Nonlinear stochastic wave equation driven by rough noise

Yaozhong Hu
University of Alberta at Edmonton

Jointly with
Shuhui Liu (Shandong University)
Xiong Wang (John Hopkins University)

International Seminar on SDEs and Related Topics
Shandong University
11/11/22

Based on

Joint work with Shuhui Liu and Xiong Wang

Nonlinear stochastic wave Equation driven by rough noise.

Journal of Differential Equations 331(2022), 99-161.

Outline of the talk

1. Problem
2. Main results
3. Difficulty
4. Background
5. Some ideas

1. Problem

$$\frac{\partial^2 u(t,x)}{\partial t^2} = \Delta u(t,x) + \sigma(t,x,u(t,x))\dot{W}, \quad t > 0, x \in \mathbb{R}.$$

- $\Delta = \dfrac{\partial^2}{\partial x^2}$ is the Laplacian and $\sigma : \mathbb{R} \to \mathbb{R}$ is a nice function (Lipschitz).
- initial condition $u(0,x) = u_0(x)$ and $\frac{\partial}{\partial} u(0,x) = v(x)$ are nice.
- $\dot{W} = \frac{\partial^2 W}{\partial t \partial x}$ is centered Gaussian field with covariance

  $$\mathbb{E}(\dot{W}(s,x)\dot{W}(t,y)) \;=\; \delta(s-t)\,|x-y|^{2H-2}.$$

  Here $1/4 < H < 1/2$
- The product $\sigma(u)\dot{W}$ is taken in Skorohod sense.

Stochastic integral

For a function $\phi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, the Marchaud fractional derivative D_-^β is defined as:

$$\begin{aligned} D_-^\beta \phi(t,x) &= \lim_{\varepsilon \downarrow 0} D_{-,\varepsilon}^\beta \phi(t,x) \\ &= \lim_{\varepsilon \downarrow 0} \frac{\beta}{\Gamma(1-\beta)} \int_\varepsilon^\infty \frac{\phi(t,x) - \phi(t,x+y)}{y^{1+\beta}} dy. \end{aligned}$$

The Riemann-Liouville fractional integral is defined by

$$I_-^\beta \phi(t,x) = \frac{1}{\Gamma(\beta)} \int_x^\infty \phi(t,y)(y-x)^{\beta-1} dy.$$

Set

$$\mathbb{H} = \{\phi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \mid \exists \psi \in L^2(\mathbb{R}_+ \times \mathbb{R}) \; s.t. \; \phi(t,x) = I_-^{\frac{1}{2}-H} \psi(t,x)\}.$$

Proposition

$\mathbb{H}$ is a Hilbert space equipped with the scalar product

$$\begin{aligned} \langle \phi, \psi \rangle_{\mathbb{H}} &= c_{1,H} \int_{\mathbb{R}_+ \times \mathbb{R}} \mathcal{F}\phi(s,\xi) \overline{\mathcal{F}\psi(s,\xi)} |\xi|^{1-2H} d\xi ds \\ &= c_{2,H} \int_{\mathbb{R}_+ \times \mathbb{R}} D_-^{\frac{1}{2}-H} \phi(t,x) D_-^{\frac{1}{2}-H} \psi(t,x) dx dt \\ &= c_{3,\beta}^2 \int_{\mathbb{R}^2} [\phi(x+y) - \phi(x)][\psi(x+y) - \psi(x)] |y|^{2H-2} dx dy \,, \end{aligned}$$

where

$$\begin{aligned}
c_{1,H} &= \frac{1}{2\pi}\Gamma(2H+1)\sin(\pi H)\,; \\
c_{2,H} &= \left[\Gamma\left(H+\frac{1}{2}\right)\right]^2 \left(\int_0^\infty \left[(1+t)^{H-\frac{1}{2}} - t^{H-\frac{1}{2}}\right]^2 dt + \frac{1}{2H}\right)^{-1}\,; \\
c_{3,\beta}^2 &= (\frac{1}{2}-\beta)\beta c_{2,\frac{1}{2}-\beta}^{-1}\,.
\end{aligned}$$

The space $D(\mathbb{R}_+ \times \mathbb{R})$ is dense in $\mathbb{H}$.

