Recent Results in the Ruin Theory with Investments

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Essentials

- The striking feature of models with investments in contrast to the classical ruin theory is that the ruin probabilities, as the best, decay as a power function.
- The rate of decay is a positive root of the cumulant generating function of an increment of the logprice process.
- Proofs of general results are based on the Kesten–Goldie theory of distributional equations (called also implicit renewal theory) but the approach based on integro-differential equations is also of great interest.

Semimartingale Ornstein–Uhlenbeck process

- SOU $X = X^u$ is the solution of linear stochastic equation $X = u + X_- \cdot R + P$, or in traditional notations, $dX = X_- dR + dP$, $X_0 = u$, where P, R are semimartingales, $\Delta R > -1$.
- In insurance P is the"business" process, $X = X^u$ is the reserve invested in a risky asset with the price $S = \mathcal{E}(R) = e^V$. Note that $dS/S = dR$, i.e. R is the relative price process or stochastic interest rate, V, the log price is a Lévy process.
- Ruin time $\tau^u := \inf\{t\colon\ X_t^u \le 0\}$.
- Ruin probability $\Psi(u) := \mathrm{P}[\tau^u < \infty]$, $\Psi(u, T) := \mathrm{P}[\tau^u \le T]$.
- Asymptotic results are available for Lévy OU model.

Models with gBm investments

- $R_t := at + \sigma W_t$, where W is a Wiener process, $\sigma > 0$.
- P as in the Lundberg model: a compound Poisson with drift,

$$P_t = ct + \sum_{i=1}^{N_t} \xi_i, \qquad \text{i.i.d } \xi_i \text{ are exponentially distributed.}$$

- No safety loading assumption.
- 3 versions: non-life insurance, annuity payments, mixed.

Theorem (Frolova, K., Pergamenshchikov, Pukhlyakov, 02, 16, 20)

If $\beta := 2a/\sigma^2 - 1 > 0$, then $\Psi(u) \sim Cu^{-\beta}$. Otherwise, $\Psi(u) \equiv 1$.

- Proofs are based on asymptotic theory of ODE.
- If a fraction $\gamma \in]0,1]$ of the reserve is invested in the stock, the ruin with probability one will be avoided only if $2a\gamma/(\sigma\gamma)^2 > 1$, i.e. if the share of the risky investment is strictly less than $2a/\sigma^2$.

Integro-differential equation (example of annuity model)

- Let $c < 0$ and $\xi > 0$. Let $\alpha > 0$ be the intensity of N.
- It is easy to prove that if $\Psi \in C^2$, then $\mathcal{L}\Psi = 0$ where

$$\mathcal{L}\Psi(u) := \frac{1}{2}\sigma^2 u^2 \Psi''(u) + (au + c)\Psi'(u) + \alpha \int (\Psi(u+y) - \Psi(u)) dF_\xi(y).$$

- Smoothness of Ψ was proven in K.-Pergamenshchikov (2016) and K.-Pukhlyakov (2022) for the mixed model.
- Equation $\mathcal{L}\Psi = 0$ holds always in viscosity sense, Belkina-K. 2015. It can be obtained for LOU model.
- An asymptotic theory for IDEs does not exist.

From IDE to ODE

If jumps are exponentially distributed, by taking the derivative of the IDE, we get an equation (of higher order) with the same integral. It can be eliminated and we get get a linear ODE for which there are results on the asymptotic behavior.
In particular, for the non-life insurance for $G = \Phi'$:

$$G'' + p(u)G' + q(u)G = 0,$$

where

$$p(u) = 1/\mu + 2(1 + a/\sigma^2)u^{-1} + \ldots u^{-2}, \quad q(u) = 2a/(\mu\sigma^2)u^{-1} + \ldots u^{-2}.$$

One can deduce from the asymptotic theory of linear DE that

$$\Psi(u) = C_0 + C_1\Phi_1(u) + C_2\Phi_2(u).$$

with $\Phi_1(u) \sim e^{-\beta u}$, $\Phi_2(u) \sim u^{-\beta}$.
It remains to check that if $\beta > 0$, then $C_0 = 0$ and $C_2 > 0$. This was done using approach suggested by V. Kalashnikov.

Ruin problem for LOU and the implicit renewal theory, 1

Let us consider the process X satisfying linear non-homogeneous equation $dX = X_- dR + dP$, $X_0 = u$ where R and P are independent Lévy processes, $\Delta R > -1$, $S := \mathcal{E}(R) =: e^V$. Then $[R, P] = 0$ (!) and we have the "Cauchy" formula:

$$X = S(u + S_-^{-1} \cdot P).$$

Indeed, $[S, u + S_-^{-1} \cdot P] = 0$ and, therefore,

$$d(S(u + S_-^{-1} \cdot P)) = (u + S_-^{-1} \cdot P_-)dS + S_- d(u + S_-^{-1} \cdot P) = X_- dR + P.$$

Put $Y := -e^{-V_-} \cdot P$. Then $X^u = e^V(u - Y)$. Obviously, $\tau^u = \inf\{t \geq 0 : \ Y_t \geq u\}$. Assuming that the (finite) limit Y_∞ exists (this requires rather mild hypotheses), we can consider the tail of its distribution $\bar{G}(u) := \mathrm{P}(Y_\infty > u)$.

