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Stochastic Modeling

Exercises 11/03/2002
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(1) Assume the probability space $([0,1], \mathcal{B}([0,1]), l)$.

(a) Check if the functions

$$f_n(x) := \begin{cases} e^n & 0 \le x \le \frac{1}{n}, \\ 0 & \frac{1}{n} < x \le 1 \end{cases}$$

converge in L_p $(0 < p < \infty)$ to 0.

(b) Define the functions
$f_n : [0,1] \to \mathbb{R}$, $n = 1, 2, \ldots$ with

$$f_n(x) := \begin{cases} 1 - \frac{1}{n^x} & \text{if } x \in [0,1] \text{ is irrational,} \\ n^x & \text{if } x \in [0,1] \text{ is rational.} \end{cases}$$

Does $f_n(x)$ converge to $f(x) \equiv 1$ if n tends to infinity

i. almost surely,
ii. in probability,
iii. in L_p $(0 < p < \infty)$?

(2) Let $(f_i)_{i=0}^\infty$ be a homogeneous Markov chain with state space $X = \{0, 1, 2\}$ and transition matrix $(0 < \varepsilon < 1)$

$$T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \varepsilon & 1-\varepsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) One determines the closed and irreducible subsets of X.

(b) One computes the stationary distributions of $(f_i)_{i=0}^\infty$.

(c) Given an initial distribution (q_0, q_1, q_2) one determines the limiting distribution

$$\lim_{n \to \infty} (q_0, q_1, q_2) \circ T^n.$$

(3) Let $(f_i)_{i=0}^\infty$ be a homogeneous Markov chain with state space X and assume $S = (s_0, s_1, \ldots)$ is a stationary distribution. Show that if one starts from a stationary distribution, all finite-dimensional distributions are time invariant, i.e. it holds for every $n \ge 1$ and all $k_1, \ldots, k_n \in X$ and $m = 0, 1, \ldots$

$$\mathbb{P}_S(f_n = k_n, \ldots, f_0 = k_0) = \mathbb{P}_S(f_{n+m} = k_n, \ldots, f_m = k_0),$$

where $\mathbb{P}_S(A) := \sum_{l \in X} \mathbb{P}(A | f_0 = l) s_l$.

(4) Given a homogeneous Markov chain with state space X and with communicating states $k, l \in X$ one shows: If k is persistent, then l is persistent as well.