Definition

An elementary process g is a process of the following form

$$g(t,x) = \sum_{i=1}^{n} \sum_{j=1}^{m} X_{i,j} \mathbf{1}_{(a_i,b_i]}(t) \mathbf{1}_{(h_j,l_j]}(x),$$

where n and m are finite positive integers, $-\infty < a_1 < b_1 < \cdots < a_n < b_n < \infty$, $h_j < l_j$ and $X_{i,j}$ are $\mathcal{F}_{a_i}$-measurable random variables for $i = 1, \ldots, n$. The stochastic integral of such an elementary process with respect to W is defined as

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} g(t,x) W(dx,dt) = \sum_{i=1}^{n} \sum_{j=1}^{m} X_{i,j} W(\mathbf{1}_{(a_i,b_i]} \otimes \mathbf{1}_{(h_j,l_j]})$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} X_{i,j} \big[W(b_i,l_j) - W(a_i,l_j) - W(b_i,h_j) + W(a_i,h_j) \big].$$

Definition

Let Λ_H be the space of predictable processes g defined on $\mathbb{R}_+ \times \mathbb{R}$ such that almost surely $g \in \mathbb{H}$ and $\mathbb{E}[\|g\|_{\mathbb{H}}^2] < \infty$. Then, the space of elementary processes defined as above is dense in Λ_H.

For $g \in \Lambda_H$, the stochastic integral $\int_{\mathbb{R}_+ \times \mathbb{R}} g(t,x) W(dx,dt)$ is defined as the $L^2(\Omega)$-limit of stochastic integrals of the elementary processes approximating $g(t,x)$ in Λ_H, and we have the following isometry equality

$$\mathbb{E}\left(\left[\int_{\mathbb{R}_+ \times \mathbb{R}} g(t,x) W(dx,dt)\right]^2\right) = \mathbb{E}\left(\|g\|_{\mathbb{H}}^2\right)$$
$$= c_{3,H}^2 \int_0^\infty \int_{\mathbb{R}^2} \mathbb{E}|g(t,x+y) - g(t,x)|^2 |y|^{2H-2} dxdydt\,.$$

Definition (Strong solution)

$u(t,x)$ is a *strong (mild random field) solution* if for all $t \in [0,T]$ and $x \in \mathbb{R}$ the process $\{G_{t-s}(x-y)\sigma(u(s,y))\mathbf{1}_{[0,t]}(s)\}$ is integrable with respect to W, where $G_t(x) := \frac{1}{2} I_{\{|x|<t\}}$ is heat kernel, and

$$u(t,x) = I_0(t,x) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(s,y,u(s,y))W(dy,ds) \tag{1}$$

almost surely, where

$$\begin{aligned} I_0(t,x) :=& \frac{\partial}{\partial t} G_t * u_0(x) + G_t * v_0(x) \\ =& \frac{1}{2}\int_{x-t}^{x+t} v_0(y)dy + \frac{1}{2}[u_0(x+t) + u_0(x-t)] \,. \end{aligned}$$

Definition (Weak solution)

We say the spde has a *weak solution* if there exists a probability space with a filtration $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{P}}, \widetilde{\mathcal{F}}_t)$, a Gaussian noise $\widetilde{W}$ identical to W in law, and an adapted stochastic process $\{u(t,x)\,, t \geq 0\,, x \in \mathbb{R}\}$ on this probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{P}}, \widetilde{\mathcal{F}}_t)$ such that $u(t,x)$ is a strong (mild) solution with respect to $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{P}}, \widetilde{\mathcal{F}}_t)$ and $\widetilde{W}$.

Want to study the existence and uniqueness of the solution (strong or weak).

2. Main results

Solution space

Let $(B, \|\cdot\|_B)$ be a Banach space with the norm $\|\cdot\|_B$ (we take $\beta = \frac{1}{2} - H$). Let $\beta \in (0,1)$ be a fixed number. For any function $f : \mathbb{R} \to B$ denote

$$\mathcal{N}_\beta^B f(x) := \left(\int_{\mathbb{R}} \|f(x+h) - f(x)\|_B^2 |h|^{-1-2\beta} dh \right)^{\frac{1}{2}}, \qquad (2)$$

if the above quantity is finite. When $B = \mathbb{R}$, we abbreviate the notation as $\mathcal{N}_\beta f$. With this notation, the norm of the homogeneous Sobolev space $\dot{H}^\beta$ can be given by using $\mathcal{N}_\beta f$: $\|f\|_{\dot{\mathcal{H}}_\beta} = \|\mathcal{N}_\beta f\|_{L^2(\mathbb{R})}$.

