Some extensions of the C^1–Itô's formula and their applications in finance and optimal control

Xiaolu Tan

The Chinese University of Hong Kong

Based on joint works with

Bruno Bouchard, Grégoire Loeper, Jixin Wang

12th January 2024

International Seminar on SDEs and Related Topics (online)

Outline

1. Introduction
   - Itô's formula and some of its applications
   - Some extensions of Itô's formula
2. The C^1-Itô's formulas
   - Itô's calculus via regularization and C^1-Itô's formula
   - An extension to the path-dependent setting
   - An extension to the measure-valued functional setting
3. Applications of the C^1-Itô's formula
   - Option pricing and replication
   - Super-hedging in the incomplete market
   - The verification theorem

Itô's formula

• Let $X = (X_t)_{t\geq 0}$ be a (continuous) semi-martingale with decomposition $X = A + M$, and $u : \mathbb{R}_+ \times \mathbb{R}^d \longrightarrow \mathbb{R}$ be in $C^{1,2}$, then

$$\begin{aligned} u(t, X_t) = u(0, X_0) &+ \int_0^T Du(s, X_s) dM_s^X \\ &+ \int_0^t \partial_t u(s, X_s) dt + \int_0^t Du(s, X_s) dA_s^X \\ &+ \int_0^t \frac{1}{2} D^2 u(s, X_s) d\langle X \rangle_s. \end{aligned}$$

Application 1 : Option pricing and replication

- Financial market with underlying $B = (B_t)_{t\in[0,T]}$, which is a Brownian motion, the interest rate $r = 0$. We consider a derivative option with payoff $g(B_T)$.

- Option pricing and replication : let

$$u(t,x) \;:=\; \mathbb{E}\big[g(B_T)\big|B_t = x\big], \;(t,x) \in [0,T] \times \mathbb{R}^d,$$

it solves the heat equation

$$\partial_t u + \frac{1}{2}\Delta^2 u = 0.$$

Application 1 : Option pricing and replication

Theorem

Assume that $u(t,x) := \mathbb{E}[g(B_T)|B_t = x] \in C^{1,2}([0,T]\times\mathbb{R})$. Then one can replicate the option $g(B_T)$ with initial wealth $u(0,B_0)$ and dynamic trading strategy $Du(t,B_t)$.

Then by Itô's formula,

$$\begin{aligned} du(t,B_t) &= Du(t,B_t)dB_t + \Big(\partial_t u + \frac{1}{2}\Delta u\Big)(t,B_t)dt \\ &= Du(t,B_t)dB_t. \end{aligned}$$

Then one obtain a self-financial portfolio to replicate the option :

$$g(B_T) = u(0,B_0) + \int_0^T H_t dB_t, \quad H_t = Du(t,B_t).$$

Application 2 : Optimal control and verification theorem

- An optimal control problem

$$\sup_{\alpha} \mathbb{E}\Big[\int_0^T L(X_t, \alpha_t) dt + g(X_T) \Big],$$

with

$$dX_t = \alpha_t dt + dW_t.$$

- The value function $v : [0, T] \times \mathbb{R}^d$ solves the HJB equation

$$\partial_t v(t,x) + \frac{1}{2} \Delta v(t,x) + H\big(x, Dv(t,x)\big) = 0, \quad v(T, \cdot) = g(\cdot),$$

with Hamiltonian

$$H(x,p) := \sup_a \big(L(x,a) + a \cdot p \big).$$

Application 2 : Optimal control and verification theorem

Theorem (Verification Theorem)

Let $v : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}$ be a smooth solution to the HJB equation, and $\hat{a}(t,x)$ be the optimizer in the definition of the Hamiltonian, then $\hat{a}(t,X_t)$ is an optimal (feedback) control.

