The inhomogeneous Poisson process and the mixed Poisson process

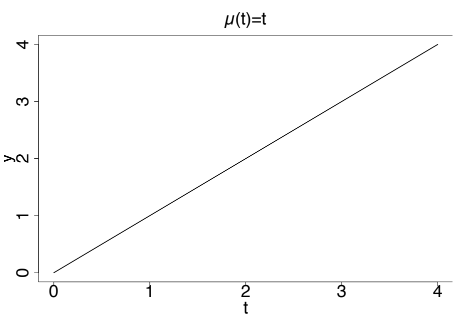
Definition 2.3.1. Let $\mu : [0,\infty) \to [0,\infty)$ be a function such that

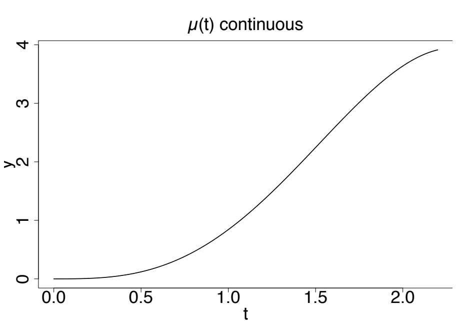
(1) $\mu(0) = 0$

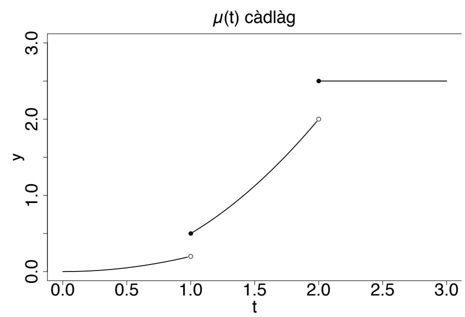
(2) μ is non-decreasing, i.e. $0 \le s \le t \Rightarrow \mu(s) \le \mu(t)$

(3) μ is càdlàg.

Then the function μ is called a **mean-value function**.







Definition 2.3.2 (Inhomogeneous Poisson process)**.** A stochastic process $N = N(t)_{t\in[0,\infty)}$ is an **inhomogeneous Poisson process** if and only if it has the following properties:

(P1) $N(0) = 0$ a.s.

(P2) N has independent increments, i.e. if $0 = t_0 < t_1 < ... < t_n$, $(n \geq 1)$, it holds that $N(t_n) - N(t_{n-1}), N(t_{n-1}) - N(t_{n-2}), ..., N(t_1) - N(t_0)$ are independent.

$(P_{inh.}3)$ There exists a mean-value function μ such that for $0 \leq s < t$

$$\mathbb{P}(N(t) - N(s) = m) = e^{-(\mu(t)-\mu(s))}\frac{(\mu(t) - \mu(s))^m}{m!},$$

where $m = 0, 1, 2, ...,$ and $t > 0$.

(P4) The paths of N are càdlàg a.s.

Theorem 2.3.3 (Time change for the Poisson process)**.** *If μ denotes the mean-value function of an inhomogeneous Poisson process N and $\tilde{N}$ is a homogeneous Poisson process with $\lambda = 1$, then*

(1)
$$(N(t))_{t\in[0,\infty)} \stackrel{d}{=} (\tilde{N}(\mu(t)))_{t\in[0,\infty)}$$

(2) *If μ is continuous, increasing and* $\lim_{t\to\infty}\mu(t) = \infty$*, then*

$$N(\mu^{-1}(t))_{t\in[0,\infty)} \stackrel{d}{=} (\tilde{N}(t))_{t\in[0,\infty)}.$$

Here $\mu^{-1}(t)$ denotes the inverse function of μ and $f \stackrel{d}{=} g$ means that the two random variables f and g have the same distribution (but one can *not* conclude that $f(\omega) = g(\omega)$ for $\omega \in \Omega$).

Definition 2.3.4 (Mixed Poisson process)**.** Let $\hat{N}$ be a homogeneous Poisson process with intensity $\lambda = 1$ and μ be a mean-value function. Let $\theta : \Omega \to \mathbb{R}$ be a random variable such that $\theta > 0$ a.s., and θ is independent of $\hat{N}$. Then

$$N(t) := \hat{N}(\theta\mu(t)),\ \ t \geq 0$$

is a **mixed Poisson process** with mixing variable θ.

Proposition 2.3.5. It holds

$$\mathrm{var}(\hat{N}(\theta\mu(t))) = \mathbb{E}\hat{N}(\theta\mu(t))\left(1 + \frac{\mathrm{var}(\theta)}{\mathbb{E}\theta}\mu(t)\right).$$

Proof. We recall that $\mathbb{E}\hat{N}(t) = \mathrm{var}(\hat{N}(t)) = t$ and therefore $\mathbb{E}\hat{N}(t)^2 = t + t^2$. We conclude

$$\begin{aligned}
\mathrm{var}(\hat{N}(\theta\mu(t))) &= \mathbb{E}\hat{N}(\theta\mu(t))^2 - \left[\mathbb{E}\hat{N}(\theta\mu(t))\right]^2 \\
&= \mathbb{E}\left(\theta\mu(t) + \theta^2\mu(t)^2\right) - (\mathbb{E}\theta\mu(t))^2 \\
&= \mu(t)\left(\mathbb{E}\theta + \mathrm{var}\theta\mu(t)\right).
\end{aligned}$$

□

The property $\mathrm{var}(N(t)) > \mathbb{E}N(t)$ is called **over-dispersion**. If N is an inhomogeneous Poisson process, then

$$\mathrm{var}(N(t)) = \mathbb{E}N(t).$$

Chapter 3

The total claim amount process $S(t)$

3.1 The renewal model and the Cramér-Lundberg-model

Definition 3.1.1. The **renewal model** (or Sparre-Anderson-model) considers the following setting:

(1) Claims happen at the claim arrival times $0 \leq T_1 \leq T_2 \leq ...$ of a **renewal process**

$$N(t) = \#\{i \geq 1 : T_i \leq t\}, \ t \geq 0.$$

(2) At time T_i the claim size X_i happens and it holds that the sequence $(X_i)_{i=1}^{\infty}$ is i.i.d., $X_i \geq 0$.

(3) The processes $(T_i)_{i=1}^{\infty}$ and $(X_i)_{i=1}^{\infty}$ are independent.

The renewal model is called **Cramér-Lundberg-model** if the claims happen at the claim arrival times $0 < T_1 < T_2 < ...$ of a **Poisson process**

3.2 Properties of the total claim amount process $S(t)$

Definition 3.2.1. The total claim amount process is defined as

$$S(t) := \sum_{i=1}^{N(t)} X_i, \ t \geq 0.$$

The insurance company needs information about $S(t)$ in order to determine a **premium** which covers the losses represented by $S(t)$. In general, the **distribution** of $S(t)$, i.e.

$$\mathbb{P}(\{\omega \in \Omega : S(t,\omega) \le x\}), \ x \ge 0,$$

can only be **approximated** by numerical methods or simulations while $\mathbb{E}S(t)$ and $\operatorname{var}(S(t))$ are easy to compute exactly. One can establish principles which use only $\mathbb{E}S(t)$ and $\operatorname{var}(S(t))$ to calculate the premium. This will be done in chapter 4.

Proposition 3.2.2. *One has that*

$$\begin{aligned} \mathbb{E}S(t) &= \mathbb{E}X_1 \mathbb{E}N(t), \\ \operatorname{var}(S(t)) &= \operatorname{var}(X_1)\mathbb{E}N(t) + \operatorname{var}(N(t)(\mathbb{E}X_1)^2. \end{aligned}$$

Consequently, one obtains the following relations:

(1) **Cramér-Lundberg-model***: It holds*

  (i) $\mathbb{E}S(t) = \lambda t \mathbb{E}X_1$,

  (ii) $\operatorname{var}(S(t)) = \lambda t \mathbb{E}X_1^2$.

(2) **Renewal model***: Let* $\mathbb{E}W_1 = \frac{1}{\lambda} \in (0,\infty)$ *and* $\mathbb{E}X_1 < \infty$.

  (i) Then $\lim_{t\to\infty} \frac{\mathbb{E}S(t)}{t} = \lambda \mathbb{E}X_1$.

  (ii) If $\operatorname{var}(W_1) < \infty$ *and* $\operatorname{var}(X_1) < \infty$, *then*

$$\lim_{t\to\infty} \frac{\operatorname{var}(S(t))}{t} = \lambda \left(\operatorname{var}(X_1) + \operatorname{var}(W_1)\lambda^2 (\mathbb{E}X_1)^2 \right).$$

Proof. (a) Since

$$1 = \mathbb{1}_\Omega(\omega) = \sum_{k=0}^{\infty} \mathbb{1}_{\{N(t)=k\}},$$

by a direct computation,

$$\begin{aligned} \mathbb{E}S(t) &= \mathbb{E}\sum_{i=1}^{N(t)} X_i \\ &= \mathbb{E}\sum_{k=0}^{\infty} \left((\sum_{i=1}^{k} X_i) \mathbb{1}_{\{N(t)=k\}} \right) \\ &= \sum_{k=0}^{\infty} \underbrace{\mathbb{E}(X_1 + ... + X_k)}_{=k\mathbb{E}X_1} \underbrace{\mathbb{E}\mathbb{1}_{\{N(t)=k\}}}_{=\mathbb{P}(N(t)=k)} \\ &= \mathbb{E}X_1 \sum_{k=0}^{\infty} k \mathbb{P}(N(t)=k) \end{aligned}$$

$$= \mathbb{E}X_1\mathbb{E}N(t).$$

In the Cramér-Lundberg-model we have $\mathbb{E}N(t) = \lambda t$. For the general case we use the Elementary Renewal Theorem (Thereom 2.2.4) to get the assertion.

(b) We continue with

$$\begin{aligned}
\mathbb{E}S(t)^2 &= \mathbb{E}\left(\sum_{i=1}^{N(t)} X_i\right)^2 = \mathbb{E}\left(\sum_{k=0}^{\infty}\left(\sum_{i=1}^{k} X_i\right)\mathbb{1}_{\{N(t)=k\}}\right)^2 \\
&= \mathbb{E}\sum_{k=0}^{\infty}\left(\sum_{i=1}^{k} X_i\right)^2 \mathbb{1}_{\{N(t)=k\}} \\
&= \sum_{k=0}^{\infty}\sum_{i,j=1}^{k} \mathbb{E}\left(X_iX_j\mathbb{1}_{\{N(t)=k\}}\right) \\
&= \mathbb{E}X_1^2\sum_{k=0}^{\infty} k\mathbb{P}(N(t)=k) + (\mathbb{E}X_1)^2\sum_{k=1}^{\infty} k(k-1)\mathbb{P}(N(t)=k) \\
&= \mathbb{E}X_1^2\,\mathbb{E}N(t) + (\mathbb{E}X_1)^2(\mathbb{E}N(t)^2 - \mathbb{E}N(t)) \\
&= \mathrm{var}(X_1)\mathbb{E}N(t) + (\mathbb{E}X_1)^2\mathbb{E}N(t)^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathrm{var}(S(t)) &= \mathbb{E}S(t)^2 - (\mathbb{E}S(t))^2 \\
&= \mathbb{E}S(t)^2 - (\mathbb{E}X_1)^2(\mathbb{E}N(t))^2 \\
&= \mathrm{var}(X_1)\mathbb{E}N(t) + (\mathbb{E}X_1)^2\mathrm{var}(N(t)).
\end{aligned}$$

For the Cramér-Lundberg-model it holds $\mathbb{E}N(t) = \mathrm{var}(N(t)) = \lambda t$, hence we have $\mathrm{var}(S(t)) = \lambda t(\mathrm{var}(X_1) + (\mathbb{E}X_1)^2) = \lambda t\mathbb{E}X_1^2$. For the renewal model we get

$$\lim_{t\to\infty}\frac{\mathrm{var}(X_1)\mathbb{E}N(t)}{t} = \mathrm{var}(X_1)\lambda.$$

The relation

$$\lim_{t\to\infty}\frac{\mathrm{var}(N(t))}{t} = \frac{\mathrm{var}(W_1)}{(\mathbb{E}W_1)^3}.$$

is shown in [6, Theorem 2.5.2]. □

Theorem 3.2.3 (SLLN and CLT for renewal model)**.**

(1) *SLLN for* $(S(t))$*: If* $\mathbb{E}W_1 = \frac{1}{\lambda} < \infty$ *and* $\mathbb{E}X_1 < \infty$*, then*

$$\lim_{t\to\infty}\frac{S(t)}{t} = \lambda\mathbb{E}X_1 \quad a.s.$$

(2) *CLT for* $(S(t))$*: If* $\mathrm{var}(W_1) < \infty$*, and* $\mathrm{var}(X_1) < \infty$*, then*

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left(\frac{S(t) - \mathbb{E}S(t)}{\sqrt{\mathrm{var}(S(t))}} \le x \right) - \Phi(x) \right| \overset{t\to\infty}{\to} 0,$$

where Φ *is the distribution function of the standard normal distribution,*

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy.$$

Proof. (1) We follow the proof of [7, Theorem 3.1.5]. We have shown that

$$\lim_{t\to\infty} \frac{N(t)}{t} = \lambda \quad a.s.$$

and therefore it holds

$$\lim_{t\to\infty} N(t) = \infty \quad a.s.$$

Because of $S(t) = X_1 + X_2 + ... + X_{N(t)}$ and, by the SSLN

$$\lim_{n\to\infty} \frac{X_1 + ... + X_n}{n} = \mathbb{E}X_1 \quad a.s.,$$

we get

$$\lim_{t\to\infty} \frac{S(t)}{t} = \lim_{t\to\infty} \frac{N(t)}{t} \lim_{t\to\infty} \frac{S(t)}{N(t)} = \lambda \mathbb{E}X_1 \quad a.s.$$

(2) See [5, Theorem 2.5.16]. □

Chapter 4

Premium calculation principles

The standard problem in insurance is to determine that amount of premium such that the losses $S(t)$ are covered. On the other hand, the price of the premiums should be low enough to be competitive and attract customers. In the following we let

$$\rho(t) \in [0, \infty) \text{ be the cumulative premium income up to time } t \in [0, \infty)$$

in our stochastic model. Below we review some premium calculation principles.

4.1 Classical premium calculation principles

First approximations of $S(t)$ are given by $\mathbb{E}S(t)$ and $\mathrm{var}(S(t))$, and the classical principles are based on these quantities. Intuitively, we have:

$$\begin{array}{lcl} \rho(t) < \mathbb{E}S(t) & \Rightarrow & \text{insurance company loses on average} \\ \rho(t) > \mathbb{E}S(t) & \Rightarrow & \text{insurance company gains on average} \end{array}$$

4.1.1 Net principle

The **Net Principle**

$$\rho_{\mathrm{NET}}(t) = \mathbb{E}S(t)$$

defines the premium to be a *fair market premium*. However, this usually leads to the ruin for the company as we will see later.

4.1.2 Expected value principle

In the **Expected Value Principle** the premium is calculated by

$$p_{\mathrm{EV}}(t) = (1 + \rho)\mathbb{E}S(t)$$

where $\rho > 0$ is called the **safety loading**.