We are particularly interested in the case $B = L^p(\Omega)$.

$$\mathcal{N}_{\beta,p} f(x) := \left(\int_{\mathbb{R}} \|f(x+h) - f(x)\|^2_{L^p(\Omega)} |h|^{-1-2\beta} dh \right)^{\frac{1}{2}} .$$

Definition of the solution space $\mathcal{Z}^p(T)$.

It consists of all continuous functions f from $[0, T] \times \mathbb{R}$ to $L^p(\Omega)$ such hat the following norm is finite:

$$\begin{aligned} \|f\|_{\mathcal{Z}^p(T)} &= \|f\|_{\mathcal{Z}_1^p(T)} + \|f\|_{\mathcal{Z}_2^p(T)} \\ &:= \sup_{t\in[0,T]} \left\|f(t,\cdot)\right\|_{L^p_\lambda(\Omega\times\mathbb{R})} + \sup_{t\in[0,T]} \mathcal{N}^*_{\frac{1}{2}-H} f(t) , \end{aligned} \tag{3}$$

where $\left\|f(t,\cdot)\right\|_{L^p_\lambda(\Omega\times\mathbb{R})} = \left[\int_{\mathbb{R}} \mathbb{E}[|f(t,x)|^p] dx\right]^{1/p}$ and

$$\mathcal{N}^*_{\frac{1}{2}-H} f(t) := \left[\int_{\mathbb{R}} \left\|f(t,\cdot+h) - f(t,\cdot)\right\|^2_{L^p(\Omega\times\mathbb{R})} |h|^{2H-2} dh \right]^{\frac{1}{2}} .$$

It is proved that $\mathcal{Z}_p(T)$ is a Banach space.

(1). $\sigma(t,x,u)$ is jointly continuous over $[0,T]\times\mathbb{R}^2$,
$\sigma(t,x,0)=0$.

(2). Assume

$$\sup_{t\in[0,T],x\in\mathbb{R},u\in\mathbb{R}}\left|\frac{\partial}{\partial u}\sigma(t,x,u)\right|\le C\,;$$

$$\sup_{t\in[0,T],x\in\mathbb{R},u\in\mathbb{R}}\left|\frac{\partial^2}{\partial x\partial u}\sigma(t,x,u)\right|\le C\,;$$

$$\sup_{t\in[0,T],x\in\mathbb{R}}\left|\frac{\partial}{\partial u}\sigma(t,x,u_1)-\frac{\partial}{\partial u}\sigma(t,x,u_2)\right|\le C|u_1-u_2|$$

Theorem

Assume that $\sigma(t,x,u)$ satisfies the above hypothesis and that $I_0(t,x)$ is in $\mathcal{Z}^p(T)$ for some $p > \frac{2}{4H-1}$. Then the nonlinear SWE has a unique strong solution with sample paths in $\mathcal{C}([0,T]\times\mathbb{R})$ almost surely. Moreover, for any $\gamma < H - \frac{1}{p}$, the process $u(t,x)$ is almost surely Hölder continuous of exponent γ with respect to t and x on any compact sets in $[0,T]\times\mathbb{R}$.

Theorem

If the hyperbolic Anderson model has a solution in $\mathcal{Z}^p(T)$ for some $p \geq 2$ and for some $T \geq 0$, then the Hurst parameter H must satisfy $H > 1/4$.