- Proof : By Itô's formula

$$v(T,X_T) = v(0,X_0) + \int_0^T \partial_t v(t,X_t) + \frac{1}{2}\Delta v(t,X_t) + Dv(t,X_t)\cdot\alpha_t dt + Dv(t,X_t)dW_t.$$

$$\Rightarrow \; v(0,X_0) = g(X_T) + \int_0^T L(X_t,\alpha_t)dt - \int_0^T Dv(t,X_t)dW_t + \int_0^T \Big(H(\cdot, Dv(\cdot)) - \alpha_t Dv - L(\cdot,\alpha_t)\Big)(t,X_t)dt.$$

Application 3 : Super-hedging in incomplete market

• We consider an uncertain volatility model : let $\Omega := C([0,T],\mathbb{R})$ be the canonical space with canonical process X,

$$\mathcal{M}(t,x) := \{\mathbb{P} \ : X_t = x,\ dX_s = \sigma_s dW_s,\ \underline{a} \le \sigma_s^2 \le \overline{a},\ \mathbb{P}\text{-a.s.}\}.$$

Let $g(X_T)$ be the payoff of some derivatives, we define

$$v(t,x) \ := \sup_{\mathbb{P}\in\mathcal{M}(t,x)} \mathbb{E}^{\mathbb{P}}[g(X_T)].$$

Then v solves the HJB equation

$$\partial_t v + H(D^2 v) = 0, \ \text{ with } \ H(D^2 v) := \frac{1}{2} \sup_{\underline{a}\le a\le\overline{a}} a\ D^2 v.$$

Application 3 : Super-hedging in incomplete market

Theorem (Pricing-hedging duality)

One has the pricing-hedging duality :

$$v(0,x_0) = \inf\left\{ y \ : y + \int_0^T H_s dX_s \ge g(X_T),\ \mathbb{P}\text{-}a.s.\ \forall \mathbb{P} \in \mathcal{M}(0,x_0) \right\}.$$

and the optimal (dynamic) super-replication strategy is $Dv(t,X_t)$.

- By Itô's formula,

$$\begin{aligned} g(X_T) = & v(T,X_T) = v(0,x_0) + \int_0^T Dv(t,X_t) dX_t \\ & + \int_0^T \left(\partial_t v + \frac{1}{2}\sigma_t^2 D^2 v(t,X_t) \right) dt \\ \le & \ v(0,X_0) + \int_0^T Dv(t,X_t) dX_t \end{aligned}$$

Itô's formula of path-dependent functionals

• Let $\Omega := D(\mathbb{R}_+, \mathbb{R}^d)$ be the space of càdlàg paths, for a path-dependent functional $F : \mathbb{R}_+ \times \Omega \longrightarrow \mathbb{R}$, we introduce the horizontal derivative

$$\partial_t F(t,\omega) \;:=\; \lim_{h \searrow 0} \frac{F(t+h, \omega_{t\wedge \cdot}) - F(t,\omega)}{h},$$

and the vertical derivative

$$\partial_\omega F(t,\omega) \;:=\; \lim_{x \to 0} \frac{F(t, \omega \oplus_t x) - F(t,\omega)}{x},$$

and the similarly the second order vertical derivative $\partial^2_{\omega\omega} F(t,\omega)$.

Itô's formula of path-dependent functionals

- Let $X = A^X + M^X$ be a continuous semi-martingale and $F : \Omega \longrightarrow \mathbb{R}$ be in $C^{1,2}$. Then

$$dF(t,X) = \partial_t F(t,X)dt + \partial_\omega F(t,X)dA_t^X + \frac{1}{2}\partial^2_{\omega\omega}F(t,X)d\langle M^X\rangle_t + \partial_\omega F(t,X)dM_t^X.$$

- Dupire (2009), Cont and Fournié (2013), etc.

- Peng (2010), Ekren, Keller, Touzi, Zhang (2014), etc.