4.1.3 The variance principle

The **Variance Principle** is given by

$$p_{\mathrm{VAR}}(t) = \mathbb{E}S(t) + \alpha \mathrm{var}(S(t)), \ \alpha > 0.$$

This principle is in the renewal model asymptotically the same as $p_{EV}(t)$, since by Proposition 3.2.2 we have that

$$\lim_{t\to\infty} \frac{p_{\mathrm{EV}}(t)}{p_{\mathrm{VAR}}(t)} = \lim_{t\to\infty} \frac{(1+\rho)\mathbb{E}S(t)}{\mathbb{E}S(t) + \alpha \mathrm{var}(S(t))} = \frac{(1+\rho)}{1 + \alpha \lim_{t\to\infty} \frac{\mathrm{var}(S(t))}{\mathbb{E}S(t)}}$$

is a constant. This means that α plays the role of the safety loading ρ.

4.1.4 The standard deviation principle

This principle is given by

$$p_{\mathrm{SD}}(t) = \mathbb{E}S(t) + \alpha\sqrt{\mathrm{var}(S(t))}, \ \alpha > 0.$$

4.1.5 The constant income principle

Here we simply

$$\rho_{\mathrm{const}}(t) := ct, \ c > 0.$$

In the case of the Cramér-Lundberg-modelthis principle coincides with the *expected value principle* by setting

$$c := (1+\rho)\lambda\mathbb{E}X_1.$$

In the case of the renewal model it is asymptotically the *expected value principle* as by Proposition 3.2.2 we have

$$\mathbb{E}S(t) \sim [\lambda\mathbb{E}X_1]t.$$

In Definition 5.1.4 we introduce the *Net Profit Condition* that gives the necessary range for c (or ρ).

4.2 Modern premium calculation principles

In the following principles the expected value $\mathbb{E}(g(S(t))$ needs to be computed for certain functions $g(x)$ in order to compute $\rho(t)$. This means it is not enough to know $\mathbb{E}S(t)$ and $\mathrm{var}(S(t))$, the **distribution** of $S(t)$ is needed as well.

4.2.1 The Exponential Principle

The Exponential Principle is defined as

$$\rho_{\text{exp}}(t) := \frac{1}{\delta} \log \mathbb{E} e^{\delta S(t)},$$

for some $\delta > 0$, where δ is the risk aversion constant. The function $p_{exp}(t)$ is motivated by the so-called *utility theory*. By JENSEN's inequality one checks that

$$\rho_{\text{exp}}(t) = \frac{1}{\delta} \log \mathbb{E} e^{\delta S(t)} \geq \mathbb{E} S(t)$$

as the function $x \mapsto e^x$ is convex.

4.2.2 The Esscher Principle

The Esscher principle is defined as

$$\rho_{\text{Ess}}(t) := \frac{\mathbb{E} S(t) e^{\delta S(t)}}{\mathbb{E} e^{\delta (S(t))}}, \ \delta > 0.$$

As a homework we show that

$$\rho_{\text{Ess}}(t) = \frac{\mathbb{E} S(t) e^{\delta S(t)}}{\mathbb{E} e^{\delta (S(t))}} \geq \mathbb{E} S(t)$$

.

4.2.3 The Quantile Principle

Denote by $F_t(x) := \mathbb{P}(\{\omega : S(t, \omega) \leq x\})$, $x \in \mathbb{R}$, the distribution function of $S(t)$. Given $0 < \varepsilon < 1$, we let

$$\rho_{\text{quant}}(t) := \min\{x \geq 0 : \mathbb{P}(\{\omega \in \Omega : S(t, \omega) \leq x\}) \geq 1 - \varepsilon\}.$$

This principle is called $(1 - \varepsilon)-$**quantile principle**. Note that

$$\mathbb{P}(S(t) > \rho_{\text{quant}}(t)) \leq \varepsilon.$$

This setting is related to the theory of *Value at Risk*.

4.3 Reinsurance treaties

Reinsurance treaties are mutual agreements between different insurance companies to reduce the risk in a particular insurance portfolio. Reinsurances can be considered as insurance for the insurance company. Reinsurances are used if there is a risk of rare but huge claims. Examples of these usually involve a catastrophe such as earthquake, nuclear power station disaster, industrial fire, war, tanker accident, etc.

According to Wikipedia, the world's largest reinsurance company in 2009 is Munich Re, based in Germany, with gross written premiums worth over $31.4 billion, followed by Swiss Re (Switzerland), General Re (USA) and Hannover Re (Germany).

There are two different types of reinsurance:

4.3.1 Random walk type reinsurance

1. Proportional reinsurance: The reinsurer pays an agreed proportion p of the claims,
$$R_{prop}(t) := pS(t).$$
2. Stop-loss reinsurance: The reinsurer covers the losses that exceed an agreed amount of K,
$$R_{SL}(t) := (S(t) - K)^+,$$
where $x^+ = \max\{x, 0\}$.
3. Excess-of-loss reinsurance: The reinsurer covers the losses that exceed an agreed amount of D for each claim separately,
$$R_{ExL} := \sum_{i=1}^{N(t)} (X_i - D)^+,$$
where D is the deductible.

4.3.2 Extreme value type reinsurance

Extreme value type reinsurances cover the largest claims in a portfolio. Mathematically, these contracts are investigated with *extreme value theory* techniques. The ordering of the claims $X_1, ..., X_{N(t)}$ is denoted by
$$X_{(1)} \leq ... \leq X_{(N(t))}.$$

1. Largest claims reinsurance: The largest claims reinsurance covers the k largest claims arriving within time frame $[0, t]$,
$$R_{LC}(t) := \sum_{i=1}^{k} X_{(N(t)-i+1)}.$$
2. ECOMOR reinsurance: (*Excédent du coût moyen relatif* means *excess of the average cost*). Define $k = \lfloor \frac{N(t)+1}{2} \rfloor$. Then
$$\begin{aligned} R_{ECOMOR}(t) &= \sum_{i=1}^{N(t)} (X_{(N(t)-i+1)} - X_{(N(t)-k+1)})^+ \\ &= \sum_{i=1}^{k-1} X_{(N(t)-i+1)} - (k-1)X_{(N(t)-k+1)} \end{aligned}$$

Treaties of **random walk type** can be handled like before. For example,

$$\mathbb{P}(\underbrace{R_{SL}(t)}_{(S(t)-K)^+} \leq x) = \mathbb{P}(S(t) \leq K) + \mathbb{P}(K < S(t) \leq x + K),$$

so if $F_{S(t)}$ is known, so is $F_{R_{SL}(t)}$.

Chapter 5

Probability of ruin: small claim sizes

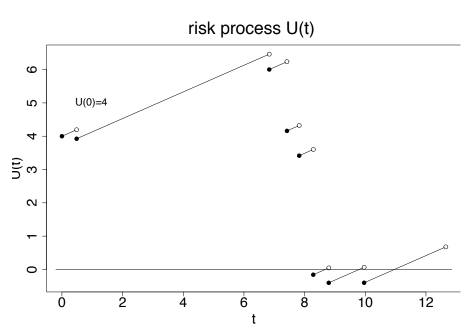
5.1 The risk process

In this chapter, if not stated differently, we use the following assumptions and notation:

- The **renewal model** is assumed.
- **Total claim amount process**: $S(t) := \sum_{i=1}^{N(t)} X_i$ with $t \geq 0$.
- **Premium income function**: $\rho(t) = ct$ where $c > 0$ is the **premium rate**.
- The **risk process** or **surplus process** is given by

  $$U(t) := u + \rho(t) - S(t), \ t \geq 0,$$

  where $U(t)$ is the insurer's capital balance at time t and u is the **initial capital**.



Definition 5.1.1 (Ruin, ruin time, ruin probability)**.** We let

$$\begin{aligned} \text{ruin}(u) &:= \{\omega \in \Omega : U(t,\omega) < 0 \text{ for some } t > 0\} \\ &= \text{the event that } U \text{ ever falls below zero,} \\ \textbf{Ruin time} \quad T &:= \inf\{t > 0 : U(t) < 0\} \\ &= \text{the time when the process falls below zero} \\ & \text{for the first time.} \end{aligned}$$

The **ruin probability** is given by

$$\psi(u) = \mathbb{P}(\text{ruin}(u)) = \mathbb{P}(T < \infty).$$

Remark 5.1.2.

(1) $T : \Omega \to \mathbb{R} \cup \{\infty\}$ is an **extended random variable** (i.e. T can also take the value ∞).

(2) In the literature $\psi(u)$ is often written as

$$\psi(u) = \mathbb{P}(\text{ruin}|U(0) = u)$$

to indicate the dependence on the initial capital u.

(3) Ruin can only occur at the times $t = T_n,\ n \geq 1$. This implies

$$\begin{aligned} \text{ruin}(u) &= \{\omega \in \Omega : T(\omega) < \infty\} \\ &= \{\omega \in \Omega : \inf_{t>0} U(t,\omega) < 0\} \\ &= \{\omega \in \Omega : \inf_{n\geq 1} U(T_n(\omega),\omega) < 0\} \\ &= \{\omega \in \Omega : \inf_{n\geq 1} (u + cT_n - S(T_n)) < 0\}, \end{aligned}$$

where the last equation yields from the fact that $U(t) = u + ct - S(t)$. Since in the renewal model it was assumed that $W_i > 0$, it follows that

$$N(T_n) = \#\{i \geq 1 : T_i \leq T_n\} = n$$

and

$$S(T_n) = \sum_{i=1}^{N(T_n)} X_i = \sum_{i=1}^{n} X_i,$$

where

$$T_n = W_1 + ... + W_n,$$

which imply that

$$\begin{aligned} &\left\{\omega \in \Omega : \inf_{n\geq 1} \Big(u + cT_n - S(T_n)\Big) < 0\right\} \\ = &\left\{\omega \in \Omega : \inf_{n\geq 1} \left(u + cT_n - \sum_{i=1}^{n} X_i\right) < 0\right\}. \end{aligned}$$

By setting

$$Z_n := X_n - cW_n,\ n \geq 1$$

and

$$G_n := Z_1 + ... + Z_n,\ n \geq 1,\ G_0 := 0,$$

it follows that

$$\begin{aligned} \{\omega \in \Omega : T(\omega) < \infty\} &= \Big\{\omega \in \Omega : \inf_{n\geq 1}(-G_n) < -u\Big\} \\ &= \Big\{\omega \in \Omega : \sup_{n\geq 1} G_n > u\Big\} \end{aligned}$$

and for the ruin probability the equality it holds that

$$\psi(u) = \mathbb{P}\Big(\sup_{n\geq 1} G_n(\omega) > u\Big).$$

First we state the theorem that justifies the *Net Profit Condition* introduced below:

Theorem 5.1.3. *If* $\mathbb{E}W_1 < \infty$, $\mathbb{E}X_1 < \infty$, *and*

$$\mathbb{E}Z_1 = \mathbb{E}X_1 - c\mathbb{E}W_1 \geq 0,$$

then $\psi(u) = 1$ *for all* $u > 0$, *i.e. ruin occurs with probability one independent from the initial capital* u.

Proof. (a) $\mathbb{E}Z_1 > 0$: By the Strong Law of Large Numbers,

$$\lim_{n\to\infty} \frac{G_n}{n} = \mathbb{E}Z_1 \text{ almost surely.}$$

Because we assumed $\mathbb{E}Z_1 > 0$, one gets that

$$G_n \overset{a.s.}{\to} \infty,\ n \to \infty,$$

because $G_n \approx n\mathbb{E}Z_1$ for large n. This means ruin probability $\psi(u) = 1$ for all $u > 0$.

(b) The case $\mathbb{E}Z_1 = 0$ we show under the additional assumption that $\mathbb{E}Z_1^2 < \infty$. Let

$$\bar{A}_m := \left\{\limsup_{n\to\infty} \frac{Z_1 + ... + Z_n}{\sqrt{n}} \geq m\right\}.$$

Notice that for fixed $\omega \in \Omega$ and $n_0 \geq 1$ we have

$$\limsup_{n\to\infty} \frac{Z_1(\omega) + ... + Z_n(\omega)}{\sqrt{n}} \geq m$$

iff

$$\limsup_{n\geq n_0} \frac{Z_{n_0}(\omega) + ... + Z_n(\omega)}{\sqrt{n}} \geq m.$$

Hence

$$\bar{A}_m \subseteq \bigcap_{n_0=1}^{\infty} \sigma(Z_{n_0}, Z_{n_0+1}, ...).$$

The sequence $(Z_n)_{n\geq 1}$ consists of independent random variables. By the $0-1$ law of Kolmogorov (see [4, Proposition 2.1.6]) we conclude that

$$\mathbb{P}(\bar{A}_m) \in \{0, 1\}.$$

Since

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{Z_1 + ... + Z_n}{\sqrt{n}} = \infty\right) = \lim_{m\to\infty} \mathbb{P}(\bar{A}_m)$$

it suffices to show $\mathbb{P}(\bar{A}_m) > 0$. We have

$$\begin{aligned} \bar{A}_m &= \left\{\limsup_{n\to\infty} \frac{Z_1 + ... + Z_n}{\sqrt{n}} \geq m\right\} \\ &\supseteq \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \left\{\frac{Z_1 + ... + Z_k}{\sqrt{k}} \geq m\right\}. \end{aligned}$$

By Fatou's Lemma and the Central Limit Theorem,

$$\begin{aligned} \mathbb{P}(\bar{A}_m) &\geq \limsup_{k\to\infty} \mathbb{P}\left(\frac{Z_1 + ... + Z_k}{\sqrt{k}} \geq m\right) \\ &= \int_m^{\infty} e^{-\frac{x^2}{2\sigma^2}} \frac{dx}{\sqrt{2\pi\sigma^2}} > 0 \end{aligned}$$

where $\sigma^2 = \mathbb{E}Z_1^2$. □

Definition 5.1.4 (Net profit condition)**.** The renewal model satisfies the **net profit condition (NPC)** if and only if $\mathbb{E}X_1 < \infty$, $\mathbb{E}W_1 < \infty$, and

$$\mathbb{E}Z_1 = \mathbb{E}X_1 - c\mathbb{E}W_1 < 0. \quad (NPC)$$

The consequence of (NPC) is that on average more premium flows into the portfolio of the company than claim sizes flow out: We have

$$\begin{aligned} G_n &= -\rho(T_n) + S(T_n) \\ &= -c(W_1 + ... + W_n) + X_1 + ... + X_n \end{aligned}$$

which implies

$$\mathbb{E}G_n = n\mathbb{E}Z_1 < 0.$$

Theorem 5.1.3 implies that any insurance company should choose the premium $\rho(t) = ct$ in such a way that $\mathbb{E}Z_1 < 0$. In that case there is hope that the ruin probability is less than 1.

5.2 Lundberg inequality and Cramér's ruin bound

Before we start: we wish to recall the following basic facts from probability: We assume a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a random variable $A : \Omega \to \mathbb{R}$. The *distribution function* $F_A : \mathbb{R} \to [0,1]$ of A was given by

$$F_A(u) := \mathbb{P}(\{\omega \in \Omega : A(\omega) \le u\}).$$

The law of A is the probability measure $\mathbb{P}_A$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that

$$\mathbb{P}_A(B) := \mathbb{P}(\{\omega \in \Omega : A(\omega) \in B\}) \quad \text{for} \quad B \in \mathcal{B}(\mathbb{R}).$$

We consider $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbb{P}_A)$ as probability space, assume a random variable $\Phi : \mathbb{R} \to \mathbb{R}$ and use the following *notation*

$$\int_{\mathbb{R}} \Phi(x) dF_A(x) := \int_{\mathbb{R}} \Phi(x) d\mathbb{P}_A(x).$$

So instead of writing $d\mathbb{P}_A(x)$ we write, as often in the literature, $dF_A(x)$. For us, this is only a notation. The real background is the notion of the Riemann-Stieltjes integral (but we do not go into this).