3. Difficulty

Naive application of Picard iteration ($v = u^{n+1}$ and $u = u^n$):

$$v(t,x) = I_0(t,x)) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(s,y,u(s,y))W(dy,ds)$$

Then following isometry equality

$$\begin{aligned}
&\mathbb{E}\left(v^2(t,x)\right) = \xi_t^2(x) \\
&\qquad + c_{3,H}^2 \int_0^t \int_{\mathbb{R}^2} \mathbb{E}|G_{t-s}(x-y-z)\sigma(s,y+z,u(s,y+z)) \\
&\qquad - G_{t-s}(x-y)\sigma(s,y,u(s,y))|^2 |z|^{2H-2} dydzds \\
&\leq \cdots + \\
&c_{3,H}^2 \int_0^t \int_{\mathbb{R}^2} \mathbb{E} G_{t-s}^2(x-y)|u(s,y+z) - u(s,y)|^2 |z|^{2H-2} dydzds
\end{aligned}$$

This means that to make the Picard iteration work we need to bound

$$\int_0^t \int_{\mathbb{R}^2} \mathbb{E} G_{t-s}^2(x-y)|u(s,y+z)-u(s,y)|^2|z|^{2H-2} dydzds$$

as well.

If $\sigma(t,x,u)=\sigma(u)$, then we may bound $\mathbb{E}\left(|v(t,x+h)-v(x)|^2\right)$ by

$$\sigma(v(s,y+h+z))-\sigma(v(s,y+h))-\sigma(v(s,y+z))+\sigma(v(s,y))$$

If we want to consider $\mathbb{E}\left(|v_1(t,x+h)-v_1(x)-v_2(t,x+h)+v_2(x)|^2\right)$ then we need to consider

the difference of

$$\sigma(v_1(s,y+h+z))-\sigma(v_1(s,y+h))-\sigma(v_1(s,y+z))+\sigma(v_1(s,y))$$

and

$$\sigma(v_2(s,y+h+z))-\sigma(v_2(s,y+h))-\sigma(v_2(s,y+z))+\sigma(v_2(s,y))$$

One difficulty is that we cannot no longer bound $|\sigma(x_1)-\sigma(x_2)-\sigma(y_1)+\sigma(y_2)|$ by a multiple of $|x_1-x_2-y_1+y_2|$ (which is possible only in the affine case).

4. Background

When $H > 1/2$ or when the noise is more regular, the equation was studied by many researchers.

R. Dalang, M. Sanz, C. Mueller, D. Nualart,

$\sigma(u) = au + b$: $H > 1/4$.

Balan, R.; Jolis, M. and Quer-Sardanyons, L.

SPDEs with affine multiplicative fractional noise in space with index $\frac{1}{4} < H < \frac{1}{2}$.

Electronic Journal of Probability 20 (2015).

Jian Song, Xiaoming Song, and Fangjun Xu

Fractional stochastic wave equation driven by a Gaussian noise rough in space.

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + \sigma(t,x,u(t,x))\dot{W}, \quad t>0, x\in\mathbb{R}\,.$$

General $\sigma(u)$ but with $\sigma(0)=0$.

Hu, Yaozhong; Huang, Jingyu; Le, Khoa; Nualart, David; Tindel, Samy

Stochastic heat equation with rough dependence in space.

Ann. Probab. 45 (2017), 4561-4616.

The condition $\sigma(t, x, 0) = 0$ is removed for the stochastic heat equation in

Hu, Y. and Wang, X.

Stochastic heat equation with general rough noise.

Ann. Inst. Henri Poincaré Probab. Stat. 58 (2022), no. 1, 379-423.

Let $u_{\mathrm{aff}}(t, x)$ be the solution to the stochastic heat equation with $\sigma(t, x, u) = 1$ and $u_0(x) = 0$:

$$\frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\Delta u(t,x) + \dot{W}, \quad t > 0, x \in \mathbb{R}\,.$$

Then, there are two positive constants c_H and C_H, independent of T and L, such that

$$c_H\,\rho(T,L) \le \mathbb{E}\left(\sup_{\substack{0\le t\le T\\ -L\le x\le L}} u_{\mathrm{aff}}(t,x)\right) \le \mathbb{E}\left(\sup_{\substack{0\le t\le T\\ -L\le x\le L}} |u_{\mathrm{aff}}(t,x)|\right) \le C_H\,\rho(T,L)\,,$$

where

$$\rho(T,L) = \begin{cases} T^{\frac{H}{2}} + T^{\frac{H}{2}}\sqrt{\log_2\left[\frac{L}{\sqrt{T}}\right]} & \text{if } L^2 > T, \\ T^{\frac{H}{2}} & \text{if } L^2 \le T\,. \end{cases}$$

5. Some ideas

We need to to use localization argument.

$$\tau_k = \inf \left\{ t \in [0, T] \, ; \sup_{0 \le s \le t \, , x \in \mathbb{R}} \left| \mathcal{N}_{\frac{1}{2}-H,p} u(s,x) \right| \ge k \right\} .$$

We need $\tau_k \uparrow T$ as $k \to \infty$. This imposes $\sigma(t,x,0) = 0$.