Itô's formula along flows of conditional measures

• Let X be semi-martingale with $dX_t = \alpha_t dt + \sigma dW_t + \sigma_0 dB_t$, and define

$$\rho_t \ := \ \mathcal{L}(X_t|B), \ \ t \ge 0.$$

Let $F : \mathbb{R}_+ \times \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{R}$ be in $C^{1,2}$. Then

$$\begin{aligned} dF(t,\rho_t) = & \Big(\partial_t F(t,\rho_t) + \mathbb{E}\big[\alpha_t \cdot D_m F(t,\rho_t,X_t) \big| B\big] \\ & \quad + \ \frac{1}{2}(\sigma^2 + \sigma_0^2)\rho_t\big(\partial_x D_m F(t,\rho_t,x)\big) \\ & \quad + \ \frac{1}{2} \ \sigma_0^2 \ \rho_t \otimes \rho_t \big(D_m^2 F(t,\rho_t,x,x')\big) \ \Big) dt \\ & + \ \mathbb{E}\big[\sigma_0 D_m F(t,\rho_t,X_t) \big| B\big] \, dB_t. \end{aligned}$$

See e.g. Buckdahn, Li, Peng and Rainer (2017), Carmona and Delarue (2018).

Outline

1. Introduction
   - Itô's formula and some of its applications
   - Some extensions of Itô's formula
2. The C^1-Itô's formulas
   - Itô's calculus via regularization and C^1-Itô's formula
   - An extension to the path-dependent setting
   - An extension to the measure-valued functional setting
3. Applications of the C^1-Itô's formula
   - Option pricing and replication
   - Super-hedging in the incomplete market
   - The verification theorem

Itô calculus via regularization (Russo, Vallois, etc.)

• Let X be a càdlàg process, $H \in L^1([0,T])$, the forward integral of H w.r.t. X is defined by

$$\int_0^t H_s dX_s := \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_0^t H_s (X_{(s+\varepsilon)\wedge t} - X_s) ds, \quad t \geq 0.$$

• Let X and Y be two càdlàg processes, the co-quadratic variation $[X, Y]$ is defined by

$$[X, Y]_t := \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_0^t (X_{(s+\varepsilon)\wedge t} - X_s)(Y_{(s+\varepsilon)\wedge t} - Y_s) ds.$$

• The limits are defined in sense of "uniformly on compacts in probability" (u.c.p.).

When X and Y are càdlàg semimartingales and H is càdlàg and adapted, they are well defined and coincide with the usual Itô integral.

Itô calculus via regularization (Russo, Vallois, etc.)

- Weak Dirichlet process :
  - A càdlàg process A is called is called (martingale) orthogonal (with weak zero energy), if $[A, N] = 0$ for all continuous martingale N.
  - A càdlàg process X is called a weak Dirichlet process if it has the decomposition

$$X_t = X_0 + M_t + A_t,$$

where M is a local martingale, A is (martingale) orthogonal.

Theorem (e.g. Gozzi and Russo (2006))

Let $X = M + A$ be a continuous weak Dirichlet process and $f \in C^{0,1}([0, T] \times \mathbb{R}^d)$, then $f(t, X_t)$ is also a weak Dirichlet process with the (unique) decomposition

$$f(t, X_t) = \int_0^t Df(s, X_s) \cdot dM_s + \Gamma_t.$$

Extension 1 : A C^1-functional Itô's formula

Theorem (Bouchard, Loeper and Tan (2022))

Let $X = M + A$ be a continuous weak Dirichlet process and $F \in C^{0,1}([0,T] \times \Omega)$. Under an additional continuity condition, $F(t,X)$ is also a weak Dirichlet process with the (unique) decomposition

$$F(t,X) = \int_0^t \partial_\omega F(s,X) dM_s + \Gamma_t^F.$$

• Motivation and applications (in mathematical finance) : pricing, hedging and super-hedging of path-dependent options.

• For càdlàg weak Dirichlet processes, a C^1-Itô's formula is provided in Bouchard and Vallet (2021).