Now, let us continue with the lecture:

Definition 5.2.1. For a random variable $f : \Omega \to \mathbb{R}$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ the function

$$m_f(h) = \mathbb{E}e^{hf},$$

is called the **moment-generating function** if it is finite for $h \in (-h_0, h_0)$ for some $h_0 > 0$.

Remark 5.2.2. (1) The map

$$h \mapsto \mathbb{E}e^{-hf}$$

is called **two-sided Laplace transform**.

(2) If $\mathbb{E}|f|^k e^{hf} < \infty$ on $(-h_0, h_0)$ for all $k = 0, \ldots, m$, then $m_f(h)$ exists, is m-times differentiable, and one has

$$\frac{d^m}{dh^m} m_f(h) = \mathbb{E}f^m e^{hf}.$$

Therefore

$$\frac{d^m}{dh^m} m_f(0) = \mathbb{E}f^m.$$

Definition 5.2.3 (Small claim size condition and Lundberg coefficient)**.**

(1) Given a claim size distribution $X_1 : \Omega \to (,\infty)$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we say that the **small claim size condition** with parameter $h_0 > 0$ is satisfied if

$$m_{X_1}(h) = \mathbb{E}e^{hX_1} < \infty \quad \text{for all} \quad h \in (-\infty, h_0).$$

(2) Assume that X_1 satisfies the small claim size condition with parameter $h_0 > 0$ and assume the independent waiting time $W_1 : \Omega \to (0, \infty)$, we call an $r \in (0, h_0)$ **Lundberg coefficient** if

$$\mathbb{E}e^{r(X_1 - cW_1)} = 1.$$

Remark 5.2.4. Some remarks about the Lundberg coefficient:

(1) The small claim size condition with parameter $h_0 > 0$ and the existence of the Lundberg coefficient $r \in (0, h_0)$ implies automatically that $\mathbb{E}e^{-rcW_1} \in (0, \infty)$, because

$$1 = \mathbb{E}e^{rX_1 - rcW_1} = \mathbb{E}e^{rX_1}\mathbb{E}e^{-rcW_1} \quad \text{and} \quad \mathbb{E}e^{rX_1} \in (0, \infty).$$

(2) If m_{Z_1} exists in $(-h_0, h_0)$ for some $h_0 > 0$, then

$$\mathbb{P}(Z_1 \geq \lambda) = \mathbb{P}(e^{\varepsilon Z_1} \geq e^{\varepsilon\lambda}) \leq e^{-\varepsilon\lambda} m_{Z_1}(\varepsilon)$$

and

$$\mathbb{P}(-Z_1 \geq \lambda) \leq e^{-\varepsilon\lambda} m_{-Z_1}(\varepsilon) = e^{-\varepsilon\lambda} m_{Z_1}(-\varepsilon)$$

for all $\varepsilon \in (-h_0, h_0)$. This implies

$$\mathbb{P}(|Z_1| \geq \lambda) \leq e^{-\varepsilon\lambda}[m_{Z_1}(\varepsilon) + m_{Z_1}(-\varepsilon)].$$

(3) If the Lundberg coefficient r exists, then it is unique. First we observe that which follows from the fact that m_{Z_1} is convex: We have

$$e^{(1-\theta)r_0 Z_1 + \theta r_1 Z_1} \leq (1-\theta)e^{r_0 Z_1} + \theta e^{r_1 Z_1}.$$

Moreover, $m_{Z_1}(0) = 1$ and by Jensen's inequality,

$$m_{Z_1}(h) = \mathbb{E}e^{Z_1 h} \geq e^{\mathbb{E}Z_1 h}$$

such that (assuming (NPC) holds) we get

$$\lim_{h \to -\infty} m_{Z_1}(h) \geq \lim_{h \to -\infty} e^{-\mathbb{E}Z_1(-h)} = \infty.$$

If m_{Z_1} exists in $(-\varepsilon, \varepsilon)$ and $m_{Z_1}(h) = 1$ for some $h \in \{r, s\} \subseteq (0, \varepsilon]$ then, by convexity,

$$m_{Z_1}(h) = 1 \quad \forall h \in [0, r \vee s].$$

From (1) we have

$$\mathbb{P}(|Z_1| > \lambda) \leq ce^{-\frac{\lambda}{c}} \quad \text{for some } c > 0$$

and it holds

$$\mathbb{E}|Z_1|^n = \int_0^\infty \mathbb{P}(|Z_1|^n > \lambda) d\lambda \;\;=\;\; n\int_0^\infty \mathbb{P}(|Z_1| > \lambda)\lambda^{n-1} d\lambda$$

$$\begin{aligned}
&\leq n\int_0^\infty ce^{-\lambda/c}\lambda^{n-1}d\lambda \\
&\leq nc^n\int_0^\infty e^{-\lambda/c}\left(\frac{\lambda}{c}\right)^{n-1}d\lambda \\
&\leq nc^{n+1}\int_0^\infty e^{-\lambda}\lambda^{n-1}d\lambda \\
&= n!c^{n+1}.
\end{aligned}$$

Because of

$$\sum_{n=0}^\infty \frac{h^n}{n!}\mathbb{E}|Z_1|^n \leq \sum_{n=0}^\infty \frac{h^n}{n!}n!c^{n+1} < \infty$$

for $|hc| < 1$, the function

$$m_{Z_1}(h) = \mathbb{E}e^{hZ_1} = \mathbb{E}\sum_{n=0}^\infty \frac{(hZ_1)^n}{n!}$$

is infinitely often differentiable for $|h| < 1/c$. Moreover the function is constant on $[0, r \vee s]$ so that, for $0 < h < \min\{1/c, r \vee s\}$, one has

$$0 = \frac{d^2}{dh^2}m_{Z_1}(h) = \mathbb{E}Z_1^2 e^{hZ_1}.$$

This implies $\mathbb{E}Z_1^2 e^{hZ_1} = 0$ and $Z_1 = 0$ a.s.

(4) In practice, r is hard to compute from the distributions of X_1 and W_1. Therefore it is often approximated numerically or by Monte Carlo methods.

Theorem 5.2.5 (Lundberg inequality in the renewal model)**.** *We assume*

(1) *the renewal model with (NPC),*

(2) *the small claim size condition with parameter* $h_0 > 0$*,*

(3) $r \in (0, h_0)$ *is the Lundberg coefficient,*

(4) $\rho(t) = ct = (1 + \rho)\mathbb{E}S(t)$*.*

Then for all $u \geq 0$ *it holds that*

$$\psi(u) \leq e^{-ru}.$$

The result implies, that if the small claim condition holds and the initial capital u is large, then the ruin probability decays *exponentially*, which is remarkable.

Proof of Theorem 5.2.5. We use $Z_n = X_n - cW_n$ and set $G_k := Z_1 + ... + Z_k$. We consider

$$\psi_n(u) := \mathbb{P}\big(\max_{1\le k\le n} G_k > u\big), \quad u > 0.$$

Because of $\psi_n(u) \uparrow \psi(u)$ for $n \to \infty$ it is sufficient to show

$$\psi_n(u) \le e^{-ru}, \quad n \ge 1, u > 0.$$

For $n = 1$ we get the inequality by

$$\psi_1(u) = \mathbb{P}(Z_1 > u) = \mathbb{P}(e^{rZ_1} > e^{ru}) \le e^{-ru}\mathbb{E}e^{rZ_1} = e^{-ru}.$$

Now we assume that the assertion holds for n. We have

$$\begin{aligned} \psi_{n+1}(u) &= \mathbb{P}\big(\max_{1\le k\le n+1} G_k > u\big) \\ &= \mathbb{P}(Z_1 > u) + \mathbb{P}\big(\max_{1\le k\le n+1} G_k > u, Z_1 \le u\big) \\ &= \mathbb{P}(Z_1 > u) + \mathbb{P}\big(\max_{2\le k\le n+1} (G_k - Z_1) > u - Z_1, Z_1 \le u\big) \\ &= \mathbb{P}(Z_1 > u) + \int_{-\infty}^{u} \mathbb{P}\big(\max_{1\le k\le n} G_k > u - x\big) dF_{Z_1}(x) \end{aligned}$$

where we have used for the last line that $\max_{2\le k\le n+1}(G_k - Z_1)$ and Z_1 are independent. We estimate the first term

$$\mathbb{P}(Z_1 > u) = \int_{(u,\infty)} dF_{Z_1}(x) \le \int_{(u,\infty)} e^{r(x-u)} dF_{Z_1}(x),$$

and proceed with the second term as follows:

$$\begin{aligned} \int_{(-\infty,u]} \mathbb{P}\big(\max_{1\le k\le n} G_k > u - x\big) dF_{Z_1}(x) &= \int_{(-\infty,u]} \psi_n(u-x) dF_{Z_1}(x) \\ &\le \int_{(-\infty,u]} e^{-r(u-x)} dF_{Z_1}(x). \end{aligned}$$

Consequently,

$$\psi_{n+1}(u) \le \int_{(u,\infty)} e^{r(x-u)} dF_{Z_1}(x) + \int_{(-\infty,u]} e^{-r(u-x)} dF_{Z_1}(x) = e^{-ru}.$$

□

We consider an example where it is possible to compute the Lundberg coefficient:

Example 5.2.6. Let $X_1, X_2, ... \sim \text{Exp}(\gamma)$ and $W_1, W_2, ... \sim \text{Exp}(\lambda)$. Then

$$m_{Z_1}(h) = \mathbb{E}e^{h(X_1 - cW_1)} = \mathbb{E}e^{hX_1}\mathbb{E}e^{-hcW_1} = \frac{\gamma}{\gamma - h}\frac{\lambda}{\lambda + ch}$$

for $-\frac{\lambda}{c} < h < \gamma$ since

$$\mathbb{E}e^{hX_1} = \int_0^\infty e^{hx}\gamma e^{-\gamma x}dx = \frac{\gamma}{\gamma - h}.$$

The (NPC) condition reads as

$$0 > \mathbb{E}Z_1 = \mathbb{E}X_1 - c\mathbb{E}W_1 = \frac{1}{\gamma} - \frac{c}{\lambda} \quad \text{or} \quad c\gamma > \lambda.$$

Hence m_{Z_1} exists on $(-\frac{\lambda}{c}, \gamma)$ and for $r > 0$ we get

$$\begin{aligned} & \frac{\gamma}{\gamma - r}\frac{\lambda}{\lambda + cr} = 1 \\ \iff \quad & \gamma\lambda = \gamma\lambda + \gamma cr - \lambda r - cr^2 \\ \iff \quad & r = \gamma - \frac{\lambda}{c}. \end{aligned}$$

Consequently,

$$\psi(u) \le e^{-ru} = e^{-(\gamma - \frac{\lambda}{c})u}.$$

Applying the expected value principle $\rho(t) = (1+\rho)\mathbb{E}S(t) = (1+\rho)\lambda\mathbb{E}X_1 t$ we get

$$\gamma - \frac{\lambda}{c} = \gamma - \frac{\lambda}{(1+\rho)\frac{\lambda}{\gamma}} = \gamma\frac{\rho}{1+\rho}.$$

This implies

$$\psi(u) \le e^{-ru} = e^{-u\gamma\frac{\rho}{1+\rho}},$$

where one should notice that even $\rho \to \infty$ does not change the ruin probability considerably!

The following theorem considers the special case, the Cramér-Lundberg-model:

Theorem 5.2.7 (Cramér's ruin bound in the Cramér-Lundberg-model)**.** *We assume*

(1) *the Cramér-Lundberg-model with (NPC),*

(2) *the small claim size condition with parameter* $h_0 > 0$*,*

(3) $r \in (0, h_0)$ *is the Lundberg coefficient,*

(4) $\rho(t) = ct = (1+\rho)\mathbb{E}S(t)$*.*

Then one has

$$\lim_{u\to\infty} e^{ru}\psi(u) = \rho\frac{\mathbb{E}X_1}{r}\left(\int_0^\infty xe^{rx}\mathbb{P}(X_1 > x)dx\right)^{-1}.$$

To prove this theorem introduce in the next section the fundamental integral equation for the survival probability.

5.3 Fundamental integral equation for the survival probability

We introduce the **survival probability**

$$\varphi(u) = 1 - \psi(u).$$

Theorem 5.3.1 (Fundamental integral equation for survival probability)**.** *We assume*

(1) *the Cramér-Lundberg-model with (NPC),*

(2) $\rho(t) = ct = (1+\rho)\mathbb{E}S(t)$.

Then one has

$$\varphi(u) = \varphi(0) + \frac{1}{(1+\rho)\mathbb{E}X_1} \int_0^u \mathbb{P}(X_1 > x)\varphi(u-x)dx. \qquad (1)$$

Remark 5.3.2. Let the assumptions of Theorem 5.3.1 hold.

(1) The assertion can be reformulated as follows. Let

$$F_{X_1,I}(x) := \frac{1}{\mathbb{E}X_1} \int_0^x \bar{F}_{X_1}(y)dy, \ x \geq 0.$$

The function $F_{X_1,I}$ is a distribution function since

$$\begin{aligned}\lim_{x\to\infty} F_{X_1,I}(x) &= \frac{1}{\mathbb{E}X_1}\int_0^\infty \bar{F}_{X_1}(y)dy \\ &= \frac{1}{\mathbb{E}X_1}\int_0^\infty \mathbb{P}(X_1 > y)dy = 1.\end{aligned}$$

Hence equation (1) can be written as

$$\varphi(u) = \varphi(0) + \frac{1}{1+\rho}\int_0^u \varphi(u-x)dF_{X_1,I}(x).$$

(2) It holds that $\lim_{u\to\infty}\varphi(u) = 1$. This can be seen as follows:

$$\begin{aligned}\lim_{u\to\infty}\varphi(u) &= \lim_{u\to\infty}(1-\psi(u)) \\ &= \lim_{u\to\infty}\left(1 - \mathbb{P}\left(\sup_{k\geq 1} G_k > u\right)\right) \\ &= \lim_{u\to\infty}\mathbb{P}\left(\sup_{k\geq 1} G_k \leq u\right)\end{aligned}$$

where $G_k = Z_1 + ... + Z_k$. Since $\mathbb{E}Z_1 < 0$ the SLLN implies

$$\lim_{k\to\infty} G_k = -\infty \quad a.s.$$

Therefore we have $\sup_{k\geq 1} G_k < \infty$ a.s. and

$$\lim_{u\to\infty} \mathbb{P}\Big(\sup_{k\geq 1} G_k \leq u\Big) = 1.$$

(3) It holds that $\varphi(0) = \frac{\rho}{1+\rho} \to 1$ for $\rho \uparrow \infty$. Indeed, because of (1), and (2), and Theorem 5.3.1 we may conclude that

$$\begin{aligned} 1 &= \varphi(0) + \frac{1}{1+\rho} \lim_{u\to\infty} \int_0^\infty \mathbb{1}_{[0,u]}(y)\varphi(u-y)dF_{X_1,I}(y) \\ &= \varphi(0) + \frac{1}{1+\rho} \int_0^\infty \lim_{u\to\infty} \big(\mathbb{1}_{[0,u]}(y)\varphi(u-y)\big)dF_{X_1,I}(y) \\ &= \varphi(0) + \frac{1}{1+\rho} \int_0^\infty dF_{X_1,I}(y) \\ &= \varphi(0) + \frac{1}{1+\rho}. \end{aligned}$$

The interpretation of $\varphi(0)$ is the survival probability when starting with 0 Euro initial capital.