To show this we need to bound

$$\mathbb{E}\left[\sup_{0\le t\le T, x\in\mathbb{R}} |\mathcal{N}_{\frac{1}{2}-H,p} u(s,x|^p\right].$$

Because the equation satisfied by the mild solution u, we need to bound

$$\mathbb{E}\left[\sup_{0\le t\le T, x\in\mathbb{R}} |\int_0^s G_{s-r}(x-y)\sigma(r,y,u(r,y))W(dr,dy)|^p\right].$$

But

$$\Phi(s,x) = \int_0^s G_{s-r}(x-y)\sigma(r,y,u(r,y))W(dr,dy)$$

is not a martingale. We cannot use the Burkholder-Davis-Gundy inequality.

In the case of heat equation one can use the semigroup property of the heat kernel.

$$\Phi(t,x) = \frac{\sin(\pi\alpha)}{\pi} \int_0^t \int_{\mathbb{R}} (t-r)^{\alpha-1} G_{t-r}(x-z) Y(r,z) dz dr \,,$$

with

$$Y(r,z) = \int_0^r \int_{\mathbb{R}} (r-s)^{-\alpha} G_{r-s}(z-y) v(s,y) W(ds,dy) \,,$$

Then

$$|\Phi(t,x)| \leq C \left(\int_0^t \left(\int_{\mathbb{R}} (t-r)^{p(\alpha-1)} G_{t-r}^p(x-z) dz \right) dr \right)^{1/p} \left(\int_0^t \int_{\mathbb{R}} |Y(s,z)|^q dz dr \right)^{1/q} .$$

Of course we need to bound something like

$$|\Phi(t, x + h) - \Phi(t, x)|$$

The wave kernel $G_t(x) = \frac{1}{2}\mathbf{1}_{\{|x|<t\}}$ can be expressed as

$$
\begin{aligned}
G_{t-s}(x-y) = & \int_{\mathbb{R}} \mathcal{C}_\beta(t-r, x-z)\mathcal{S}_{1-\beta}(r-s, z-y)dz \\
& + \int_{\mathbb{R}} \mathcal{S}_\alpha(t-r, x-z)\mathcal{C}_{1-\alpha}(r-s, z-y)dz \\
& + \int_{\mathbb{R}} \mathcal{S}(t-r, x-z)\mathcal{E}(r-s, z-y)dz \\
& + \int_{\mathbb{R}} \mathcal{E}(t-r, x-z)\mathcal{S}(r-s, z-y)dz\,,
\end{aligned}
\tag{4}
$$

where $\alpha, \beta \in (0,1)$, $\mathcal{S}(t,x) = \mathcal{S}_1(t,x) = G_t(x) = \frac{1}{2}\mathbf{1}_{\{|x|<t\}}$ and

$$\begin{cases} \mathcal{E}(t,x) := \dfrac{1}{\pi}\dfrac{t}{t^2+x^2}, \\ \mathcal{S}_\alpha(t,x) := \dfrac{\Gamma(1-\alpha)}{2\pi}\cos\left[\dfrac{\alpha\pi}{2}\right]\Big((t+|x|)^{\alpha-1} \\ \qquad\qquad + \operatorname{sgn}(t-|x|)\big|t-|x|\big|^{\alpha-1}\Big), \\ \mathcal{C}_{1-\alpha}(t,x) := \dfrac{\Gamma(\alpha)}{2\pi}\Bigg[\cos\left[\dfrac{\alpha\pi}{2}\right]\left[\big|t+|x|\big|^{-\alpha} + \big|t-|x|\big|^{-\alpha}\right] \\ \qquad\qquad - 2\cos\left[\alpha\tan^{-1}\left[\dfrac{|x|}{t}\right]\right][t^2+x^2]^{-\frac{\alpha}{2}}\Bigg]. \end{cases} \tag{5}$$