The key to the implicit renewal theory

Lemma (Paulsen)

If $Y_t \to Y_\infty$ a.s. where Y_∞ is unbounded from above, then

$$\bar{G}(u) \le \Psi(u) = \frac{\bar{G}(u)}{\mathrm{E}\left[\bar{G}(X_{\tau^u}) \,|\, \tau^u < \infty\right]} \le \frac{\bar{G}(u)}{\bar{G}(0)}.$$

In particular, if $\Delta P \ge 0$, then $\Psi(u) = \bar{G}(u)/\bar{G}(0)$.

Proof of the Paulsen lemma

Let $\tau \in \mathcal{T}$ and let

$$Y_{\tau,\infty} := -\lim_{N\to\infty} \int_{(\tau,\tau+N]} e^{-(V_{t-}-V_\tau)} dP_t.$$

On the set $\{\tau < \infty\}$

$$Y_{\tau,\infty} = e^{V_\tau}(Y_\infty - Y_\tau) = X_\tau^u + e^{V_\tau}(Y_\infty - u).$$

Let ξ be a $\mathcal{F}_\tau^{P,R}$-measurable r.v. Since the Lévy process V starts afresh at τ, the conditional law of $Y_{\tau,\infty}$ given $(\tau,\xi) = (t,x)$ is the same as the law of Y_∞. It follows that $\mathrm{P}\left[Y_{\tau,\infty} > \xi,\ \tau < \infty\right] = \mathrm{E}\left[\bar{G}(\xi)\, I_{\{\tau<\infty\}}\right]$. Thus, if $\mathrm{P}[\tau < \infty] > 0$, then

$$\mathrm{P}\left[Y_{\tau,\infty} > \xi,\ \tau < \infty\right] = \mathrm{E}\left[\bar{G}(\xi)\,|\,\tau < \infty\right]\, \mathrm{P}[\tau < \infty].$$

Noting that $\Psi(u) := \mathrm{P}\left[\tau^u < \infty\right] \geq \mathrm{P}\left[Y_\infty > u\right] > 0$, we get that

$$\begin{aligned}\bar{G}(u) &= \mathrm{P}\left[Y_\infty > u,\ \tau^u < \infty\right] = \mathrm{P}\left[Y_{\tau^u,\infty} > X_{\tau^u}^u,\ \tau^u < \infty\right] \\ &= \mathrm{E}\left[\bar{G}(X_{\tau^u}^u)\,|\,\tau^u < \infty\right]\, \mathrm{P}[\tau^u < \infty] \geq \bar{G}(0)\,\mathrm{P}[\tau^u < \infty].\end{aligned}$$

The structure of the process Y

We have:

$$
\begin{aligned}
Y_n &= -\sum_{k=1}^{n}\int_{(k-1,k]} e^{-V_{s-}}dP_s = -\sum_{k=1}^{n} e^{-V_{k-1}}\int_{(k-1,k]} e^{-(V_{s-}-V_{k-1})}dP_s \\
&= Q_1 + M_1Q_2 + M_1M_2Q_3 + ... + M_1M_2\dots M_{n-1}Q_n,
\end{aligned}
$$

where the two-dimensional random variables

$$
Q_i := -\int_{(k-1,k]} e^{-(V_{s-}-V_{k-1})}dP_s, \quad M_k := e^{-(V_k-V_{k-1})}
$$

form an i.i.d. sequence.
It is easy to prove that if $\mathrm{E}[|Q_k|^p] < \infty$ and $\mathrm{E}[M_k^p] < 1$ for some $p > 0$, then $Y_n \to Y_\infty$ a.s. where Y_∞ is finite (one can take $p < 1$ and notice that the series is absolutely converging in L^p). Note also that $Y_\infty = Q_1 + M_1Y_{1,\infty}$ where $Y_{1,\infty} := Q_2 + M_2Q_3 + \dots$ has the same law as Y_∞.

Implicit renewal theory

We are given r.v. (M, Q) (in fact, only the law of (M, Q)).
Let $M > 0$ be such that $\mathcal{L}(\ln\ M)$ is non-arithmetic and

$$\mathrm{E}\,[M^\beta] = 1\,, \quad \mathrm{E}\,[M^\beta\,(\ln\ M)^+] < \infty \qquad \text{for some } \beta > 0.$$

Then $\ln \mathrm{E}[M] \in (-\infty, 0[$ and $\kappa := \mathrm{E}[M^\beta(\ln M)^+] \in]0, \infty[$.