Proof of Theorem 5.3.1. (a) We first show that

$$\varphi(u) = \frac{\lambda}{c} e^{\frac{\lambda u}{c}} \int_{[u,\infty)} e^{\frac{-\lambda y}{c}} \int_{[0,y]} \varphi(y-x)dF_{X_1}(x)dy. \tag{2}$$

To do this, we consider

$$\begin{aligned} &\varphi(u) \\ = &\ \mathbb{P}\Big(\sup_{n\geq 1} G_n \leq u\Big) \\ = &\ \mathbb{P}\Big(Z_1 \leq u, G_n - Z_1 \leq u - Z_1 \text{ for } n \geq 2\Big) \\ = &\ \int_{[0,\infty)} \int_{[0,u+cw]} \mathbb{P}\Big(G_n - Z_1 \leq u - (x-cw) \text{ for } n \geq 2\Big) dF_{X_1}(x)dF_{W_1}(w) \end{aligned}$$

where we used for the last line that

$$x - cw \leq u \text{ and } x \geq 0 \iff 0 \leq x \leq u + cw.$$

We use that $G_n - Z_1 \sim Z_1 + ... + Z_{n-1}$ and substitute $y := u + cw$ in order to obtain

$$\varphi(u) = \int_{[0,\infty)} \int_{[0,u+cw]} \mathbb{P}\Big(G_n \leq u - (x-cw) \text{ for } n \geq 1\Big) dF_{X_1}(x)\lambda e^{-\lambda w}dw$$

$$= \int_{[0,\infty)} \int_{[0,u+cw]} \varphi(u-x+cw)dF_{X_1}(x)\lambda e^{-\lambda w}dw$$
$$= \int_{[u,\infty)} \int_{[0,y]} \varphi(y-x)dF_{X_1}(x)\lambda e^{-\lambda\frac{y-u}{c}}d\frac{y}{c}.$$

(b) Differentiation of (2) leads to

$$\begin{aligned}\varphi'(u) &= \frac{\lambda}{c}\varphi(u) - \int_{[0,y]} \varphi(y-x)dF_{X_1}(x)\frac{\lambda}{c}e^{-\lambda\frac{y-u}{c}}\bigg|_{y=u} \\ &= \frac{\lambda}{c}\varphi(u) - \frac{\lambda}{c}\int_{[0,u]} \varphi(u-x)dF_{X_1}(x),\end{aligned}$$

so that

$$\begin{aligned}&\varphi(t)-\varphi(0)-\frac{\lambda}{c}\int_0^t \varphi(u)du \\ &= -\frac{\lambda}{c}\int_0^t \int_{[0,u]} \varphi(u-x)dF_{X_1}(x)du \\ &= -\frac{\lambda}{c}\int_0^t \left[\varphi(u-x)F_{X_1}(x)\Big|_0^u + \int_{[0,u]} \varphi'(u-x)F_{X_1}(x)dx\right]du \\ &= -\frac{\lambda}{c}\int_0^t \left[\varphi(0)F_{X_1}(u) - \varphi(u)F_{X_1}(0) + \int_{[0,u]} \varphi'(u-x)F_{X_1}(x)dx\right]du \\ &= -\frac{\lambda}{c}\varphi(0)\int_0^t F_{X_1}(u)du - \frac{\lambda}{c}\int_0^t \int_{[x,t]} \varphi'(u-x)F_{X_1}(x)dudx \\ &= -\frac{\lambda}{c}\int_0^t \varphi(t-x)F_{X_1}(x)dx.\end{aligned}$$

This implies

$$\varphi(t)-\varphi(0) = \frac{\lambda}{c}\int_0^t \varphi(t-x)(1-F_{X_1}(x))dx.$$

Using that

$$\mathbb{E}S(t) = \lambda t\mathbb{E}X_1 \quad \text{and} \quad ct = (1+\rho)\mathbb{E}S(t) = (1+\rho)\lambda t\mathbb{E}X_1$$

gives $\frac{\lambda}{c} = \frac{1}{(1+\rho)\mathbb{E}X_1}$ which yields the assertion. □

From the fundamental integral equation for survival probability we deduce a method for its computation. It can be considered as some sort of Monte-Carlo method where we have to simulate *independent* random variables $X_{I,1}, X_{I,2}, \ldots : \Omega \to [0,\infty)$ that have the distribution function $F_{X_{I,1}}$, i.e.

$$\mathbb{P}(X_{I,n} \le x) = F_{X_{I,1}}(x) \quad \text{for} \quad x \in \mathbb{R}.$$

Theorem 5.3.3. *We assume*

(1) *the Cramér-Lundberg-model with (NPC),*

(2) $\rho(t) = ct = (1+\rho)\mathbb{E}S(t)$.

(3) *independent random variables* $X_{I,1}, X_{I,2}, \ldots : \Omega \to [0,\infty)$ *that have the distribution function* $F_{X_{I,1}}$, *i.e.*

$$\mathbb{P}(X_{I,n} \le x) = F_{X_{I,1}}(x) \quad \textit{for} \quad x \in \mathbb{R}.$$

Define

$$f(x) := \frac{\rho}{1+\rho}\left[1 + \sum_{n=1}^{\infty}(1+\rho)^{-n}\mathbb{P}(X_{I,1} + ... + X_{I,n} \le x)\right] \quad \textit{for} \quad x \ge 0$$

and $f(x) = 0$ *if* $x < 0$. *Then* f *is the unique solution to*

$$f(x) = f(0) + \frac{1}{1+\rho}\int_0^x f(x-y)dF_{X_1,I}(y)$$

for $x \ge 0$ *in the class*

$$\mathcal{G} \;:=\; \Big\{ g : \mathbb{R} \to [0,\infty) : \textit{ non-decreasing, bounded,} \\ \textit{right-continuous with } g(x) = \left\{ \begin{array}{ll} 0 & \textit{for } x < 0 \\ \frac{\rho}{1+\rho} & \textit{for } x = 0 \end{array} \right\}.$$

Consequently, we obtain for the ruin probability $\varphi(u) = f(u)$ *for* $u \ge 0$.

Proof. (a) Uniqueness: Assume f_1, f_2 are solutions and $\Delta f = f_1 - f_2$. Then

$$\begin{aligned} \Delta f(x) &= \frac{1}{1+\rho}\int_0^x \Delta f(u-y)dF_{X_1,I}(y) \\ &= \frac{1}{1+\rho}\int_0^x \Delta f(u-y)\frac{\overline{F}_{X_1}(y)}{\mathbb{E}X_1}dy \\ &= \frac{1}{(1+\rho)\mathbb{E}X_1}\int_0^x \Delta f(y)\overline{F}_{X_1}(u-y)dy \end{aligned}$$

and

$$|\Delta f(x)| \le \frac{1}{(1+\rho)\mathbb{E}X_1}\int_0^x |\Delta f(y)|dy.$$

Gronwall's Lemma implies that $|\Delta f(x)| = 0$ for $x \in \mathbb{R}$.

(b) Verification that φ is a solution: Here we get

$$f(0) = \frac{\rho}{1+\rho}\left[1 + \sum_{n=1}^{\infty}(1+\rho)^{-n}\mathbb{P}(X_{I,1} + ... + X_{I,n} \le 0)\right] = \frac{\rho}{1+\rho}$$

and

$$\begin{aligned}
& f(0) + \frac{1}{1+\rho}\int_0^x f(x-y)dF_{X_1,I}(y) \\
= \; & \frac{\rho}{1+\rho} + \frac{1}{1+\rho}\int_0^x f(x-y)dF_{X_1,I}(y) \\
= \; & \frac{\rho}{1+\rho} + \frac{1}{1+\rho} \\
& \int_0^x \frac{\rho}{1+\rho}\left[1 + \sum_{n=1}^{\infty}(1+\rho)^{-n}\mathbb{P}\left(X_{I,1} + ... + X_{I,n} \le x - y\right)\right] dF_{X_1,I}(y) \\
= \; & \frac{\rho}{1+\rho} + \frac{1}{1+\rho} \\
& \frac{\rho}{1+\rho}\left[F_{X_1,I}(x) + \sum_{n=1}^{\infty}(1+\rho)^{-n}\int_0^x \mathbb{P}\left(X_{I,1} + ... + X_{I,n} + y \le x\right) dF_{X_1,I}(y)\right] \\
= \; & \frac{\rho}{1+\rho} + \frac{1}{1+\rho} \\
& \frac{\rho}{1+\rho}\left[F_{X_1,I}(x) + \sum_{n=1}^{\infty}(1+\rho)^{-n}\mathbb{P}\left(X_{I,1} + ... + X_{I,n+1} \le x\right)\right] \\
= \; & \frac{\rho}{1+\rho} + \frac{1}{1+\rho} \\
& \rho\left[(1+\rho)^{-1}\mathbb{P}(X_{I,1} \le x) + \sum_{n=1}^{\infty}(1+\rho)^{-(n+1)}\mathbb{P}\left(X_{I,1} + ... + X_{I,n+1} \le x\right)\right] \\
= \; & \frac{\rho}{1+\rho}\left[1 + \sum_{n=1}^{\infty}(1+\rho)^{-(n+1)}\mathbb{P}\left(X_{I,1} + ... + X_{I,n} \le x\right)\right] \\
= \; & f(x).
\end{aligned}$$

(c) Finally we observe that the ruin probability φ belongs to the class $\mathcal{G}$ which completes our proof. □

5.4 Proof of Cramér's ruin bound

To prove Cramér's ruin bound we transform the fundamental integral equation for the survival probability into the fundamental integral equation for the ruin probability, however after performing the Esscher transform to the integrated claim size distribution with the Lundberg coefficient as parameter. So let us first explain the notion of the Esscher transform:

Definition 5.4.1 (Esscher transform)**.** Assume that $D : \mathbb{R} \to [0,\infty)$ is the density of a probability measure μ, i.e.

$$\mu(B) = \int_B D(x)dx \quad \text{for} \quad B \in \mathcal{B}(\mathbb{R}).$$

If $h \in \mathbb{R}$ and $\int_{\mathbb{R}} e^{hx} D(x)dx < \infty$, then

$$D^{(h)}(x) := \frac{e^{hx} D(x)}{\int_{\mathbb{R}} e^{hy} D(y)dy}$$

defines a density, which is called the **Esscher transform** of D with parameter h. We let

$$\mu^{(h)}(B) := \int_B D^{(h)}(x)dx \quad \text{for} \quad B \in \mathcal{B}(\mathbb{R}).$$

If $F, F^{(h)} : \mathbb{R} \to [0,1]$ are the distribution functions of μ and $\mu^{(h)}$, respectively, then we call $F^{(h)}$ the Esscher transform of F.

Now we get the following transformed fundamental integral equation:

Theorem 5.4.2 (Smith's renewal equation)**.** *We assume*

(1) *the Cramér-Lundberg-model with (NPC),*

(2) *the small claim size condition with parameter $h_0 > 0$,*

(3) *$r \in (0, h_0)$ is the Lundberg coefficient,*

(4) *$\rho(t) = ct = (1+\rho)\mathbb{E}S(t)$.*

Then one has:

(i) *The function $F_{X_1}^{(r)} : \mathbb{R} \to [0,1]$ with*

$$F_{X_1}^{(r)}(x) := \begin{cases} \int_0^x \frac{e^{ry}}{(1+\rho)\mathbb{E}X_1} \mathbb{P}(X_1 > y)dy & : x \geq 0, \\ 0 & : x < 0, \end{cases}$$

is the Esscher transform of F_{X_1} with parameter r.

(ii) *For $x \geq 0$ it holds*

$$e^{rx}\psi(x) = \frac{1}{1+\rho} e^{rx}(1 - F_{X_1,I}(x)) + \int_0^x e^{r(x-y)}\psi(x-y)dF_{X_1}^{(r)}(y).$$

Proof. (a) $F_{X_1}^{(r)}$ is a distribution function: First we compute

$$\begin{aligned} \mathbb{E}e^{rX_1} &= \int_0^\infty \mathbb{P}(e^{rX_1} > z)dz \\ &= \int_{-\infty}^\infty \mathbb{P}(e^{rX_1} > e^{ry}) re^{ry} dy \\ &= \int_{-\infty}^0 re^{ry}dy + \int_0^\infty \mathbb{P}(X_1 > y)re^{ry}dy \\ &= 1 + \int_0^\infty \mathbb{P}(X_1 > y)re^{ry}dy. \end{aligned}$$

This implies

$$\lim_{x\to\infty} F_{X_1}^{(r)}(x) = \int_0^\infty \frac{e^{ry}}{(1+\rho)\mathbb{E}X_1}\mathbb{P}(X_1 > y)dy = \frac{1}{(1+\rho)\mathbb{E}X_1}\frac{1}{r}(\mathbb{E}e^{rX_1} - 1).$$

From $\mathbb{E}e^{r(X_1 - cW_1)} = 1$ we conclude

$$\begin{aligned}
\lim_{x\to\infty} F_{X_1}^{(r)}(x) &= \frac{1}{(1+\rho)\mathbb{E}X_1}\frac{1}{r}\left(\frac{1}{\mathbb{E}e^{-rcW_1}} - 1\right) \\
&= \frac{1}{(1+\rho)\mathbb{E}X_1}\frac{1}{r}\left(\frac{rc+\lambda}{\lambda} - 1\right) \\
&= \frac{c}{(1+\rho)\mathbb{E}X_1}\frac{1}{\lambda} = \frac{c}{(1+\rho)\mathbb{E}X_1}\frac{1}{\lambda} = 1.
\end{aligned}$$

(b) Inserting the definitions of $dF_{X_1}^{(r)}(y)$ we have to show

$$e^{rx}\psi(x) = \frac{1}{1+\rho}e^{rx}(1-F_{X_1,I}(x)) + \int_0^x e^{r(x-y)}\psi(x-y)\frac{e^{ry}}{(1+\rho)\mathbb{E}X_1}\mathbb{P}(X_1 > y)dy.$$

Using $e^{r(x-y)}e^{ry} = e^{rx}$ and dividing the equation yields to

$$\psi(x) = \frac{1}{1+\rho}(1 - F_{X_1,I}(x)) + \int_0^x \psi(x-y)\frac{1}{(1+\rho)\mathbb{E}X_1}\mathbb{P}(X_1 > y)dy,$$

what we have to prove. We use $\psi(x) = 1 - \varphi(x)$ so that the equation becomes

$$1 - \varphi(x) = \frac{1}{1+\rho}(1 - F_{X_1,I}(x)) + \int_0^x (1 - \varphi(x-y))\frac{1}{(1+\rho)\mathbb{E}X_1}\mathbb{P}(X_1 > y)dy.$$

Replacing $\varphi(x)$ on the left-hand side by the expression from Theorem 5.3.1 we get

$$1 - \varphi(0) = \frac{1}{1+\rho}(1 - F_{X_1,I}(x)) + \int_0^x \frac{1}{(1+\rho)\mathbb{E}X_1}\mathbb{P}(X_1 > y)dy.$$