Extension 2 : A C^1-Itô formula along conditional measures

Let $Y_t = A_t^Y + M_t^Y$ be a weak Dirichlet process, and $X_t = X_0 + A_t + M_t + \int_0^t \sigma_s^\circ dM_s^\circ$ be a continuous semi-martingale. Assume that the sub-filtration $(\mathcal{G}_t)_{t\geq 0}$ generated by M° satisfies the (H)-Hypothesis, and

$$m_t := \mathcal{L}(X_t|\mathcal{G}_t), \;\; t \geq 0.$$

Theorem (Bouchard, T. and Wang, 2023)

Let $F : \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \longrightarrow \mathbb{R}$ lie in $C^{0,1,1}$, and assume some local square-integrability conditions. Then $F(t, Y_t, m_t)$ is a weak Dirichlet process with the (unique) decomposition :

$$F(t, Y_t, m_t) = \int_0^t D_y F(s, Y_s, m_s) dM_s^Y + \int_0^t \mathbb{E}\big[D_m F(s, y, m_s, X_s)\sigma_s^\circ \big| \mathcal{G}_s\big]_{y=Y_s} dM_s^\circ + \Gamma_t.$$

Technical proofs

- Step 1 : define

$$\begin{aligned}\Gamma_t \ := F(t, Y_t, m_t) &- \int_0^t D_y F(s, Y_s, m_s) dM_s^Y \\ &- \int_0^t \mathbb{E}\Big[D_m F(s, y, m_s, X_s)\sigma_s^\circ \Big| \mathcal{G}_s \Big]_{y=Y_s} dM_s^\circ .\end{aligned}$$

Step 2 : check that, for any continuous martingale N,

$$[\Gamma, N]_t := \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_0^t (\Gamma_{s+\varepsilon} - \Gamma_s)(N_{s+\varepsilon} - N_s) ds = 0.$$

Outline

1. Introduction
    - Itô's formula and some of its applications
    - Some extensions of Itô's formula
2. The C^1-Itô's formulas
    - Itô's calculus via regularization and C^1-Itô's formula
    - An extension to the path-dependent setting
    - An extension to the measure-valued functional setting
3. Applications of the C^1-Itô's formula
    - Option pricing and replication
    - Super-hedging in the incomplete market
    - The verification theorem

Application 1 : Option pricing and replication

Theorem (Replication of options)

Assume that $u(t,x) := \mathbb{E}[g(B_T)|B_t = x] \in C^{0,1}([0,T] \times \mathbb{R})$. Then one can replicate the option $g(B_T)$ with initial wealth $u(0,B_0)$ and dynamic trading strategy $Du(t,B_t)$.

Proof : 1. By C^1-Itô's formula,

$$u(t,B_t) = u(0,B_0) + \int_0^t Du(s,B_s)dB_s + \Gamma_t,$$

where Γ satisfies $[\Gamma, N] = 0$ for any continuous martingale N.

2. By its definition, $(u(t,B_t))_{t\in[0,T]}$ is a martingale, so $\Gamma_t \equiv 0$.

Application 3 : Super-hedging in incomplete market

- Recall that $\Omega := C([0,T],\mathbb{R})$ is the canonical space, X is the canonical process, and

$$\mathcal{M}(t,x) := \left\{\mathbb{P} \ : X_t = x,\ dX_s = \sigma_s dW_s,\ \underline{a} \le \sigma_s^2 \le \bar{a},\ \mathbb{P}\text{-a.s.}\right\}.$$

With the payoff $g(X_T)$ of the derivative, we define

$$v(t,x) \ := \sup_{\mathbb{P}\in\mathcal{M}(t,x)} \mathbb{E}^{\mathbb{P}}[g(X_T)].$$

Theorem (Pricing-hedging duality)

If $v \in C^{0,1}$, then

$$v(0,x_0) = \inf\left\{y \ : y + \int_0^T H_s dX_s \ge g(X_T),\ \mathbb{P}\text{-a.s.}\ \forall \mathbb{P}\in\mathcal{M}(0,x_0)\right\}.$$

and the optimal (dynamic) super-replication strategy is $Dv(t,X_t)$.