For any $\theta \in (0,1)$ and $i = 1,2,3,4$, set

$$J_\theta^{\mathcal{K}_i}(r,z) := \int_0^r \int_{\mathbb{R}} (r-s)^{-\theta} \mathcal{K}_i(r-s, z-y) v(s,y) W(dy, ds)\,, \tag{6}$$

where

$$\mathcal{K}_1 = \mathcal{C}_\alpha,\ \mathcal{K}_2 = \mathcal{S}_\alpha,\ \mathcal{K}_3 = \mathcal{S},\ \text{and } \mathcal{K}_4 = \mathcal{E}.$$

And we define $\bar{\mathcal{K}}_i$ to be the complements of $\mathcal{K}_i$ according to (4), namely,

$$\bar{\mathcal{K}}_1 = \mathcal{S}_{1-\alpha},\ \bar{\mathcal{K}}_2 = \mathcal{C}_{1-\alpha},\ \bar{\mathcal{K}}_3 = \mathcal{E},\ \text{and } \bar{\mathcal{K}}_4 = \mathcal{S}.$$

$$\begin{aligned}
\Phi(t,x) &= \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)v(s,y)W(ds,dy) \\
&= \frac{\sin(\theta\pi)}{\pi} \int_0^t \int_{\mathbb{R}} \int_s^t (t-r)^{\theta-1}(r-s)^{-\theta} dr \qquad (7) \\
&\qquad \times G_{t-s}(x-y)v(s,y)W(dy,ds) \\
&= \sum_{i=1}^{4} \frac{\sin(\theta\pi)}{\pi} \int_0^t \int_{\mathbb{R}} \int_s^t \int_{\mathbb{R}} (t-r)^{\theta-1}(r-s)^{-\theta} \bar{\mathcal{K}}_i(t-r,x-z) \\
&\qquad \times \mathcal{K}_i(r-s,z-y) dz dr \times v(s,y)W(dy,ds) \\
&= \sum_{i=1}^{4} \frac{\sin(\theta\pi)}{\pi} \int_0^t \int_{\mathbb{R}} (t-r)^{\theta-1} \bar{\mathcal{K}}_i(t-r,x-z) J_\theta^{\mathcal{K}_i}(r,z) dz dr \\
&= \sum_{i=1}^{4} \Phi_i(t,x)\,, \qquad (8)
\end{aligned}$$

where we have applied the identity

$$\int_s^t (t-r)^{\theta-1}(r-s)^{-\theta}dr = \frac{\pi}{\sin(\theta\pi)}\,, \quad \theta \in (0,1)\,, 0 \le s \le t\,.$$

By the Hölder inequality with $1/p + 1/q = 1$

$$
\begin{aligned}
&\sup_{0\le t\le T, x\in\mathbb{R}} |\Phi_i(t,x)| \\
&\lesssim \sup_{t,x} \int_0^t (t-r)^{\theta-1} \left[\int_{\mathbb{R}} |\bar{\mathcal{K}}_i(t-r, x-z)|^q dz\right]^{\frac{1}{q}} \\
&\qquad\qquad\qquad\qquad \times \|J_\theta^{\mathcal{K}_i}(r,z)\|_{L^p(\mathbb{R})} dr \\
&\lesssim \left[\sup_t \int_0^t \int_{\mathbb{R}} r^{q(\theta-1)} |\bar{\mathcal{K}}_i(r,z)|^q dz dr\right]^{\frac{1}{q}} \\
&\qquad\qquad\qquad \times \left[\int_0^T \|J_\theta^{\mathcal{K}_i}(r,z)\|^p_{L^p(\mathbb{R})} dr\right]^{\frac{1}{p}} \\
&= (I_i^{(1)})^{1/q} \times (I_i^{(2)})^{1/p}, \qquad (9)
\end{aligned}
$$

$$
\begin{aligned}
I_1^{(1)} &= \sup_t \int_0^t \int_{\mathbb{R}} r^{q(\theta-1)} |\mathcal{S}_{1-\alpha}(r,z)|^q dz dr \\
&\lesssim \left(\sup_t \int_0^t r^{q\left[\theta-1-\alpha+\frac{1}{q}\right]} dr \right) \\
&\quad \times \int_0^\infty \left| [1+|z|]^{-\alpha} + \operatorname{sign}(1-|z|) \big|1-|z|\big|^{-\alpha} \right|^q dz\,.
\end{aligned}
$$