By the definition of $F_{X_1,I}(x)$ this becomes

$$1 - \varphi(0) = \frac{1}{1+\rho}$$

which is true because of $\varphi(0) = \frac{\rho}{1+\rho}$. □

Proof of Theorem 5.2.7. From Theorem 5.4.2 we know that

$$e^{rx}\psi(x) = \frac{1}{1+\rho}e^{rx}(1 - F_{X_1,I}(x)) + \int_0^x e^{r(x-y)}\psi(x-y)dF_{X_1}^{(r)}(y)$$

for $x \geq 0$. With the notation

$$R(x) := e^{rx}\psi(x),$$

$$\begin{aligned} k(x) &:= \frac{e^{rx}}{1+\rho}\bar{F}_{X_1,I}(x), \\ H(y) &:= F_{X_1}^{(r)}(y) \end{aligned}$$

this equation turns into

$$R(x) = k(x) + \int_0^x R(x-y)dH(y).$$

The function $k : [0,\infty) \to [0,\infty)$ is continuous because $x \mapsto F_{X_1,I}(x)$ is continuous. Moreover, from Theorem 5.4.2 we have

$$F_{X_1}^{(r)}(x) = \int_0^x \frac{e^{ry}}{(1+\rho)\mathbb{E}X_1}\mathbb{P}(X_1 > y)dy \to 1$$

as $x \to \infty$, so that

$$k(x) = \frac{e^{rx}}{(1+\rho)\mathbb{E}X_1}\int_x^\infty \mathbb{P}(X_1 > y)dy \le \int_x^\infty \frac{e^{ry}}{(1+\rho)\mathbb{E}X_1}\mathbb{P}(X_1 > y)dy \to 0$$

for $x \to \infty$ with $x \ge 0$. From Smith's key renewal lemma [9, pp. 202] we know that

$$\lim_{u\to\infty} R(u) = \frac{1}{\int_{\mathbb{R}} xdH(x)}\int_0^\infty k(x)dx.$$

Therefore, with $\alpha := \int_{\mathbb{R}} xdF_{X_1}^{(r)}(x)$, we get

$$\begin{aligned} \lim_{u\to\infty} e^{ru}\psi(u) &= \frac{1}{\alpha}\int_0^\infty \frac{e^{rx}}{1+\rho}\bar{F}_{X_1,I}(x)dx \\ &= \frac{1}{\alpha}\int_0^\infty \frac{e^{rx}}{1+\rho}\left[1 - \frac{1}{\mathbb{E}X_1}\int_0^x \bar{F}_{X_1}(y)dy\right]dx \\ &= \frac{1}{\alpha}\int_0^\infty \frac{e^{rx}}{1+\rho}\frac{1}{\mathbb{E}X_1}\int_x^\infty \bar{F}_{X_1}(y)dydx \\ &= \frac{1}{\alpha(1+\rho)\mathbb{E}X_1}\int_0^\infty\int_0^y e^{rx}dx\bar{F}_{X_1}(y)dy \\ &= \frac{1}{\alpha(1+\rho)}\frac{1}{r}\left[\frac{\int_0^\infty e^{ry}\bar{F}_{X_1}(y)dy}{\mathbb{E}X_1} - \frac{\int_0^\infty \bar{F}_{X_1}(y)dy}{\mathbb{E}X_1}\right] \\ &= \frac{1}{\alpha(1+\rho)}\frac{1}{r}\left[(1+\rho) - 1\right] \\ &= \frac{1}{\alpha r}\frac{\rho}{1+\rho}, \end{aligned}$$

where for $\frac{\int_0^\infty e^{ry}\bar{F}_{X_1}(y)dy}{\mathbb{E}X_1} = 1+\rho$ one looks at the proof of Theorem 5.4.2. Finally

$$\alpha = \int_{\mathbb{R}} xdF^{(r)}(x) = \frac{1}{(1+\rho)\mathbb{E}X_1}\int_0^\infty xe^{rx}\bar{F}_{X_1}(x)dx$$

implies

$$\lim_{u\to\infty} e^{ru}\psi(u) = \frac{\rho\mathbb{E}X_1}{r}\frac{1}{\int_0^\infty xe^{rx}\bar{F}_{X_1}(x)dx}.$$

□

Chapter 6

Probability of ruin: large claim sizes

6.1 Tails of claim size distributions

So far we did not discuss how to separate small and large claim sizes, and how to choose the distributions to model the claim sizes (X_i)? If one analyzes data of claim sizes that have happened in the past, for example by a histogram or a QQ-plot (see Chapter 7), it turns out that the distribution is either *light-tailed* or *heavy-tailed*, the latter case is more often the case. Let us recall that for probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and we associate to a random variable $X : \Omega \to \mathbb{R}$ the distribution function $F_X : \mathbb{R} \to [0, 1]$,

$$F_X(x) := \mathbb{P}(\{\omega \in \Omega : X(\omega) \leq x\}).$$

Definition 6.1.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X : \Omega \to \mathbb{R}$ be a random variable with $f(\omega) \geq 0$ for all $\omega \in \Omega$.

(1) The distribution function F_X or the random variable X is called **light-tailed** if and only if there is some $\alpha > 0$ such that one has

$$\sup_{u \geq 0} e^{\alpha x} \mathbb{P}(X > x) = \sup_{x \geq 0} e^{\alpha x} [1 - F_X(x)] < \infty.$$

(2) The distribution function F_X or the random variable X is called **heavy-tailed** if and only if for all $\alpha > 0$ one has

$$\lim_{x \to \infty} e^{\alpha x} \mathbb{P}(X > x) = \lim_{x \to \infty} e^{\alpha u} [1 - F_X(x)] = \infty.$$

Remark 6.1.2. The distribution function F_X or the random variable X is heavy-tailed if and only if for all $\beta > 0$ one has

$$\inf_{x \geq 0} e^{\beta x} \mathbb{P}(X > x) = \inf_{x \geq 0} e^{\beta x} [1 - F_X(x)] > 0.$$

In fact, assume for X the latter condition and $\alpha > 0$. Choose $\beta \in (0, \alpha)$. Then

$$\begin{aligned}\liminf_{x\to\infty} e^{\alpha x}\mathbb{P}(X > x) &= \liminf_{x\to\infty} e^{(\alpha-\beta)x}[e^{\beta}\mathbb{P}(X > x) \\ &\geq \liminf_{x\to\infty} e^{(\alpha-\beta)x} \inf_{y\geq 0}[e^{\beta y}\mathbb{P}(X > y)] = \infty.\end{aligned}$$

Therefore X is heavy-tailed. Conversely, assume X to be heavy-tailed. Then $\inf_{x\geq 0} e^{\alpha x}\mathbb{P}(X > x) > 0$ follows from $\lim_{x\to\infty} e^{\alpha x}\mathbb{P}(X > x) = \infty$.

Example 6.1.3. (1) The exponential distribution with parameter $\lambda > 0$ is light-tailed, since for $\alpha := \lambda$ one has

$$e^{\alpha x}\mathbb{P}(X > x) = e^{\alpha x}e^{-\lambda x}\mathbb{P}(X > x) = 1$$

if X is a random variable with an exponential distribution with parameter $\lambda > 0$.

(2) The **Pareto distribution** is heavy-tailed: For Type I the distribution function is

$$F_{\alpha,b}(x) = 1 - \frac{b^a}{x^a} \quad \text{for} \quad x \geq b > 0, a > 0,$$

and for for Type II,

$$F_{\alpha,\kappa}(x) = 1 - \frac{\kappa^\alpha}{(\kappa + x)^\alpha} \quad \text{for} \quad x \geq 0,\ \alpha > 0, \kappa > 0.$$

Proposition 6.1.4. *If $X : \Omega \to \mathbb{R}$ is light-tailed, then there is an $h_0 > 0$, such that the moment generating function*

$$h \to \mathbb{E}e^{hX}$$

is finite on $(-h_0, h_0)$.

Proof. It is sufficient to find an $h_0 > 0$ such that $\mathbb{E}e^{h_0 X} < \infty$. We get

$$\begin{aligned}\mathbb{E}e^{h_0 X} &= \int_0^\infty \mathbb{P}(e^{h_0 X} > u)du \\ &= 1 + \int_{(1,\infty)} \mathbb{P}(e^{h_0 X} > u)du \\ &= 1 + \int_{(1,\infty)} \mathbb{P}\left(X > \frac{1}{h_0}\log(u)\right)du \\ &= 1 + \int_{(1,\infty)} \left[1 - F\left(\frac{1}{h_0}\log(u)\right)\right]du \\ &\leq 1 + C\int_{(1,\infty)} \left[e^{-\lambda_0\left(\frac{1}{h_0}\log(u)\right)}\right]du\end{aligned}$$

$$\begin{aligned} &\le 1 + C\int_{(1,\infty)} \left[u^{-\frac{\lambda_0}{h_0}}\right] du \\ &< \infty \end{aligned}$$

for $0 < h_0 < \lambda_0$. □

Proposition 6.1.4 means that a light-tailed claim size distribution satisfies the small claim size condition from Definition 5.2.3.

6.2 Subexponential distributions

6.2.1 Definition and basic properties

In Theorem 6.3.1 below we need subexponential distributions:

Definition 6.2.1. A distribution function $F : \mathbb{R} \to [0,1]$ such that $F(0) = 0$ and $F(x) < 1$ for all $x > 0$ is called **subexponential** if and only if for i.i.d. $(X_i)_{i=1}^\infty$ and $\mathbb{P}(X_i \le u) = F(u)$, $u \in \mathbb{R}$, it holds that

$$\lim_{x\to\infty} \frac{\mathbb{P}(X_1 + \cdots + X_n > x)}{\mathbb{P}(\max_{1\le k\le n} X_k > x)} = 1 \quad \text{for all} \quad n \ge 2.$$

We denote the class of subexponential distribution functions by $\mathcal{S}$.

We start with an equivalence that can be used to define $\mathcal{S}$ as well.

Proposition 6.2.2 (Equivalent conditions for $\mathcal{S}$, part I)**.** *Assume i.i.d.* $(X_i)_{i=1}^\infty$ *with* $\mathbb{P}(X_i \le u) = F(u)$, $u \in \mathbb{R}$. *Then* $F \in \mathcal{S}$ *if and only if*

$$\lim_{x\to\infty} \frac{\mathbb{P}(X_1 + \cdots + X_n > x)}{\mathbb{P}(X_1 > x)} = n \quad \textit{for all} \quad n \ge 1.$$

Proof. It holds for $S_n = X_1 + \cdots + X_n$ that

$$\begin{aligned} \frac{\mathbb{P}(S_n > x)}{\mathbb{P}(\max_{1\le k\le n} X_k > x)} &= \frac{\mathbb{P}(S_n > x)}{1 - \mathbb{P}(\max_{1\le k\le n} X_k \le x)} \\ &= \frac{\mathbb{P}(S_n > x)}{1 - \mathbb{P}(X_1 \le x)^n} \\ &= \frac{\mathbb{P}(S_n > x)}{1 - (1 - \mathbb{P}(X_1 > x))^n} \\ &= \frac{\mathbb{P}(S_n > x)}{\mathbb{P}(X_1 > x) n (1 + o(1))}. \end{aligned}$$

□

Next we continue with properties of subexponential distributions that motivate the name *subexponential*:

Proposition 6.2.3. *Assume i.i.d. random variables* $(X_i)_{i=1}^{\infty}$ *and a random variable* X *with* $\mathbb{P}(X \le u) = \mathbb{P}(X_i \le u) = F(u)$, $u \in \mathbb{R}$. *Then one has the following assertions:*

(1) *For* $F \in \mathcal{S}$ *it holds*

$$\lim_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} = 1 \quad \textit{for all} \quad y > 0.$$

(2) *If* $F \in \mathcal{S}$, *then for all* $\varepsilon > 0$ *it holds*

$$e^{\varepsilon x}\mathbb{P}(X > x) \to \infty \textit{ for } x \to \infty.$$

(3) *If* $F \in \mathcal{S}$, *then for all* $\varepsilon > 0$ *there exists a* $K > 0$ *such that*

$$\frac{\mathbb{P}(S_n > x)}{\mathbb{P}(X_1 > x)} \le K(1+\varepsilon)^n \quad \forall n \ge 2 \textit{ and } x \ge 0.$$

For the proof we need the concept of *slowly varying functions*:

Definition 6.2.4 (Slowly varying functions)**.** A measurable function $L : [0,\infty) \to (0,\infty)$ is called **slowly varying** if

$$\lim_{\xi\to\infty} \frac{L(c\xi)}{L(\xi)} = 1 \quad \text{for all} \quad c > 0.$$

Proposition 6.2.5 (Karamata's representation)**.** *Any slowly varying function can be represented as*

$$L(\xi) = c_0(\xi) \exp\left(\int_{\xi_0}^{\xi} \frac{\varepsilon(t)}{t} dt \right) \quad \textit{for all} \quad \xi \ge \xi_0$$

for some $\xi_0 > 0$ *where* $c_0, \varepsilon : [\xi_0, \infty) \to \mathbb{R}$ *are measurable functions with*

$$\lim_{\xi\to\infty} c_0(\xi) = c_0 > 0 \quad \textit{and} \quad \lim_{t\to\infty} \varepsilon(t) = 0.$$

Corollary 6.2.6. *For any slowly varying function* L *it holds*

$$\lim_{\xi\to\infty} \xi^{\delta} L(\xi) = \infty \quad \textit{for all} \quad \delta > 0.$$

Proof. We can enlarge ξ_0 such that $\sup_{t\ge\xi_0} |\varepsilon(t)| < \delta$. With this choice we get

$$\begin{aligned} & \lim_{\xi\to\infty} \xi^{\delta} c_0(\xi) \exp\left(\int_{\xi_0}^{\xi} \frac{\varepsilon(t)}{t} dt \right) \\ = & \lim_{\xi\to\infty} c_0(\xi) \exp\left(\delta \log \xi + \int_{\xi_0}^{\xi} \frac{\varepsilon(t)}{t} dt \right) \end{aligned}$$

$$
\begin{aligned}
&\geq \lim_{\xi\to\infty} c_0(\xi)\exp\left(\delta\log\xi - \sup_{t\geq\xi_0}|\varepsilon(t)|\int_{\xi_0}^{\xi}\frac{dt}{t}\right)\\
&\geq \lim_{\xi\to\infty} c_0(\xi)\exp\left(\delta\log\xi - (\log\xi-\log\xi_0)\sup_{t\geq\xi_0}|\varepsilon(t)|\right)\\
&= \infty. \quad \square
\end{aligned}
$$

Proof of Proposition 6.2.3. (1) For $0 \leq y \leq x < \infty$ we have

$$
\begin{aligned}
\frac{\mathbb{P}(X_1+X_2>x)}{\mathbb{P}(X_1>x)} &= \frac{\int_{\mathbb{R}}\mathbb{P}(t+X>x)dF(t)}{\mathbb{P}(X_1>x)}\\
&= \frac{\mathbb{P}(X_1>x)+\int_{(-\infty,y]}\mathbb{P}(t+X>x)dF(t)}{\mathbb{P}(X_1>x)}\\
&\quad + \frac{\int_{(y,x]}\mathbb{P}(t+X>x)dF(t)}{\mathbb{P}(X_1>x)}\\
&\geq 1+F(y)+\frac{\overline{F}(x-y)}{\overline{F}(x)}(F(x)-F(y)).
\end{aligned}
$$

We choose x large enough such that $F(x)-F(y)>0$ and observe that

$$1 \leq \frac{\overline{F}(x-y)}{\overline{F}(x)} \leq \left(\frac{\mathbb{P}(X_1+X_2>x)}{\mathbb{P}(X_1>x)} - 1 - F(y)\right)\frac{1}{F(x)-F(y)} \to 1$$

as $x\to\infty$.