Application 3 : Super-hedging in incomplete market

• Step 1 : By C^1-Itô's formula, for any $\mathbb{P} \in \mathcal{M}(0, x_0)$,

$$v(t, X_t) = v(0, x_0) + \int_0^t Dv(s, X_s) dX_s + \Gamma_t, \ \ t \in [0, T] \ \mathbb{P}\text{-a.s.}$$

• Step 2 : By dynamic programming principle, $(v(t, X_t))_{t \in [0,T]}$ is a $\mathbb{P}$-super-martingale for all $\mathbb{P} \in \mathcal{M}(0, x_0)$, so that by Doob-Meyer decomposition,

$$v(t, X_t) = v(0, x_0) + M_t - A_t, \ \ t \in [0, T], \mathbb{P}\text{-a.s.}$$

Therefore, $\Gamma_t = -A_t \leq 0$, $\mathbb{P}$-a.s. for all $\mathbb{P} \in \mathcal{M}(0, x_0)$, and hence

$$g(X_T) \ \leq \ v(0, x_0) + \int_0^T Dv(s, X_s) dX_s, \ \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \mathcal{M}(0, x_0).$$

Applications : replication and super-replication of path-dependent options

- The above two theorems can extended to the path-dependent setting with the C^1-functional Itô's formula, in order to find the replication or super-replication strategy of the path-dependent options.

- This is also the initial intuition and motivation in Dupire (2009) to introduce the horizontal and vertical derivatives of the path-dependent functionals.

Application 2 : Verification theorem by C^1-Itô's formula

- A classical optimal control problem under weak formulation :

$$\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\Big[\int_t^T L(t, X_t, \alpha_t^{\mathbb{P}})dt + g(X_T)\Big],$$

with

$$dX_t = \alpha_t^{\mathbb{P}} dt + dW_t^{\mathbb{P}}, \ \mathbb{P}\text{-a.s.}$$

- Define the value function $V(t,x)$ by

$$V(t,x) \ := \sup_{\mathbb{P}\in\mathcal{P}(t,x)} \mathbb{E}^{\mathbb{P}}\Big[\int_t^T L(t, X_t, \alpha_t^{\mathbb{P}})dt + g(X_T)\Big].$$

Under mild conditions, V is a (viscosity) solution to the corresponding HJB equation.

Application 2 : Verification theorem by C^1-Itô's formula

Theorem

Assume that $V \in C^{0,1}$, *and* $\widehat{\mathbb{P}}$ *be such that* $\alpha_t^{\widehat{\mathbb{P}}} = \hat{a}(t, X_t)$, *where*

$$\hat{a}(t,x) := \arg\max_a \big(L(t,x,a) + a \cdot DV(t,x) \big).$$

Then $\widehat{\mathbb{P}}$ *is an optimal control.*

- Dual Formulation 1 :

$$D_1 := \inf \Big\{ v_0 \ : v_0 + \int_0^T \phi_t dX_t \ge g(X_T) + \int_0^T \big(\alpha_s^{\mathbb{P}} \phi_s + L(\alpha_s^{\mathbb{P}}, X_s)\big) ds, \quad \mathbb{P}\text{-a.s. for all } \mathbb{P} \Big\},$$

- Dual Formulation 2 :

$$D_2 := \inf \Big\{ v_0 \ : v_0 + \int_0^T \phi_t dX_t \ge g(X_T) + \int_0^T H(t, X_t, \phi_t) dt, \ \text{a.s.} \Big\}.$$

