Models in Financial Mathematics

Exercises for
07.11.06 16:10-17:55 (MaD 380)
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Finance

(1) Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{T}, (\mathcal{F}_t)_{t=0}^T, (S_t)_{t=0}^T)$ be a market model. What is the definition of:

(a) a viable market,
(b) an American option?

(2) Assume the CRR-model with $S_0 = 1$, $\mathbb{T} = \{0, 1, 2\}$,

$$S_{t+1} = \begin{cases} 5S_t & \text{with probability } \frac{1}{5}, \\ \frac{1}{5}S_t & \text{with probability } \frac{4}{5}, \end{cases}$$

and $r = \frac{2}{5}$.

(a) Is the given measure the EMM $\mathbb{Q}$?
(b) Compute the 'fair price' C_0 of the European put-option

$$H = (K - S_2)^+$$

with strike price $K = 2$.
(c) Compute the 'fair price' U_0 of the American put-option

$$Z_t = (K - S_t)^+, \qquad t = 0, 1, 2$$

with strike price $K = 2$.
Hints: Use

$$\begin{aligned} U_2 &= Z_2, \\ \tilde{U}_1 &= \max\{\tilde{Z}_1, \mathbb{E}_{\mathbb{Q}}[\tilde{U}_2 | \mathcal{F}_1]\}, \\ U_0 &= \max\{Z_0, \mathbb{E}_{\mathbb{Q}}\tilde{U}_1\}. \end{aligned}$$

To compute $\mathbb{E}_{\mathbb{Q}}[\tilde{U}_2|\mathcal{F}_1]$ write $\tilde{U}_2$ like

$$\begin{aligned} \tilde{U}_2 &= \frac{(K - S_2)^+}{S_2^0} \\ &= \frac{1}{S_2^0}\left\{(K - 5\frac{S_2}{S_1})^+ \mathbb{1}_{\{S_1=5\}} + (K - \frac{1}{5}\frac{S_2}{S_1})^+ \mathbb{1}_{\{S_1=\frac{1}{5}\}}\right\} \end{aligned}$$

and use independence ($\frac{S_2}{S_1}$ and $\mathcal{F}_1$) and measurability (S_1 and $\mathcal{F}_1$). It follows

$$\tilde{U}_1 = \frac{25}{49}\frac{3}{4}\mathbb{1}_{\{S_1=5\}} + \frac{9}{7}\mathbb{1}_{\{S_1=\frac{1}{5}\}}.$$

Comparing $\mathbb{E}_{\mathbb{Q}}\tilde{U}_1$ and Z_0 leads to $U_0 = \mathbb{E}_{\mathbb{Q}}\tilde{U}_1$.