(2) Let $L(\xi) := \overline{F}(\log\xi)$. It follows from (1) that for all $c>0$ one has

$$\lim_{\xi\to\infty}\frac{L(c\xi)}{L(\xi)} = \lim_{\xi\to\infty}\frac{\overline{F}(\log c+\log\xi)}{\overline{F}(\log\xi)} = 1.$$

By definition, L is slowly varying. Therefore,

$$\lim_{\xi\to\infty}\xi^{\delta}\overline{F}(\log\xi) = \lim_{x\to\infty}e^{\delta x}\overline{F}(x) = \infty$$

where we use Corollary 6.2.6.

(3) The proof can be found in [5][Lemma 1.3.5]. $\square$

6.2.2 Examples

In order to consider fundamental examples we need the next lemma:

Lemma 6.2.7. *Let X_1, X_2 be independent positive random variables such that for some $\alpha > 0$*

$$\overline{F}_{X_i}(x) = \frac{L_i(x)}{x^{\alpha}}$$

where L_1, L_2 are slowly varying. Then

$$\overline{F}_{X_1+X_2}(x) = x^{-\alpha}(L_1(x)+L_2(x))(1+o(1)).$$

Proof. For $0 < \delta < \frac{1}{2}$ we have

$$\begin{aligned}\{X_1 + X_2 > x\} &\subseteq \{X_1 > (1-\delta)x\} \cup \{X_2 > (1-\delta)x\} \\ &\quad \cup \{X_1 > \delta x, X_2 > \delta x\}\end{aligned}$$

and hence

$$\begin{aligned}
& \mathbb{P}(X_1 + X_2 > x) \\
\leq\ & \overline{F}_{X_1}((1-\delta)x) + \overline{F}_{X_2}((1-\delta)x) + \overline{F}_{X_1}(\delta x)\overline{F}_{X_2}(\delta x) \\
\leq\ & [\overline{F}_{X_1}((1-\delta)x) + \overline{F}_{X_2}((1-\delta)x)]\left[1 + \overline{F}_{X_1}(\delta x)\frac{\overline{F}_{X_2}(\delta x)}{\overline{F}_{X_2}((1-\delta)x)}\right] \\
=\ & [\overline{F}_{X_1}((1-\delta)x) + \overline{F}_{X_2}((1-\delta)x)][1 + o(1)] \\
=\ & \left[\frac{L_1((1-\delta)x)}{((1-\delta)x)^\alpha} + \frac{L_2((1-\delta)x)}{((1-\delta)x)^\alpha}\right][1 + o(1)] \\
=\ & \left[\overline{F}_{X_1}(x)\frac{L_1((1-\delta)x)}{L_1(x)} + \overline{F}_{X_2}(x)\frac{L_2((1-\delta)x)}{L_2(x)}\right][1 + o(1)](1-\delta)^{-\alpha}.
\end{aligned}$$

From this we get

$$\begin{aligned}
\limsup_{x\to\infty} \frac{\mathbb{P}(X_1 + X_2 > x)}{\overline{F}_{X_1}(x) + \overline{F}_{X_1}(x)} &= \limsup_{x\to\infty} \frac{\mathbb{P}(X_1 + X_2 > x)}{\overline{F}_{X_1}(x)\frac{L_1((1-\delta)x)}{L_1(x)} + \overline{F}_{X_2}(x)\frac{L_2((1-\delta)x)}{L_2(x)}} \\
&\leq (1-\delta)^{-\alpha}.
\end{aligned}$$

As this is true for all $0 < \delta < \frac{1}{2}$, we may conclude

$$\limsup_{x\to\infty} \frac{\mathbb{P}(X_1 + X_2 > x)}{\overline{F}_{X_1}(x) + \overline{F}_{X_1}(x)} \leq 1.$$

On the other hand,

$$\begin{aligned}
\mathbb{P}(X_1 + X_2 > x) &\geq \mathbb{P}(\{X_1 > x\} \cup \{X_2 > x\}) \\
&= \mathbb{P}(X_1 > x) + \mathbb{P}(X_2 > x) - \mathbb{P}(X_1 > x)\mathbb{P}(X_2 > x) \\
&= \overline{F}_{X_1}(x) + \overline{F}_{X_2}(x) - \overline{F}_{X_1}(x)\overline{F}_{X_2}(x) \\
&\geq \left[\overline{F}_{X_1}(x) + \overline{F}_{X_2}(x)\right]\left[1 - \overline{F}_{X_1}(x)\right]
\end{aligned}$$

and hence

$$\liminf_{x\to\infty} \frac{\mathbb{P}(X_1 + X_2 > x)}{\overline{F}_{X_1}(x) + \overline{F}_{X_2}(x)} \geq 1.$$

Consequently,

$$\lim_{x\to\infty} \frac{\mathbb{P}(X_1 + X_2 > x)}{\overline{F}_{X_1}(x) + \overline{F}_{X_2}(x)} = 1.$$

□

Definition 6.2.8. If there exists a slowly varying function L and some $\alpha > 0$ such that for a positive random variable X it holds

$$\overline{F}_X(x) = \frac{L(x)}{x^\alpha},$$

then F_X is called **regularly varying** with index α or of Pareto type with exponent α.

Proposition 6.2.9. *If F_X is regularly varying with index α, then F_X is subexponential.*

Proof. An iteration of Lemma 6.2.7 implies

$$\frac{\overline{F}_{X_1+...+X_n}(x)}{\overline{F}_X(x)} \sim \frac{L(x)+...+L(x)}{L(x)} = n.$$

□

Example 6.2.10. (1) The exponential distribution with parameter $\lambda > 0$ is not subexponential.

(2) The Pareto distribution

$$F(x) = 1 - \frac{\kappa^\alpha}{(\kappa+x)^\alpha}, \; x \geq 0, \; \alpha > 0, \kappa > 0$$

is subexponential.

(3) The Weibull distribution

$$F(x) = 1 - e^{-cx^r}, \quad 0 < r < 1, x \geq 0,$$

is subexponential.

Proof. (1) The relation (1) of Proposition 6.2.3 is not satisfied.

(2) We define $L(x)$ by

$$\mathbb{P}(X > x) = \frac{1}{x^\alpha}\frac{(x\kappa)^\alpha}{(\kappa+x)^\alpha} =: \frac{1}{x^\alpha}L(x)$$

and conclude

$$\begin{aligned} \frac{L(cx)}{L(x)} &= \left(\frac{cx\kappa}{\kappa+cx}\frac{x+\kappa)}{\kappa x}\right)^\alpha \\ &= \left(c\frac{x+\kappa}{\kappa+cx}\right)^\alpha \to 1 \quad \text{for } x \to \infty. \end{aligned}$$

Now we apply Proposition 6.2.9.

(3) See [5, Sections 1.4.1 and A3.2]. □

6.2.3 Another characterization of subexponential distributions

There is the following extension Proposition 6.2.2:

Proposition 6.2.11. *Assume independent positive random variables* $X_1, X_2 : \Omega \to (0,\infty)$ *such that*

$$\mathbb{P}(X_1 \le x) = \mathbb{P}(X_2 \le x) = F(x) \quad \text{for all} \quad x \in \mathbb{R}.$$

Then the following assertions are equivalent:

(1) $F \in \mathcal{S}$.

(2) $\lim_{x\to\infty} \frac{\mathbb{P}(X_1+X_1>x)}{\mathbb{P}(\max\{X_1,X_2\}>x)} = 1$.

(3) $\lim_{x\to\infty} \frac{\mathbb{P}(X_1+X_1>x)}{\mathbb{P}(X_1>x)} = 2$.

Proof. We only show (1) $\iff$ (3). The implication (1) $\iff$ (3) follows from Proposition 6.2.2. To check the implication (3) $\iff$ (1) we show by induction that

$$\lim_{x\to\infty} \frac{\mathbb{P}(X_1 + \cdots + X_n > x)}{\mathbb{P}(X_1 > n)} = n$$

implies

$$\lim_{x\to\infty} \frac{\mathbb{P}(X_1 + \cdots + X_{n+1} > x)}{\mathbb{P}(X_1 > n)} = n+1.$$

Then we start with $n = 2$, which is true according to Proposition 6.2.2, and get the assertion for all $n \ge 2$. So we assume that

$$\lim_{x\to\infty} \frac{\mathbb{P}(X_1 + \cdots + X_n > x)}{\mathbb{P}(X_1 > n)} = n.$$

Hence there exists for all $\varepsilon \in (0,n)$ an $x_0 > 0$ such that

$$(n-\varepsilon)\mathbb{P}(X_1 > x) \le \mathbb{P}(X_1 + \cdots + X_n > x) \le (n+\varepsilon)\mathbb{P}(X_1 > x)$$

for $x \ge x_0$. For the following computation we remark that for $y \ge 0$ one has $x - y \ge x_0$ if and only if $0 \le y \le x - x_0$. We estimate

$$\begin{aligned} & \frac{\mathbb{P}(X_1 + \cdots + X_n + X_{n+1} > x)}{\mathbb{P}(X_1 > x)} \\ = \quad & 1 + \frac{\int_0^x \mathbb{P}(X_1 + \cdots + X_n > x - y) dF_X(y)}{\mathbb{P}(X_1 > x)} \\ = \quad & 1 + \frac{\int_0^{x-x_0} \mathbb{P}(X_1 + \cdots + X_n > x - y) dF_X(y)}{\mathbb{P}(X_1 > x)} \\ & + \frac{\int_{x-x_0}^x \mathbb{P}(X_1 + \cdots + X_n > x - y) dF_X(y)}{\mathbb{P}(X_1 > x)} \end{aligned}$$

$$\leq \quad 1+(n+\varepsilon)\frac{\int_0^{x-x_0}\mathbb{P}(X_1>x-y)dF_X(y)}{\mathbb{P}(X_1>x)} + \frac{\int_{x-x_0}^{x}dF_X(y)}{\mathbb{P}(X_1>x)}$$

One can show that (see Proposition 6.2.3) that

$$\lim_{x\to\infty}\frac{\int_{x-x_0}^{x}dF_X(y)}{\mathbb{P}(X_1>x)} = \lim_{x\to\infty}\frac{\mathbb{P}(X_1>x)-\mathbb{P}(X_1>x-x_0)}{\mathbb{P}(X_1>x)} = 0.$$

Moreover,

$$\begin{aligned}\lim_{x\to\infty}\frac{\int_0^{x}\mathbb{P}(X_1>x-y)dF_X(y)}{\mathbb{P}(X_1>x)} &= \lim_{x\to\infty}\frac{\mathbb{P}(X_1+X_2>x)}{\mathbb{P}(X_1>x)}-1\\ &= 2-1=1\end{aligned}$$

which implies

$$\lim_{x\to\infty}\frac{\mathbb{P}(X_1+\cdots+X_n+X_{n+1}>x)}{\mathbb{P}(X_1>x)} \leq n+1.$$

The other inequality can be shown similarly. □

6.3 An asymptotics for the ruin probability for large claim sizes

We proceed with the main result of this chapter.

Theorem 6.3.1. *We assume*

(1) *the Cramér-Lunberg-model with (NPC),*

(2) $\rho(t)=ct=(1+\rho)\mathbb{E}S(t)$,

(3) *that the distribution function*

$$F_{X_1,I}(y) := \frac{1}{\mathbb{E}X_1}\int_0^y \overline{F}_{X_1}(x)dx \quad \textit{with} \quad y\geq 0$$

is subexponential.

Then one has

$$\lim_{u\to\infty}\frac{\psi(u)}{1-F_{X_1,I}(u)} = \frac{1}{\rho}.$$

Remark 6.3.2. (1) The left-hand side of the assertion of Theorem 6.3.1 can be written more explicitly as

$$\lim_{u\to\infty}\frac{\psi(u)}{1-F_{X_1,I}(u)}=\lim_{u\to\infty}\frac{\psi(u)}{\int_u^\infty \mathbb{P}(X_1>x)dx}.$$

(2) The right-hand side of the assertion of Theorem 6.3.1 relates to the NP condition as follows: As we use the Cramér-Lunberg model and as premium principle the expected value principle, we have (using Proposition 3.2.2)

$$\rho(t)=(1+\rho)\mathbb{E}S_t=(1+\rho)\mathbb{E}N(t)\mathbb{E}X_1=(1+\rho)\lambda t\mathbb{E}X_1=(1+\rho)\frac{\mathbb{E}X_1}{\mathbb{E}W_1}t.$$

so that $p(t)=ct$ with

$$c=(1+\rho)\frac{\mathbb{E}X_1}{\mathbb{E}W_1}.$$

On the other hand the NP condition holds if and only if

$$\mathbb{E}X_1-c\mathbb{E}W_1<0.$$

Now we get

$$\rho=c\frac{\mathbb{E}W_1}{\mathbb{E}X_1}-1=\frac{c\mathbb{E}W_1-\mathbb{E}X_1}{\mathbb{E}X_1}\quad\text{and}\quad\frac{1}{\rho}=\frac{\mathbb{E}X_1}{c\mathbb{E}W_1-\mathbb{E}X_1}.$$

This means, the larger the 'overshoot' $c\mathbb{E}W_1-\mathbb{E}X_1$ is, the smaller gets the factor $\frac{1}{\rho}$ in Theorem 6.3.1.

(3) Summarizing, one can also write

$$\lim_{u\to\infty}\frac{\psi(u)}{\int_u^\infty \mathbb{P}(X_1>x)dx}=\frac{\mathbb{E}X_1}{c\mathbb{E}W_1-\mathbb{E}X_1}$$

if the premium rate is $\rho(t)=ct$.