Application 2 : Verification theorem by C^1-Itô's formula

- Step 1 of the Proof : First, one has the weak duality :

$$V(0, x_0) \leq D_1 \leq D_2.$$

Next, $V(t, X_t) + \int_0^t L(s, X_s, \alpha_s^{\mathbb{P}}) ds$ is a $\mathbb{P}$-super-martingale by the dynamic programming principle. Its Doob-Meyer decomposition coincides with the C^1-Itô's formula, so that

$$V(0, x_0) + \int_0^T DV(t, X_t) dX_t \geq g(X_T) + \int_0^T \big(L(X_t, \alpha_t^{\mathbb{P}}) + DV(t, X_t) \cdot \alpha_t^{\mathbb{P}} \big) dt.$$

This proves the duality :

$$V(0, x_0) = D_1.$$

By the optimality of $\alpha^{\widehat{\mathbb{P}}}$, one further has

$$V(0, x_0) = D_1 = D_2.$$

Application 2 : Verification theorem by C^1-Itô's formula

• Step 2 of the Proof : One can identify that the optimizer of D_2 is given by $(v_0, \phi) = (V(0, x_0), DV(\cdot, X_\cdot))$. By a classical duality argument, one must have

$$V(0, x_0) + \int_0^T DV(t, X_t) dX_t \; = \; g(X_T) + \int_0^T H(t, X_t, DV(t, X_t)) dt$$

$$= g(X_T) + \int_0^T \big(L(t, X_t, \alpha_t^{\widehat{\mathbb{P}}}) + DV(t, X_t) \cdot \alpha_t^{\widehat{\mathbb{P}}} \big) dt, \; \widehat{\mathbb{P}}\text{-a.s.}$$

This shows that $\widehat{\mathbb{P}}$ is an optimizer, which concludes the proof.

Verification theorem by C^1-Itô's formula in the McKean-Vlasov setting

- A McKean-Vlasov optimal control problem :

$$\sup_{\alpha} \mathbb{E}\Big[\int_0^T L(t, \rho_t, \alpha_t)dt + g(\rho_T)\Big],$$

where

$$\rho_t := \mathcal{L}(X_t|B), \quad dX_t = \alpha_t dt + \sigma dW_t + \sigma_0 dB_t.$$

- The value function becomes a function of probability measures

$$(t, \mu) \in [0, T] \times \mathcal{P}(\mathbb{R}^d) \longmapsto V(t, \mu) \in \mathbb{R}.$$

Verification theorem by C^1-Itô's formula in the McKean-Vlasov setting

- The value function V satisfies the HJB equation becomes

$$\partial_t V(t,\mu) + \mathbb{L}V(t,\mu) + H\big(t,\mu,D_m V(t,\mu,\cdot)\big) = 0,$$

with Hamiltonian

$$H\big(t,\mu,D_m V(t,\mu,\cdot)\big) := \sup_a \Big\{ L(t,\mu,a) + a\mu\big(D_m V(t,\mu,\cdot)\big) \Big\}.$$

Theorem

Assume that $V \in C^{0,1}$ and satisfies some growth condition, then the optimizer $\hat{a}(t,\mu)$ in the definition of the Hamiltonian

$$H\big(t,\mu,D_m V(t,\mu,\cdot)\big) := \sup_a \Big\{ L(t,\mu,a) + a\mu\big(D_m V(t,\mu,\cdot)\big) \Big\}.$$

gives the optimal (feedback) control.

Further works : regularity of the value function

• Regularity of path-dependent functionals (solutions to the path-dependent PDEs)

- Bouchard B., Loeper G. and Tan X., Approximate viscosity solutions of path-dependent PDEs and Dupire's vertical differentiability, Annals of Applied Probability, to appear.
- Bouchard B. and Tan X., On the regularity of solutions of some linear parabolic path-dependent PDEs, in preparation.

• Regularity of functionals on Wasserstein space (solution to the master equations) : Cardaliaguet, Delarue, Lasry and Lions (2019), Gangbo, Mészáros, Mou and Zhang (2022).