Lemma (Goldie, 1991)

Let M satisfies the conditions above, $\mathrm{E}[|Q|^\beta] < \infty$. Then the distributional equation $Z \overset{d}{=} Q + M\,Z$ has a unique solution Z independent of (M,Q) and for some $C_+, C_- \in \mathbb{R}$ such that $C_+ + C_- > 0$

$$\lim_{u\to\infty} u^\beta\,\mathrm{P}[Z > u] = C_+, \qquad \lim_{u\to-\infty} u^\beta\,\mathrm{P}[Z < -u] = C_-.$$

Lemma (Guivarc'h, Le Page, 2015; Buraczewski, Damek, 2017)

$$C_+ > 0 \;\Leftrightarrow\; Z \;\; \textit{unbounded from above.}$$

Exit probabilities for Lévy OU process

For r.v. V_1 the cumulant generating function (always convex) $H(q) := \ln \mathrm{E}[e^{-qV_1}] = \ln \mathrm{E}[M_1^q]$ and $Q_1 = e^{-V_-} \cdot P_1$.

Theorem (K.,Pergamenshchikov 2020)

If H has a root $\beta > 0$, $H(\beta+) < \infty$, and $\Pi_P(|x|^\beta I_{\{|x|>1\}}) < \infty$, where Π_P is the Lévy measure of the process P, then

$$0 < \liminf_{u\to\infty} u^\beta \Psi(u) \le \limsup_{u\to\infty} u^\beta \Psi(u) < \infty.$$

If, moreover, $\Pi_P(]-\infty,0[) = 0$ and the law $\mathcal{L}(V_T)$ is non-arithmetic for some $T > 0$, then $\Psi(u) \sim C_\infty u^{-\beta}$, $C_\infty > 0$.

Thus, for the model with upward jumps we have an exact asymptotic if the distribution of the increment of log-price process is non-arithmetic, i.e. is not concentrated on the set $\mathbb{Z}d := \{\pm nd, n = 0, 1, \dots\}$, $d > 0$.

As Paulsen, it was used implicit renewal theory but more recent results.

Two comments on applications of IRT

Theorem

Suppose that (M, Q) is such that the distribution of $\ln M$ is non-arithmetic and, for some $\beta > 0$,

$$\mathrm{E}[M^\beta] = 1, \quad \mathrm{E}[M^\beta (\ln M)^+] < \infty, \quad \mathrm{E}[|Q|^\beta] < \infty.$$

Then $\limsup u^\beta \bar{G}(u) < \infty$. If Y_∞ is unbounded from above, then $\liminf u^\beta \bar{G}(u) > 0$ and in the case where $\mathcal{L}(\ln M)$ is non-arithmetic, $\bar{G}(u) \sim C_+ u^{-\beta}$ where $C_+ > 0$.

“For simplicity" we substitute the integrability condition in red by a stronger one: $\mathrm{E}[M^{\beta+\varepsilon}] < \infty$ for some $\varepsilon > 0$. But there is Kevei's result with a weaker condition leading to a different asymptotic of the tail ...

Lemma

If the random variables Q_1 and $Y_n/(M_1 \cdots M_n)$ for some $n \geq 1$ are unbounded from above, then Y_∞ is unbounded from above.

Sparre Andersen non-life insurance model with investments

In this model R is a Lévy process with $\Delta R > -1$, independent of the compound renewal process $P_t = ct + \sum_{i=1}^{N_t} \xi_i$, where $c > 0$, N is a counting renewal process with i.i.d. interarrival times $T_i - T_{i-1}$, independent of the i.i.d. sequence $\xi_i < 0$.
Now $H(q) := \ln \mathrm{E}[e^{-qV_{T_1}}] = \ln \mathrm{E}[M_1^q]$, $Q_1 := e^{-V_-} \cdot P_{T_1}$ and the idea is to use the representation $Y_\infty = \sum_i (Y_{T_i} - Y_{T_{i-1}})$.

Theorem (Ernst Eberlein, K., Thorsten Schmidt)

Suppose that there is $\beta > 0$ such that $H(\beta) = 0$, $H(\beta+) < \infty$, $\mathrm{E}\,|\xi_1|^\beta < \infty$, $\mathrm{E}\, e^{\varepsilon T_1} < \infty$ for some $\varepsilon > 0$. If $\sigma \neq 0$ or $|\xi_1|$ is unbounded,

$$0 < \liminf u^\beta \Psi(u) \le \limsup u^\beta \Psi(u) < \infty.$$

If $\sigma = 0$ and $|\xi_1|$ is bounded, the above properties also hold except the case where $0 < \Pi(|h|) < \infty$ and $\Pi(]-1,0[)\Pi(]0,\infty[) = 0$. In the latter case one needs the extra assumption $\mathrm{P}(T_1 \le t) > 0$ for any $t > 0$.

