Models in Financial mathematics 2

Exercises for
09.12.08 16:05-17:35 (MaD 380)
-6-

1. **$\int_0^t L_s dW_s$ is normal distributed for deterministic $(L_s) \in \mathcal{L}_2$:**

   Show by dominated convergence that $\int_0^t L_s dW_s$ is normal distributed with mean 0 and variance $\int_0^t L_s^2 ds$ for any non-random $(L_s) \in \mathcal{L}_2$.

   Hint: Use the exercises -5- 3. and 4.

2. Let $(W_t)_{t\in[0,\infty)}$ be a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$. Is it true that

   $$\int_0^t W_s ds \quad \text{is distributed like} \quad \frac{t^{\frac{3}{2}}}{\sqrt{3}} W_1?$$

   Hint: Use exercise -6- 1. and exercise -5- 5.

3. Let $(W_t)_{t\in[0,\infty)}$ be a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{Q}, \mathbb{F})$. Assume that the interest rate process $(r(s))_{s\in[0,T])}$ is given by the Vasicek model. Compute the fair price $p(t,T)$ of the bond at time t for the special case of a Vasicek model with $\alpha = 0, \beta = 0$ and $\gamma = 1$.

   Hint: Notice that $\int_t^T W_s - W_t \, ds$ is independent from $\mathcal{F}_t$.

4. Assume $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. Let H be a random variable such that $\mathbb{P}(\{\omega \in \Omega : H(\omega) > 0\}) = 1$. Define a probability measure $\mathbb{Q}$ by

   $$\mathbb{Q}(A) := \mathbb{E}(H 1\!\!1_A), \quad A \in \mathcal{F}$$

   Show that for all $A \in \mathcal{F}$ it holds

   (a) $\mathbb{P}(A) = 0$ implies $\mathbb{Q}(A) = 0$.

   (b) $\mathbb{Q}(A) = 0$ implies $\mathbb{P}(A) = 0$.

   Hint: For (b) define the sets $B_n := \{\omega \in \Omega : H(\omega) > \frac{1}{n}\}$ and use that $\bigcup_{n=1}^{\infty} B_n = \Omega$ and $H 1\!\!1_A \geq \frac{1}{n} 1\!\!1_{B_n \cap A}$.