Proof of Theorem 6.3.1. From Theorem 5.3.1 we know that the survival probability solves

$$\varphi(u)=\varphi(0)+\frac{1}{(1+\rho)\mathbb{E}X_1}\int_0^u\varphi(u-y)d\overline{F}_{X_1,I}(y)$$

The function φ is bounded, non-decreasing and right-continuous, since

$$\varphi(u)=\mathbb{P}(\sup_{k\geq 1}G_k\leq u).$$

Therefore we can apply Theorem 5.3.3 and get

$$\varphi(u)=\frac{\rho}{1+\rho}\left[1+\sum_{n=1}^\infty(1+\rho)^{-n}\mathbb{P}(X_{I,1}+...+X_{I,n}\leq u)\right]$$

and

$$\psi(u) = \frac{\rho}{1+\rho}\sum_{n=1}^{\infty}(1+\rho)^{-n}\mathbb{P}(X_{I,1}+...+X_{I,n}>u)$$

since

$$\frac{\rho}{1+\rho}\sum_{n=0}^{\infty}(1+\rho)^{-n} = 1.$$

Hence

$$\frac{\psi(u)}{\overline{F}_{X_1,I}(u)} = \frac{\rho}{1+\rho}\sum_{n=1}^{\infty}(1+\rho)^{-n}\frac{\mathbb{P}(X_{I,1}+...+X_{I,n}>u)}{\overline{F}_{X_1,I}(u)}$$

By assumption,

$$\lim_{n\to\infty}\frac{\mathbb{P}(X_{I,1}+...+X_{I,n}>u)}{\overline{F}_{X_1,I}(u)} = n.$$

In order to be able to exchange summation and limit, we will use the estimate of Proposition 6.2.3

$$\frac{\mathbb{P}(X_{I,1}+...+X_{I,n}>u)}{\overline{F}_{X_1,I}(u)} \le K(1+\varepsilon)^n.$$

For $\varepsilon \in (0,\rho)$ we have

$$\frac{\rho}{1+\rho}\sum_{n=0}^{\infty}(1+\rho)^{-n}K(1+\varepsilon)^n = K\frac{\rho}{1+\rho}\sum_{n=0}^{\infty}\left(\frac{1+\varepsilon}{1+\rho}\right)^n < \infty.$$

Therefore we obtain by dominated convergence, that

$$\begin{aligned}\lim_{u\to\infty}\frac{\psi(u)}{\overline{F}_{X_1,I}(u)} &= \frac{\rho}{1+\rho}\sum_{n=1}^{\infty}(1+\rho)^{-n}\lim_{u\to\infty}\frac{\mathbb{P}(X_{I,1}+...+X_{I,n}>u)}{\overline{F}_{X_1,I}(u)}\\ &= \frac{\rho}{1+\rho}\sum_{n=1}^{\infty}(1+\rho)^{-n}n = \frac{1}{\rho}.\end{aligned}$$

□

6.4 Conditions for $F_{X,I} \in \mathcal{S}$

The main condition in Theorem 6.3.1 consists in $F_{X,I} \in \mathcal{S}$. For this reason we introduce the class $\mathcal{S}^*$ and show that $F \in \mathcal{S}^*$ implies that $F_{X,I} \in \mathcal{S}$.

Definition 6.4.1. A positive random variable X with distribution function F_X belongs to $\mathcal{S}^*$ if and only if

(1) $\mathbb{E}X = \mu \in (0, \infty)$,

(2) $\lim_{x\to\infty} \int_0^x \frac{\overline{F_X}(x-y)}{\overline{F_X}(x)} \overline{F_X}(y)dy = 2\mu$.

Proposition 6.4.2. *If $X \in \mathcal{S}^*$, then $X \in \mathcal{S}$ and $F_{X,I} \in \mathcal{S}$.*

Proof. We only prove $F_{X,I} \in \mathcal{S}$: From the definition we conclude that for all $\varepsilon > 0$ there exists a constant $x_0 > 0$ such that, for $t > x_0$,

$$2\mu(1-\varepsilon)\overline{F_X}(t) \le \int_0^t \overline{F_X}(t-y)\overline{F}(y)dy \le 2\mu(1+\varepsilon)\overline{F_X}(t)$$

Therefore, for any $x > x_0$,

$$2\mu(1-\varepsilon)\int_x^\infty \overline{F_X}(t)dt \le \int_x^\infty \int_0^t \overline{F_X}(t-y)\overline{F_X}(y)dydt \le 2\mu(1+\varepsilon)\int_x^\infty \overline{F_X}(t)dt$$

and

$$2(1-\varepsilon) \le \frac{\int_x^\infty \int_0^t \overline{F_X}(t-y)\overline{F_X}(y)dydt/\mu^2}{\int_x^\infty \overline{F_X}(t)dt/\mu} \le 2(1+\varepsilon)$$

or, in another notation,

$$2(1-\varepsilon) \le \frac{\overline{F_{I,X} * F_{I,X}}(x)}{\overline{F}_{I,X}(x)} \le 2(1+\varepsilon).$$

Proposition 6.2.11 implies that $F_{X,I} \in \mathcal{S}^*$. □

Proposition 6.4.3. *The Weibull distribution*

$$\mathbb{P}(X > x) = e^{-cx^r} \quad \textit{for} \quad x \ge 0,$$

with fixed $c > 0$ and $r \in (0,1)$ belongs to $\mathcal{S}^*$.

Proof. Let $M(x) := cx^r$. We show

$$\int_0^x \frac{\overline{F}(x-y)}{\overline{F}(x)}\overline{F}(y)dy = 2\int_0^{x/2} e^{M(x)-M(x-y)-M(y)}dy \to 2\mu$$

for $x \to \infty$ where we use that

$$\int_0^{x/2} e^{M(x)-M(x-y)-M(y)}dy = \int_{x/2}^x e^{M(x)-M(x-y)-M(y)}dy$$

because of the symmetry of $y \mapsto M(x) - M(x-y) - M(y)$ around $x/2$. For $0 < y < \frac{x}{2}$ we have

$$1 \le e^{yM'(x)} \le e^{M(x)-M(x-y)} \le e^{yM'(x/2)}$$

and hence

$$\int_0^{x/2} e^{-M(y)} dy \le \int_0^{x/2} e^{M(x)-M(x-y)-M(y)} dy \le \int_0^{x/2} e^{yM'(x/2)} e^{-M(y)} dy.$$

For the left-hand side we have

$$\lim_{x\to\infty} \int_0^{x/2} e^{-M(y)} dy = \lim_{x\to\infty} \int_0^{x/2} \mathbb{P}(X > y) dy = \int_0^{\infty} \mathbb{P}(X > y) dy = \mathbb{E}X = \mu.$$

For the right-hand side and $0 \le y \le x/2$ we observe that

$$yM'(x/2) = cry \left| \frac{x}{2} \right|^{r-1} \to 0 \quad \text{as} \quad x \to \infty$$

for all $y \ge 0$. Therefore, we can use dominated convergence on the right-hand side to derive

$$\lim_{x\to\infty} \int_0^{x/2} e^{yM'(x/2)} e^{-M(y)} dy = \mu$$

as for the left-hand side. Since the left-hand side and the right-hand side of the inequality, both converge to μ as $x \to \infty$ we get

$$\lim_{x\to\infty} \int_0^{x/2} e^{M(x)-M(x-y)-M(y)} dy = \mu.$$

□

Corollary 6.4.4. *Assume the Cramér-Lundberg-model and that the claim size distribution is Weibull distributed, i.e.*

$$\mathbb{P}(X > u) = e^{-cu^r} \quad \text{for} \quad u \ge 0,$$

where $c > 0$ and $r \in (0,1)$ are fixed. Assume that the NP condition is fulfilled. Then

$$\lim_{u\to\infty} \frac{\psi(u)}{\int_u^\infty e^{-cx^r} dx} = \frac{1}{\rho \int_0^\infty e^{-cx^r} dx}.$$

6.5 Summary

Heavy-tailed

Class $\mathcal{S}$

- $*$ Pareto type with exponent $\alpha > 0$
- $*$ Pareto with parameters $\alpha, \kappa > 0$

Class $\mathcal{S}^*$

- $*$ Pareto II with parameters $\alpha > 1$ and $\kappa > 0$
- $*$ Weibull with parameters $0 < r < 1$ and $c > 0$

Light-tailed

- $*$ Exponential distribtion with parameter $\lambda > 0$
- $*$ Weibull with parameters $r \geq 1$ and $c > 0$

Chapter 7

More facts on claim size distributions and distributions of the total claim amount

7.1 QQ-Plot

A quantile is "the inverse of the distribution function". We take the "left inverse" if the distribution function is not strictly increasing and continuous which is is defined by

$$F^{\leftarrow}(t) := \inf\{x \in \mathbb{R}, F(x) \geq t\}, \quad 0 < t < 1,$$

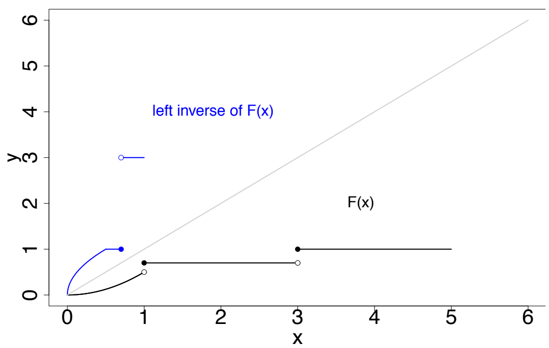
and the *empirical distribution function* of the data $X_1, ...X_n$ as

$$F_n(x) := \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{(-\infty,x]}(X_i), \ x \in \mathbb{R}.$$

It can be shown that if $X_1 \sim F$, $(X_i)_{i=1}^{\infty}$ i.i.d., then

$$\lim_{n\to\infty} F_n^{\leftarrow}(t) \to F^{\leftarrow}(t),$$

almost surely for all continuity points t of $F^{\leftarrow}$. Hence, if $X_1 \sim F$, then the plot of $(F_n^{\leftarrow}(t), F^{\leftarrow}(t))$ should give almost the straight line $y = x$.



7.2 The distribution of the total claim amount $S(t)$

7.2.1 Compound Poisson random variables

The following mixture distributions will be used to show that an independent sum of compound Poisson random variables, introduced in Definition 7.2.3 below, is a compound Poisson random variable.

Definition 7.2.1 (Mixture distributions)**.** Let $F_k,\ k = 1, ..., n$ be distribution functions and $p_k \in [0, 1]$ such that $\sum_{k=1}^n p_k = 1$. Then

$$G(x) = p_1 F_1(x) + ... + p_n F_n(x),\ x \in \mathbb{R},$$

is called the **mixture distribution** of $F_1, ..., F_n$.

Lemma 7.2.2. *Let $f_1, ..., f_n$ be random variables with distribution function $F_1, ..., F_n$, respectively. Assume that $J : \Omega \to \{1, ..., n\}$ is independent from $f_1, ..., f_n$ and $\mathbb{P}(J = k) = p_k$. Then the random variable*

$$g = \mathbb{1}_{\{J=1\}} f_1 + ... + \mathbb{1}_{\{J=n\}} f_n$$

has the mixture distribution function G.

Definition 7.2.3 (Compound Poisson random variable)**.** Let $N_\lambda \sim Pois(\lambda)$ and $(X_i)_{i=1}^\infty$ i.i.d. random variables, independent from N_λ. Then

$$Y := \sum_{i=1}^{N_\lambda} X_i$$

is called a **compound Poisson random variable**.

Proposition 7.2.4. *The sum of independent compound Poisson random variables is a compound Poisson random variable: Let $S_1, \ldots, S_n$ given by*

$$S_k = \sum_{i=1}^{N_{\lambda_k}^{(k)}} X_i^{(k)}, \ k = 1, ..., n,$$

be independent compound Poisson random variables such that $\lambda_k > 0$,

$$N_{\lambda_k}^{(k)} \sim Pois(\lambda_k), \quad \text{and} \quad (X_i^{(k)})_{i \geq 1} \text{ i.i.d.},$$

and N_k is independent from $(X_i^{(k)})_{i\geq 1}$ for all $k = 1, ..., n$. Then $S := S_1 + ... + S_n$ is a compound Poisson random variable with representation

$$S \stackrel{d}{=} \sum_{i=1}^{N_\lambda} Y_i, \ N_\lambda \sim Pois(\lambda), \ \lambda = \lambda_1 + ... + \lambda_n,$$

and $(Y_i)_{i\geq 1}$ is an i.i.d. sequence, independent from N_λ, and such that

$$Y_1 \stackrel{d}{=} \sum_{k=1}^{n} \mathbb{1}_{\{J=k\}} X_1^{(k)}, \text{ with } \mathbb{P}(J = k) = \frac{\lambda_k}{\lambda},$$

and J is independent of $(X_1^{(k)})_{k=1}^n$.

Proof. From Section 9 we know that it is sufficient to show that S and $\sum_{i=1}^{N_\lambda} Y_i$ have the same characteristic function. We start with the characteristic function of S_k and get that

$$\begin{aligned}
\varphi_{S_k}(u) &= \mathbb{E}e^{iuS_k} = \mathbb{E}e^{iu\sum_{j=1}^{N_{\lambda_k}^{[(k)}} X_j^{(k)}} \\
&= \mathbb{E}\sum_{m=0}^{\infty} e^{iu\sum_{j=1}^{m} X_j^{(k)}} \mathbb{1}_{\{N_{\lambda_k}^{(k)}=m\}} \\
&= \mathbb{E}\sum_{m=0}^{\infty} \underbrace{e^{iuX_1^{(k)}} \times ... \times e^{iuX_m^{(k)}} \mathbb{1}_{\{N_{\lambda_k}^{(k)}=m\}}}_{\text{all of these are independent}} \\
&= \sum_{m=0}^{\infty} \left(\mathbb{E}e^{iuX_1^{(k)}}\right)^m \mathbb{P}(N_{\lambda_{\lambda_k}^{(k)}} = m) \\
&= \sum_{m=0}^{\infty} \left(\varphi_{X_1^{(k)}}(u)\right)^m \mathbb{P}(N_{\lambda_k}^{(k)} = m) \\
&= \sum_{m=0}^{\infty} \left(\varphi_{X_1^{(k)}}(u)\right)^m \frac{\lambda_k^m}{m!} e^{-\lambda_k} = e^{-\lambda_k(1-\varphi_{X_1^{(k)}}(u))}.
\end{aligned}$$

Then

$$\varphi_S(u) = \mathbb{E}e^{iu(S_1+...+S_n)}$$

$$\begin{aligned}
&= \mathbb{E}e^{iuS_1} \times ... \times \mathbb{E}e^{iuS_n} \\
&= \varphi_{S_1}(u) \times ... \times \varphi_{S_n}(u) \\
&= e^{-\lambda_1(1-\varphi_{X_1^{(1)}}(u))} \times ... \times e^{-\lambda_n(1-\varphi_{X_1^{(n)}}(u))} \\
&= \exp\left(-\lambda\left(1 - \sum_{k=1}^{n} \frac{\lambda_k}{\lambda} \varphi_{X_1^{(k)}}(u)\right)\right).
\end{aligned}$$

Now we compute the characteristic function of $\xi := \sum_{l=1}^{N_\lambda} Y_l$. Then by the same computation, as we have done for $\varphi_{S_k}(u)$, we get

$$\varphi_\xi(u) = \mathbb{E}e^{iu\xi} = e^{-\lambda(1-\varphi_{Y_1}(u))}.$$

Finally,

$$\begin{aligned}
\varphi_{Y_1}(u) &= \mathbb{E}e^{iu\sum_{k=1}^{n} \mathbf{1}_{\{J=k\}} X_1^{(k)}} \\
&= \mathbb{E}\sum_{l=1}^{n}\left(e^{iu\sum_{k=1}^{n} \mathbf{1}_{\{J=k\}} X_1^{(k)}} \mathbb{1}_{\{J=l\}}\right) \\
&= \sum_{k=1}^{n} \mathbb{E}\left(e^{iuX_1^{(k)}} \mathbb{1}_{\{J=k\}}\right) \\
&= \sum_{k=1}^{n} \varphi_{X_1^{(k)}(u))} \frac{\lambda_k}{\lambda}.
\end{aligned}$$

□

7.2.2 Applications in insurance

First application

Assume that the claims arrive according to an inhomogeneous Poisson process, i.e.

$$N(t) - N(s) \sim Pois(\mu(t) - \mu(s)).$$

The **total claim amount in year** l is

$$S_l = \sum_{j=N(l-1)+1}^{N(l)} X_j^{(l)}, \; l = 1, ..., n.$$

Now, it can be seen, that

$$S_l \stackrel{d}{=} \sum_{j=1}^{N(l)-N(l-1)} X_j^{(l)}, \; l = 1, ..., n$$

and S_l is compound Poisson distributed. Proposition 7.2.4 implies that the total claim amount of the first n years is again compound Poisson distributed, where

$$S(n) \; := \; S_1 + ... + S_n \stackrel{d}{=} \sum_{i=1}^{N_\lambda} Y_i$$

$$\begin{aligned} N_\lambda &\sim Pois(\mu(n)) \\ Y_i &\overset{d}{=} \mathbb{1}_{\{J=1\}} X_1^{(1)} + ... + \mathbb{1}_{\{J=n\}} X_1^{(n)} \\ \mathbb{P}(J=i) &= \frac{\mu(i)-\mu(i-1)}{\mu(n)}. \end{aligned}$$

Hence the total claim amount $S(n)$ in the first n years (with possibly different claim size distributions in each year) has a representation as a compound Poisson random variable.