In order to make sure the above integrals converge, we need

$$
\alpha q < 1\,,\ (\alpha+1)q > 1 \quad \Leftrightarrow \quad 0 < \alpha < \frac{1}{q} = 1 - \frac{1}{p}\,, \tag{10}
$$

and also

$$
q\left[\theta - \alpha - 1 + \frac{1}{q}\right] > -1 \quad \Leftrightarrow \quad \theta > 1 - \frac{2}{q} + \alpha\,. \tag{11}
$$

Do the same for other ones.

Use notation $\mathfrak{D}_h\Phi(t,x) := \Phi(t,x+h) - \Phi(t,x)$ and same notations for $\mathfrak{D}_h\bar{\mathcal{K}}_i(t-r,z)$, $\mathfrak{D}_h J_\theta^{\mathcal{K}_i}(r,z)$. Then

$$\mathfrak{D}_h\Phi(t,x) = \frac{\sin(\theta\pi)}{\pi}\sum_i \int_0^t \int_{\mathbb{R}} (t-r)^{\theta-1}\mathfrak{D}_h\bar{\mathcal{K}}_i(t-r,x-z) J_\theta^{\mathcal{K}_i}(r,z)dzdr$$
$$\simeq \sum_i \int_0^t \int_{\mathbb{R}} (t-r)^{\theta-1}\bar{\mathcal{K}}_i(t-r,x-z)\mathfrak{D}_h J_\theta^{\mathcal{K}_i}(r,z)dzdr\,, \tag{12}$$

By Minkowski's inequality and then Hölder's inequality we get

$$\sup_{t,x}\left[\int_{\mathbb{R}} |\mathfrak{D}_h\Phi(t,x)|^2|h|^{2H-2}dh\right]^{\frac{1}{2}}$$
$$\lesssim \sup_{t,x}\sum_i \Big(\int_{\mathbb{R}} \Big| \int_0^t \int_{\mathbb{R}} (t-r)^{\theta-1}\bar{\mathcal{K}}_i(t-r,x-z) \tag{13}$$

$$\times \mathfrak{D}_h J_\theta^{\mathcal{K}_i}(r,z)dzdr\Big|^2 \cdot |h|^{2H-2}dh\Big)^{\frac{1}{2}}$$

$$\lesssim \sup_{t,x} \sum_i \int_0^t \int_{\mathbb{R}} (t-r)^{\theta-1} |\bar{\mathcal{K}}_i(t-r, x-z)|$$

$$\times \left[\int_{\mathbb{R}} \left|\mathfrak{D}_h J_\theta^{\mathcal{K}_i}(r,z)\right|^2 |h|^{2H-2}dh\right]^{\frac{1}{2}} dzdr$$

$$\lesssim \sum_i \left(\sup_t \int_0^t \int_{\mathbb{R}} r^{q(\theta-1)} \left|\bar{\mathcal{K}}_i(r,z)\right|^q dzdr\right)^{\frac{1}{q}}$$

$$\times \left[\int_0^T \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left|\mathfrak{D}_h J_\theta^{\mathcal{K}_i}(r,z)\right|^2 |h|^{2H-2}dh\right)^{\frac{p}{2}} dzdr\right]^{\frac{1}{p}}$$

$$=: (J_i^{(1)})^{\frac{1}{q}} \times (J_i^{(2)})^{\frac{1}{p}} .$$

The first factor $(J_i^{(1)})^{\frac{1}{q}}$ is finite under some appropriate choice of parameters.

The proof of the following is more involved:

$$\mathbb{E}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\left|\mathfrak{D}_h J_\theta^{\mathcal{K}_i}(r,z)\right|^2 |h|^{2H-2} dh\right)^{\frac{p}{2}} dzdr \lesssim \|v\|_{\mathcal{Z}^p(T)}^p,$$

under the conditions

$$p > \frac{1}{H},\ 1-2/q+\alpha < \theta < 2H+\alpha-1,\ \ \frac{3}{2}-2H < \alpha < 1-\frac{1}{p}.$$

THANKS