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Stochastic Modeling

Exercises 18/03/2002

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(1) Tossing one coin n times is regarded like getting the data of n i.i.d.variables $X_1, \ldots, X_n$ with distribution

$$\mathbb{P}(X_i = x) = \Theta^x (1-\Theta)^{1-x}, \quad x = 0, 1$$

We write $h(x|\Theta) = \Theta^x(1-\Theta)^{1-x}$ and assume that Θ is uniformly distributed in $[0,1]$:

$$\mathbb{P}(\Theta \in [a,b]) = b - a, \quad 0 \le a < b \le 1.$$

Then, the posterior expectation for Θ given the data $x_1, \ldots, x_n$ is

$$\int_0^1 \Theta \pi(d\Theta | x_1, \ldots, x_n) = \frac{\int_0^1 \Theta \prod_{i=1}^n h(x_i|\Theta) d\Theta}{\int_0^1 \prod_{i=1}^n h(x_i|\Theta) d\Theta}.$$

Compute the posterior expectation of Θ if the tossing yielded

(a) the data $0, 1, 1, 0$.

(b) the data $1, \ldots, 1$ (n-times).

(2) A transition matrix $T = (p(k,l))_{k,l=0}^K$ of a homogeneous Markov chain is called *double stochastic* if

$$\sum_{i=0}^K p(k,i) = \sum_{i=0}^K p(i,l) = 1 \quad \text{for all} \quad k, l = 0, ..., K.$$

For a homogeneous Markov chain with state space $X = \{0, ..., K\}$ one shows the following equivalence.

(a) The Markov chain has a uniform stationary distribution, that means $\frac{1}{K+1} = s_0 = \cdots = s_K$ is a stationary distribution.

(b) The transition matrix is double stochastic.

(3) The *standard deviation* σ of a random variable f is defined by

$$\sigma := \sqrt{\mathbb{E}\,(f - \mathbb{E}f)^2}, \quad (\text{ if } \mathbb{E}f^2 < \infty).$$

(a) Compute the expectation and the standard deviation σ_X of the uniformly on the intervall (a, b) distributed random variable X $(a < b)$, i.e. the law of X is given by

$$\mathbb{P}(X \in A) := \frac{1}{b-a} \int_a^b \mathbb{1}_A(x)\, dx, \quad A \in \mathcal{B}((a, b)).$$

(b) Assume f and g are independent and have the standard deviation σ_f and σ_g, respectively. Compute σ_{f+g}, the standard deviation of $f + g$ in terms of σ_f and σ_g.

(c) Give the expectation of $X + Z$ and the standard deviation σ_{X+Z} where X and Z are independent, X is like in (a) and $Z \sim N(m, \sigma^2)$, with $m \in \mathbb{R}$, $\sigma^2 > 0$, i.e. the law of Z is given by

$$\mathbb{P}(Z \in A) := \frac{1}{\sqrt{2\pi\sigma^2}} \int_A e^{-\frac{(x-m)^2}{2\sigma^2}}\, dx, \quad A \in \mathcal{B}(\mathbb{R}).$$