Second application

We can interpret the random variables

$$S_i = \sum_{j=1}^{N_i} X_j^{(i)}, \ N_i \sim Pois(\lambda_i), \quad i=1,\dots,n,$$

as the total claim amounts of n independent portfolios for the same fixed period of time. The $(X_j^{(i)})_{j\ge1}$ in the i-th portfolio are i.i.d, but the distributions may differ from portfolio to portfolio (one particular type of car insurance, for example). Then

$$S(n) = S_1 + ... + S_n \overset{d}{=} \sum_{i=1}^{N_\lambda} Y_i$$

is again compound Poisson distributed with

$$\begin{aligned} N_\lambda &= Pois(\lambda_1 + ... + \lambda_n) \\ Y_i &\overset{d}{=} \mathbb{1}_{\{J=1\}} X_1^{(1)} + ... + \mathbb{1}_{\{J=n\}} X_1^{(n)} \end{aligned}$$

and $\mathbb{P}(J=l) = \frac{\lambda_l}{\lambda}$.

7.2.3 The Panjer recursion: an exact numerical procedure to calculate $F_{S(t)}$

Let

$$S = \sum_{i=1}^{N} X_i,$$

$N : \Omega \to \{0,1,...\}$ and $(X_i)_{i\ge1}$ i.i.d, N and (X_i) independent. Then, setting $S_0 := 0$, $S_n := X_1 + ... + X_n$, $n \ge 1$ yields

$$\begin{aligned} \mathbb{P}(S \le x) &= \sum_{n=0}^{\infty} \mathbb{P}(S \le x, N=n) \\ &= \sum_{n=0}^{\infty} \mathbb{P}(S \le x | N=n) \mathbb{P}(N=n) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \mathbb{P}(S_n \le x)\mathbb{P}(N = n) \\
&= \sum_{n=0}^{\infty} F_{X_1}^{n*}(x)\mathbb{P}(N = n),
\end{aligned}
$$

where $F_{X_1}^{n*}(x)$ is the n-th convolution of F_{X_1}, i.e.

$$
\begin{aligned}
F_{X_1}^{2*}(x) &= \mathbb{P}(X_1 + X_2 \le x) = \mathbb{E}\mathbb{1}_{\{X_1+X_2\le x\}} \\
&\overset{X_1, X_2 \text{ independent}}{=} \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{1}_{\{x_1+x_2\le x\}}(x_1,x_2)dF_{X_1}(x_1)dF_{X_2}(x_2) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{1}_{\{x_1\le x-x_2\}}(x_1,x_2)dF_{X_1}(x_1)dF_{X_2}(x_2) \\
&= \int_{\mathbb{R}} F_{X_1}(x-x_2)dF_{X_2}(x_2)
\end{aligned}
$$

and by recursion using $F_{X_1} = F_{X_2}$,

$$
F_{X_1}^{(n+1)*}(x) := \int_{\mathbb{R}} F_{X_1}^{n*}(x-y)dF_{X_1}(y).
$$

But the computation of $F_{X_1}^{n*}(x)$ is numerically difficult. However, there is a recursion formula for $\mathbb{P}(S \le x)$ that holds under certain conditions:

Theorem 7.2.5 (Panjer recursion scheme). *Assume the following conditions:*

(C1) $X_i : \Omega \to \{0, 1, ...\}$

(C2) *for N it holds that*

$$
q_n = \mathbb{P}(N = n) = \left(a + \frac{b}{n}\right) q_{n-1}, \; n = 1, 2, ...
$$

for some $a, b \in \mathbb{R}$.

Then for $p_n := \mathbb{P}(S = n), \; n = 0, 1, 2, \ldots$ one has

$$
\begin{aligned}
p_0 &= \begin{cases} q_0 & : \mathbb{P}(X_1 = 0) = 0 \\ \mathbb{E}\mathbb{P}(X_1 = 0)^N & : \textit{otherwise} \end{cases}, && (1) \\
p_n &= \frac{1}{1 - a\mathbb{P}(X_1 = 0)} \sum_{i=1}^{n} \left(a + \frac{bi}{n}\right) \mathbb{P}(X_1 = i) p_{n-i}, \; n \ge 1. && (2)
\end{aligned}
$$

Proof. First we observe

$$
\begin{aligned}
p_0 &= \mathbb{P}(S = 0) = \mathbb{P}(S = 0, N = 0) + \mathbb{P}(S = 0, N > 0) \\
&= \underbrace{\mathbb{P}(S_0 = 0)}_{=1} \mathbb{P}(N = 0) + \mathbb{P}(S = 0, N > 0)
\end{aligned}
$$

$$
\begin{aligned}
&= \underbrace{\mathbb{P}(N=0)}_{=q_0} + \mathbb{P}(S=0, N>0) \\
&= \underbrace{q_0}_{\mathbb{P}(X_1=0)^0\mathbb{P}(N=0)} + \sum_{k=1}^{\infty} \underbrace{\mathbb{P}(X_1+...+X_k=0, N=k)}_{\mathbb{P}(X_1=0)^k \underbrace{\mathbb{P}(N=k)}_{q_k}} \\
&= \mathbb{E}\mathbb{P}(X_1=0)^N
\end{aligned}
$$

which implies (1). For $p_n,\ n \geq 1$,

$$
\begin{aligned}
p_n &= \mathbb{P}(S=n) = \sum_{k=1}^{\infty} \mathbb{P}(S_k=n) q_k \\
&\overset{(C2)}{=} \sum_{k=1}^{\infty} \mathbb{P}(S_k=n)(a+\frac{b}{k}) q_{k-1}. \qquad (3)
\end{aligned}
$$

Assume $\mathbb{P}(S_k=n)>0$. Now, because $\mathbb{Q}=\mathbb{P}(\cdot|S_k=n)$ is a probability measure the following holds.

$$
\begin{aligned}
& \sum_{l=0}^{n} \left(a+\frac{bl}{n}\right) \underbrace{\mathbb{P}(X_1=l|S_k=n)}_{\mathbb{Q}(X_1=l)} \\
&= a+\frac{b}{n}\mathbb{E}_{\mathbb{Q}}X_1 \\
&= a+\frac{b}{nk}\mathbb{E}_{\mathbb{Q}}(X_1+...+X_k) \\
&= a+\frac{b}{nk}\underbrace{\mathbb{E}_{\mathbb{Q}}S_k}_{=n} \\
&= a+\frac{b}{k}, \qquad (4)
\end{aligned}
$$

where the last equation yields from the fact that $\mathbb{Q}(S_k=n)=1$. On the other hand, we can express the term $a+\frac{b}{k}$ also by

$$
\begin{aligned}
& \sum_{l=0}^{n} \left(a+\frac{bl}{n}\right) \mathbb{P}(X_1=l|S_k=n) \\
&= \sum_{l=0}^{n} (a+\frac{bl}{n}) \frac{\mathbb{P}(X_1=l, S_k-X_1=n-l)}{\mathbb{P}(S_k=n)} \\
&= \sum_{l=0}^{n} (a+\frac{bl}{n}) \frac{\mathbb{P}(X_1=l)\mathbb{P}(S_{k-1}=n-l)}{\mathbb{P}(S_k=n)}. \qquad (5)
\end{aligned}
$$

Thanks to (4) we can now replace the term $a+\frac{b}{k}$ in (3) by the RHS of (5) which yields

$$\begin{aligned}
p_n &= \sum_{k=1}^{\infty}\sum_{l=0}^{n}\left(a+\frac{bl}{n}\right)\mathbb{P}(X_1=l)\mathbb{P}(S_{k-1}=n-l)q_{k-1}\\
&= \sum_{l=0}^{n}\left(a+\frac{bl}{n}\right)\mathbb{P}(X_1=l)\underbrace{\sum_{k=1}^{\infty}\mathbb{P}(S_{k-1}=n-l)q_{k-1}}_{\mathbb{P}(S=n-l)}\\
&= a\mathbb{P}(X_1=0)\mathbb{P}(S=n)+\sum_{l=1}^{n}\left(a+\frac{bl}{n}\right)\mathbb{P}(X_1=l)\mathbb{P}(S=n-l)\\
&= a\mathbb{P}(X_1=0)p_n+\sum_{l=1}^{n}\left(a+\frac{bl}{n}\right)\mathbb{P}(X_1=l)p_{n-l},
\end{aligned}$$

which will give the equation (2)

$$p_n = \frac{1}{1-a\mathbb{P}(X_1=0)}\sum_{l=1}^{n}\left(a+\frac{bl}{n}\right)\mathbb{P}(X_1=l)p_{n-l}.$$

□

Remark 7.2.6.

(1) The Panjer recursion only works for distributions of X_i on $\{0,1,2,...\}$ i.e. $\sum_{k=0}^{\infty}\mathbb{P}_{X_i}(k)=1$ (or, by scaling, on a lattice $\{0,d,2d,...\}$ for $d>0$ fixed).

(2) Traditionally, the distributions used to model X_i have a density, and $\int_{\{0,1,2,...\}} h_{x_i}(x)dx=0$. But on the other hand, claim sizes are expressed in terms of prices, so they take values on a lattice. The density $h_{X_i}(x)$ could be approximated to have a distribution on a lattice, but how large would the approximation error then be?

(3) N can **only** be Poisson, binomially or negative binomially distributed.

7.2.4 Approximation of $F_{S(t)}$ using the Central Limit Theorem

Assume, that the renewal model is used, and that

$$S(t)=\sum_{i=1}^{N(t)}X_i,\ t\geq 0.$$

In Theorem 3.2.3 the Central Limit Theorem is used to state that if $\mathrm{var}(W_1)<\infty$ and $\mathrm{var}(X_1)<\infty$, then

$$\sup_{x\in\mathbb{R}}\left|\mathbb{P}\left(\frac{S(t)-\mathbb{E}S(t)}{\sqrt{\mathrm{var}(S(t))}}\leq x\right)-\Phi(x)\right|\overset{t\to\infty}{\to}0.$$

Now, by setting

$$x := \frac{y - \mathbb{E}S(t)}{\sqrt{\mathrm{var}(S(t))}},$$

for large t the approximation

$$\mathbb{P}(S(t) \leq y) \approx \Phi\left(\frac{y - \mathbb{E}S(t)}{\sqrt{\mathrm{var}(S(t))}}\right)$$

can be used.

Warning: This approximation is not good enough to estimate $\mathbb{P}(S(t) > y)$ for large y, see [7], Section 3.3.4.

7.2.5 Monte Carlo approximations of $F_{S(t)}$

a) The Monte Carlo method

If the distributions of $N(t)$ and X_1 are known, then an i.i.d. sample of

$$N_1, ..., N_m, \ (N_k \sim N(t), \ k = 1, ..., m)$$

and i.i.d. samples of

$$\left.\begin{array}{l} X_1^{(1)}, ..., X_{N_1}^{(1)} \\ \dots \\ X_1^{(n)}, ..., X_{N_m}^{(n)} \end{array}\right\} X_i^{(j)} \sim X_1, \ i = 1, ..., N_j, \ j = 1, ..., m$$

can be simulated on a computer and the sums

$$S_1 = \sum_{i=1}^{N_1} X_i^1, ..., S_m = \sum_{i=1}^{N_m} X_i^m$$

calculated. Then it follows that $S_i \sim S(t)$, and the S_i's are independent. By the Strong Law of Large Numbers,

$$\hat{\rho}_m := \frac{1}{m} \sum_{i=1}^{m} 1\!\!1_A(S_i) \overset{\text{a.s.}}{\to} \mathbb{P}(S(t) \in A) = p, \quad \text{as } m \to \infty.$$

It can be shown that this does **not** work well for small values of p (see [7], section 3.3.5 for details).

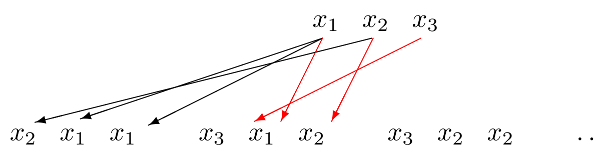
b) The bootstrap method

The bootstrap method is a statistical simulation technique, that doesn't require the distribution of X_i's. The term "bootstrap" is a reference to Münchhausen's tale, where the baron escaped from a swamp by pulling himself up by his own bootstraps. Similarly, the bootstrap method only uses the given data.

Assume, there's a sample, i.e. for some fixed $\omega \in \Omega$ we have the real numbers

$$x_1 = X_1(\omega), ..., x_n = X_n(\omega),$$

of the random variables $X_1, ..., X_n$, which are supposed to be i.i.d. Then, a **draw with replacement** can be made as illustrated in the following example: Assume $n = 3$ and $x_1 = 4, x_2 = 1, x_3 = 10$ for example. **Drawing with replacement** means we choose a sequence of triples were each triple consists of the randomly out of {1,4,10} chosen numbers. For example, we could get:



We denote the k-th triple by $X^*(k) = (X_1^*(k), X_2^*(k), X_3^*(k)),\ k \in \{1, 2, ...\}$. Then, for example, the sample mean of the k-th triple

$$\bar{X}^*(k) := \frac{X_1^*(k) + X_2^*(k) + X_3^*(k)}{3}$$

has values between $\min\{x_1, x_2, x_3\} = 1$ and $\max\{x_1, x_2, x_3\} = 10$, but the values near $\frac{x_1+x_2+x_3}{3} = 5$ are more likely than the minimum or the maximum, and it holds the SLLN

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \bar{X}^*(i) \to \frac{x_1 + x_2 + x_3}{3} \quad a.s.$$

Moreover, it holds in general

$$\mathrm{var}(\bar{X}^*(i)) = \frac{\mathrm{var}(X_1)}{n}.$$

Verifying this is left as an exercise.
In insurance, the sum of the claim sizes $X_1 + ... + X_n = n\bar{X}_n$ is the target of interest and with this, the total claim amount

$$S(t) = \sum_{i=1}^{N(t)} X_i = \sum_{n=0}^{\infty} \left(\sum_{i=1}^{n} X_i \right) \mathbb{1}_{\{N(t)=n\}}.$$

Here, the bootstrap method is used to calculate confidence bands for (the parameters of) the distributions of the X_i's and $N(t)$.

Warning

The bootstrap method doesn't always work! In general, simulation should only be used, if everything else fails. Often better approximation results can be obtained by using the Central Limit Theorem.

So all the methods represented should be used with great care, as each of them has advantages and disadvantages. After all, "nobody is perfect" also applies to approximation methods.

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