Lundberg–Cramér model with "telegraph" volatility, [K.,E.]

The price dynamics for gBm with regime switching

$$dS_t = S_t(a_{\theta_t}dt + \sigma_{\theta_t}dW_t), \qquad S_0 = 1,$$

where $a_k \in \mathbb{R}$, $\sigma_k > 0$, $k = 0, 1$, $\theta = (\theta_t)$ is a telegraph process with the transition intensity matrix Λ, $\lambda^{10} > 0$, $\lambda^{01} > 0$, $\lambda^{00} = -\lambda^{01}$, $\lambda^{11} = -\lambda^{10}$.
Let $\tau^{u,i} := \inf\{t : \ X_t^{u,i} \le 0\}$ (the instant of ruin when $\theta_0 = i$),
$\Psi_i(u) := \mathrm{P}[\tau^{u,i} < \infty]$ and $\beta_k := 2a_k/\sigma_k^2 - 1 > 0$, $k = 0, 1$.

Theorem

Let $0 < \beta_0 < \beta_1$ *and let* $\beta \in]\beta_0, \beta_1[$ *be the solution of the equation*

$$\sigma_0^2\sigma_1^2 q(\beta_0 - q)(\beta_1 - q) + 2\sigma_0^2(\beta_0 - q)\lambda^{10} + 2\sigma_1^2(\beta_1 - q)\lambda^{01} = 0.$$

If $\Pi_P(|x|^\beta) < \infty$, *then*

$$0 < \liminf_{u\to\infty} u^\beta \Psi_i(u) \le \limsup_{u\to\infty} u^\beta \Psi_i(u) < \infty.$$

Reduction in the model with a hidden Markov process

Lemma

For all $u > 0$

$$\bar{G}_i(u) \le \Psi_i(u) = \frac{\bar{G}_i(u)}{\mathrm{E}\big(\bar{G}_{\theta_{\tau^{u,i}}}(0)|\tau^{u,i} < \infty\big)} \le \frac{\bar{G}_i(u)}{\bar{G}_0(0) \wedge \bar{G}_1(0)},$$

where $\bar{G}_i(u) := \mathrm{P}(Y^i_\infty > u)$.

$$\mathrm{E}\ldots = \bar{G}_0(0)\mathrm{P}(\theta_{\tau^{u,i}} = 0|\tau^{u,i} < \infty) + \bar{G}_0(1)\mathrm{P}(\theta_{\tau^{u,i}} = 1|\tau^{u,i} < \infty)$$

Let τ_j be the consecutive jumps of θ. Put

$$f(q) := M_1^q = \mathrm{E}[e^{-qV_{\tau_2}}] = \mathrm{E}e^{-qV_{\tau_1}}\mathrm{E}[e^{-q(V_{\tau_2} - V_{\tau_1})}] = f_0(q)f_1(q),$$

the functions f_0 and f_1 admit explicit expression leading to an explicit form of the equation $f(q) = 1$. The reduction to the implicit renewal theory is done by the representation $Y_\infty = \sum_i (Y_{\tau_{2i}} - Y_{\tau_{2(i-1)}})$.

Regime switching by a finite state Markov process, [K.,P.]

Now $\theta = (\theta_t)$ is a piecewise constant right-continuous Markov process with values in the set $\{0, 1, ..., K-1\}$ and the transition intensity matrix $\Lambda = (\lambda^{ij})$ with the simple eigenvalue 0 and the initial value $\theta_0 = i$ (thus, $\theta = \theta^i$). Suppose that $2a_k/\sigma_k^2 - 1 > 0$, $k = 0, 1, ..., K-1$.

Let $\upsilon_1^i := \inf\{t > 0 \colon \theta_{t-}^i \neq i,\ \theta_t^i = i\}$ be the first return time of θ^i to the state i and let $\gamma_i > 0$ be such that

$$H_i(\gamma_i) := \ln \mathrm{E} e^{-\gamma_i V_{\upsilon_1^i}} = 0.$$

Theorem

Suppose that $\Pi_P(|x|^{\gamma_i}) := \int |x|^{\gamma_i} \Pi_P(dx) < \infty$.

$$0 < \liminf_{u \to \infty} u^{\gamma_i} \Psi_i(u) \leq \limsup_{u \to \infty} u^{\gamma_i} \Psi_i(u) < \infty.$$

Open problem: modulating process has a countable phase space ...

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